

**An Adaptive Gauss Method for Computing
Irradiance Coefficients of Galerkin Radiosity
Systems**

Reid Gershbein

Research Report CS-TR-485-95
February 1995

An Adaptive Gauss Method For Computing Irradiance Coefficients of Galerkin Radiosity Systems

Reid Gershbein
Department Of Computer Science
Princeton University

1 Abstract

Computing energy transfer between objects is the most expensive operation in radiosity systems. This energy transfer operation, known as the irradiance operator, is an integral that, in general, must be calculated numerically. We wish to increase the speed of this computation without severely compromising fidelity and perform a study of numerical integration techniques, *Quadrature Methods*. The results of our study show the strengths of *Gauss Quadrature Rules* and give us insights into greatly reducing the cost of the irradiance operator while maintaining accuracy. An adaptive method for choosing Gauss quadrature rules is presented, and our performance analysis of the new adaptive algorithm shows that it can be up to 10 times faster than previous methods.

2 Introduction

The radiosity equation can be written as a simplification of the rendering equation, introduced by Kajiyama [8], by assuming that all surfaces are perfectly diffuse (Lambertian):

$$B(x) = B^e(x) + \rho(x) \int dA_{x'} G(x, x') \pi^{-1} B(x')$$

where $B(x)$ is the radiosity at point x , consisting of emitted and reflected radiosity, $\rho(x)$ is the reflectance, and $G(x, x') = \frac{\cos\theta_x \cos\theta_{x'}}{\|x-x'\|^2} V(x, x')$ characterizes the radiant coupling between points x and x' . G accounts for relative surface orientation, distance, and visibility, $V = 0$ or $V = 1$, depending on whether x can or cannot see x' . The integral is taken over the hemisphere about x and represents the amount of energy per unit area received from other surfaces, irradiance.

Radiosity algorithms are usually based on finite element methods. It is assumed that all functions will be represented in the chosen orthonormal basis

$\{M_i\}_{i=1,\dots,n}$. We derive the linear system

$$B_i = B_i^e + \sum_{j=1}^n K_{ij} B_j$$

$$K_{ij} = \int dx \int dA_{x'} \rho(x) G(x, x') \pi^{-1} M_i(x) M_j(x') \quad (1)$$

B_i is coefficient of the radiosity projection onto the basis, B_i^e is the emission function's projection, and K_{ij} is the global irradiance projection. Here, we consider $\rho(x)$ to be constant since it can be removed from the global irradiance integral K_{ij} and applied locally as shown by Gershbein *et al.* [3]. The computation of K_{ij} is by far the most expensive operation of any radiosity system.

There have been many studies of finite-element methods for radiosity [4, 1, 6, 7, 17, 18, 5, 11, 14], and the irradiance projection. A survey of methods for computing the irradiance integral for a constant basis (called the form factor) can be found in Cohen and Wallace's *Radiosity and Realistic Image Synthesis* [2]. For non-constant bases, the method of choice has been Gauss quadrature product rules, except when there are singularities in the kernel [18, 17, 5, 12, 10]. Unfortunately, for an order N basis an $N + 1$ point Gauss rule is applied to each variable and the 4D product rule results in $(N + 1)^4$ kernel evaluations.

Gauss rules have been used for a number of reasons. A polynomial of degree $2N - 1$ can be exactly integrated with N function evaluations by a Gauss rule, and multivariate integration rules can easily be constructed from tensor products of univariate rules. When a degree N basis is used, the irradiance coupling function $G(x, x')$ is assumed to be a degree N polynomial and the integrand in (1) is at most a degree $2N$ polynomial along each axis since it is simply the product of a basis function and G . Therefore, an $N + 1$ point Gauss rule is needed for each variable in order to integrate this degree $2N$ function. Of the Gauss rules, Gauss-Legendre have been most commonly used in Galerkin Radiosity systems. A 1-D N point Gauss-Legendre rule's evaluation points are the zeros of the degree N Legendre polynomial.

We explore the efficiency of the computation of the irradiance integral by studying a number of quadrature methods. First, we describe a number of quadrature rules and the reasons why they were chosen for study. Results of using these rules when computing the irradiance integral with partial visibility and a constant basis are presented. We then show the results of these methods when evaluating unoccluded and partially occluded irradiance integrals in the presence of a non-constant basis. Based on these results we construct an adaptive method for choosing Gauss-Legendre quadrature rules and discuss its performance, which includes a factor of ten speedup in many cases.

3 Methodology

We've computed the error of a range of quadrature methods of equal and lower cost (kernel evaluations) than the typically used Gauss-Legendre product rules

in a wide range of scenarios: unoccluded/partially occluded, different element sizes, a range of distances, a variety of orientations and geometric complexities. The tests involved computing coefficients of linear, quadratic, and cubic multiwavelets (Legendre polynomials).

The quadratures studied can be broken down into three main categories: *Gauss Product Rules*, *Non-Product Rules*, and *Monte Carlo Methods*. We now list the characteristics of these categories, how and why they were used, and describe the quadratures used with quadratic multiwavelets in Figures 1 and 2.

1. Gauss Product Rules

Characteristics:

- N function evaluations integrate degree $\leq 2N - 1$ functions.
- Tensor products of univariate Gauss rules.
- Can have a different rule for each variable.
- Exponential in the dimension of the problem (radiosity is 4D).

What was tested:

- Gauss-Legendre rules were tested.
- We experimented with sampling source differently than receiver.

Methods used in figures 1 and 2.

- *FullGauss* : 3-point rules source and receiver: 81 samples
- *LinSrcGauss* : 2-point rules source, 3-point receiver: 36 samples
- *ConstSrcGauss* : 1-point rule source, 3-point receiver: 9 samples

2. Non-Product Rules

Characteristics:

- Non-exponential methods exist.

What was tested:

- Examined the possibility of using less samples than product rules with comparable accuracy.

Methods used in figures 1 and 2 (Methods presented by Stroud in [15])

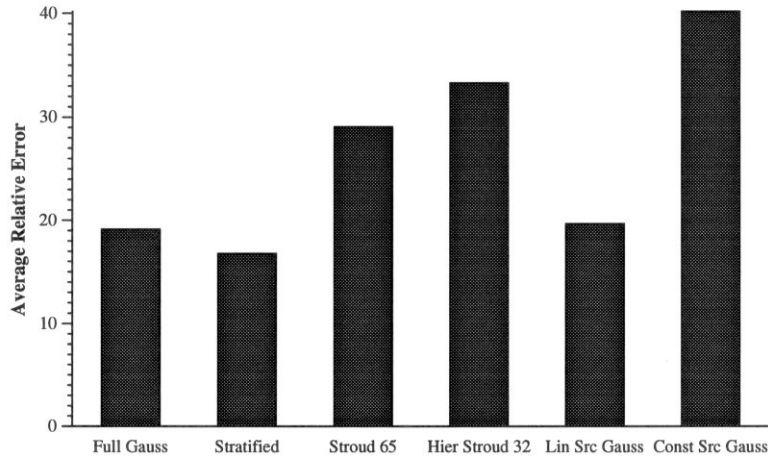


Figure 1: Relative error of the coupling of constant bases in partially visible links.

- *Stroud65* : degree 5 rule, Stroud's C_n 5-7 : 65 samples
- *HierStroud* : degree 3 rule applied to 4 receiver quadrants, Stroud's C_n 3-1: 32 samples

3. Monte Carlo Methods

Characteristics:

- Non-exponential methods exist.
- Usually perform well when function is non-smooth, discontinuous.

What was tested:

- Pure, N-Rook [13], and Stratified Sampling.

Methods tested in figures 1 and 2

- *Stratified*: each axis stratified into 3 regions : 81 samples

4 Study Results

In Figures 1 and 2, we measure error by comparing the quadrature methods' results with that of 10,000 point stratified Monte Carlo method (10 stratified zones in each dimension). This gives us an accurate and unbiased estimate of the integral in both unoccluded and partially occluded situations.

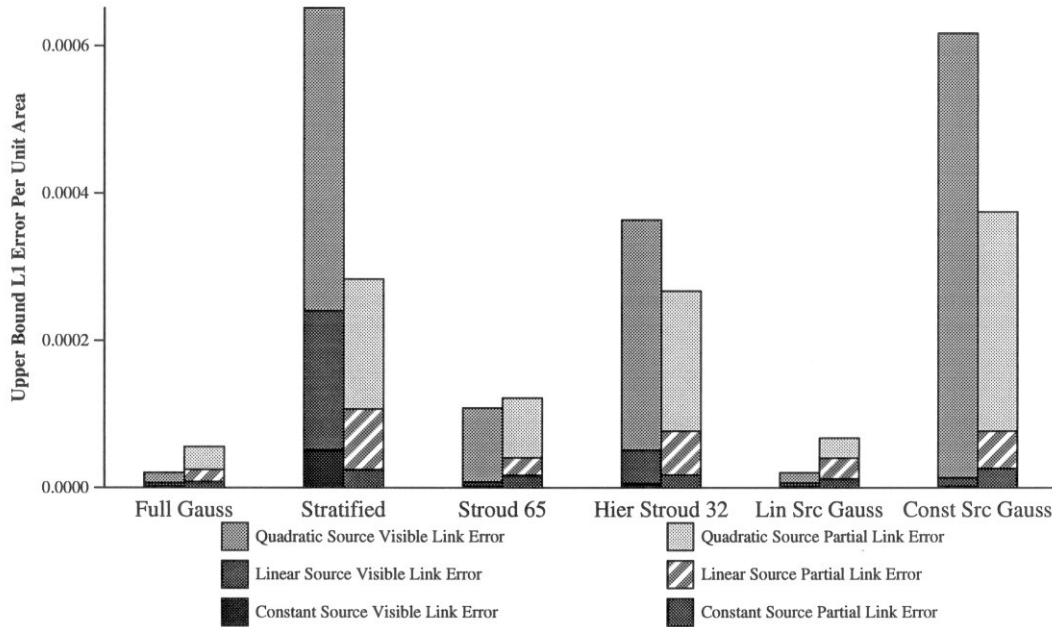


Figure 2: L1 error per unit area of partial and visible links.

Figure 1 shows the relative error between constant bases in the partially occluded links. This figure shows that in this scenario 81 stratified Monte Carlo samples perform better than other methods. These results are typical of a wide range of scenarios (different element sizes, orientations, and occlusions) that we have tried. We find that the superiority of Monte Carlo methods decreases as fewer samples are used.

For each quadrature method, figure 2 shows the upper bound on L1 error per unit area of the links in a scene that used a quadratic multiwavelet basis. The coefficients of the source’s basis functions are 1. The left bars show the error in the unoccluded links and right bars show the partially occluded links. Each bar is broken down by source basis function so, to see the amount of error of a constant light source we look at the height of the lowest bar. We see that for a quadratic basis Gauss rules work extremely well compared to the other methods we have tried, including stratified Monte Carlo. These measurements depict the amount of error when the coefficients are 1, so they are scaled by the coefficients when the coefficients change. We see that if the coefficients of the source’s high-order bases are small we can use a Gauss quadrature rule with fewer points than the *FullGauss*, such as *LinSrcGauss*, and have a negligible amount of error.

5 Adaptive Algorithm

Based on the above observations, we now present an adaptive way of choosing Gauss quadrature rules that compute the irradiance coupling between basis functions in a hierarchical Galerkin wavelet radiosity system. The magnitudes of the source’s basis functions’ coefficients and occlusion factors will be used to choose a Gauss quadrature method. The choice of quadrature method is made after we have performed subdivision and have determined that the geometric kernel G is sufficiently smooth. The oracle that makes this decision could be, for example, BF refinement described by Hanrahan *et al.* [6] or Gortler *et al.*’s polynomial fitting approximation [5]. The subdivision strategy could be a quadtree subdivision, or discontinuity meshing as presented by Lischinski *et al.* [9]. In our examples, we use a BF refinement oracle with visibility criteria based on Teller and Hanrahan’s visibility classification [16] and a quadtree subdivision. This method is convenient because the oracle is independent of the quadrature calculation method and does not require sampling of the space in order to make visibility calculations.

Once we have determined the level at which we should compute the irradiance coupling between basis functions we must choose a quadrature method. Our results show that Gauss quadrature methods work best, so we wish to reduce the cost of the computation and still use Gauss rules. This can be done by noting that if M of the highest order basis functions for a variable have very small coefficients, then any integration in the presence of these functions is also small. Therefore, in that variable’s domain the basis is essentially degree $N - M$ instead of N and we can use an $(N - M) + 1$ point rule rather than an $N + 1$ point rule. When computing this integral we know the radiosity coefficients of the basis functions of the source. We may take advantage of this fact and use the $(N - M) + 1$ rule along each variable of the source surface (possibly with a different M for each variable, M_0 and M_1) when the coefficients are sufficiently small. This results in $(N - M_0 + 1)(N - M_1 + 1)(N + 1)^2$ kernel evaluations instead of the $(N + 1)^4$ of previous methods. For example, if using a quadratic basis, $N = 3$, and only the constant basis functions of the source were significant, $M_0 = M_1 = 2$, it would require 9 kernel evaluations instead of 81.

An example of why we have fewer samples along one axis when integrating is shown in Figure 3. The function varies slowly along Y (top) and rapidly along X (right side). Therefore, it can be sampled less frequently in Y. In our case the source function is of low frequency when the higher order basis functions have small coefficients, barring discontinuities in the geometric function due to partial visibility. Similar knowledge of the receiver’s basis function is not available, so we must assume that it is a higher-frequency function.

We now construct an adaptive algorithm based on this criterion. When the basis is of degree N we use an $N + 1$ point Gauss rule along each axis of the receiver. We then choose a rule of 1 to $N + 1$ points for each axis of the source based on an error threshold, $ErrThres$, set by the user. The coefficients of the source are checked along one of its axes and we find the highest order basis

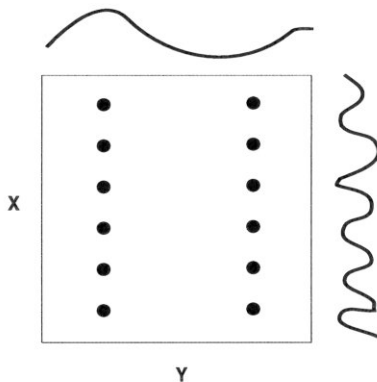


Figure 3: More samples along X than Y due to the higher frequency characteristics of the function.

function, M , that has a coefficient above $ErrThres$. An $M + 1$ point rule is then used along that axis and this process is repeated for the other axis. Finally, tensor products of the source and receiver rules are used evaluate the integral.

There is more error in partially occluded regions and this should be accounted for when choosing a quadrature rule. To reduce error this error, when the integral is partially occluded we randomly choose an axis on the source that has fewer than $N + 1$ points and increase the number of points along the axis by one. The pseudocode for this algorithm is shown in figure 4. After an iteration of the radiosity solver, we subdivide if necessary and check the coefficients of all couplings to see if we need a better quadrature rule due to the energy gain.

We have tried the adaptive algorithm in a range of scenes with linear, quadratic, and cubic multiwavelets. We find that the speedup is half the theoretical best-case maximum of 4, 9, and 16 times for linear, quadratic, and cubic bases, respectively. The images in Plate 1 show a comparison of the fixed point method and the adaptive method using a cubic multiwavelet basis. The leftmost image used the adaptive algorithm without accounting for partial visibility when choosing quadrature rules and had 12 times fewer kernel evaluations than the fixed point method. The center image is the adaptive rule accounting for partial visibility and is 8 times faster than the non-adaptive method. The rightmost is the fixed point method.

6 Summary and Conclusion

We have presented a study of quadrature methods for computing irradiance couplings in Galerkin radiosity systems. First, we saw that the best methods for computing partially occluded irradiance integrals for constant bases are Monte Carlo based. When we moved to computing partially occluded irradiance integrals with non-constant bases, Gauss product rules proved to be far superior to


```

ChooseGaussRules( Element Src, Element Rec, Double ErrThres )
  highestOrderSignificantBasisU = 0
  highestOrderSignificantBasisV = 0
  Rec.GaussRule.u = basisDegree + 1
  Rec.GaussRule.v = basisDegree + 1
  for (int i=0 ; i <= basisDegree; i++ )
    if (Src.coef.u.basis[i] > ErrThres)
      highestOrderSignificantBasisU = i
    if (Src.coef.v.basis[i] > ErrThres)
      highestOrderSignificantBasisV = i
  Src.GaussRule.u = highestOrderSignificantBasisU + 1
  Src.GaussRule.v = highestOrderSignificantBasisV + 1
  if (PartiallyOccluded(Src, Rec))
    RandomlyIncreaseAnAxis(Src)

```

Figure 4: Adaptive Gauss Quadrature Selection

the others, including Monte Carlo. However, Monte Carlo methods may become more effective as the dimensionality of the problem and number of samples used increases. Our study also shows that Gauss product rules are superior to the other quadrature methods when computing unoccluded irradiance couplings for the bases of our experiments.

Based on the results of the study, an adaptive method for choosing Gauss product rules is constructed and tested. This algorithm takes advantage of the characteristics of the source’s basis functions and accounts for partial visibility. Typically, our algorithm achieves a speedup of half it’s theoretical maximum, which can be translate into an order of magnitude or greater speedup over non-adaptive methods.

We look optimistically upon extensions of the algorithm for Galerkin radiance systems. The adaptive approach’s speedup is proportional to the basis degree since more kernel evaluations can be avoided. This situation improves for adaptive approaches applied to the 6-dimensional angularly-dependent radiance case due to the extraordinary amount of samples previously used to compute couplings, $(N + 1)^6$ for a degree N basis.

References

- [1] COHEN, M. F., AND GREENBERG, D. P. The hemi-cube: A radiosity solution for complex environments. *Computer Graphics* 19, 3 (July 1985), 31–40.
- [2] COHEN, M. F., AND WALLACE, J. R. *Radiosity and Realistic Image Synthesis*. Academic Press Professional, Boston, 1993.

- [3] GERSHBEIN, R., SCHRÖDER, P., AND HANRAHAN, P. Textures and radiosity: Controlling emission and reflection with texture maps. In *Computer Graphics Annual Conference Series 1994* (August 1994), Siggraph, pp. 51–58.
- [4] GORAL, C. M., TORRANCE, K. E., GREENBERG, D. P., AND BATTLE, B. Modelling the interaction of light between diffuse surfaces. *Computer Graphics* 18, 3 (July 1984), 212–222.
- [5] GORTLER, S., SCHRÖDER, P., COHEN, M., AND HANRAHAN, P. Wavelet radiosity. In *Computer Graphics Annual Conference Series 1993* (August 1993), Siggraph, pp. 221–230.
- [6] HANRAHAN, P., SALZMAN, D., AND AUPPERLE, L. A rapid hierarchical radiosity algorithm. *Computer Graphics* 25, 4 (July 1991), 197–206.
- [7] HECKBERT, P. S. Radiosity in flatland. *Computer Graphics Forum* 2, 3 (1992), 181–192.
- [8] KAJIYA, J. T. The rendering equation. *Computer Graphics* 20, 4 (1986), 143–150.
- [9] LISCHINSKI, D., TAMPIERI, F., AND GREENBERG, D. P. Combining hierarchical radiosity and discontinuity meshing. In *Computer Graphics Annual Conference Series 1993* (August 1993), Siggraph, pp. 199–208.
- [10] SCHRÖDER, P. Numerical integration for radiosity in the presence of singularities. In *Fourth Eurographics Workshop on Rendering* (1993).
- [11] SCHRÖDER, P., GORTLER, S. J., COHEN, M. F., AND HANRAHAN, P. Wavelet projections for radiosity. In *Fourth Eurographics Workshop on Rendering* (June 1993), Eurographics, pp. 105–114.
- [12] SCHRÖDER, P., AND HANRAHAN, P. On the form factor between two polygons. In *Computer Graphics Annual Conference Series 1993* (August 1993), Siggraph, pp. 163–164.
- [13] SHIRLEY, P. Discrepancy as a quality measure for sampling distributions. In *Eurographics '91* (September 1991), pp. 183–193.
- [14] SMITS, B., ARVO, J., AND GREENBURG, D. A clustering algorithm for radiosity in complex environments. In *Computer Graphics Annual Conference Series 1994* (August 1994), Siggraph, pp. 435–442.
- [15] STROUD, A. H. *Approximate Calculation of Multiple Integrals*. Prentice-Hall, New Jersey, 1971.
- [16] TELLER, S., AND HANRAHAN, P. Global visibility algorithms for illumination computations. In *Computer Graphics Annual Conference Series 1993* (August 1993), Siggraph, pp. 239–246.

- [17] TROUTMAN, R., AND MAX, N. Radiosity algorithms using higher-order finite elements. In *Computer Graphics Annual Conference Series 1993* (August 1993), Siggraph, pp. 209–212.
- [18] ZATZ, H. R. Galerkin radiosity: A higher-order solution method for global illumination. In *Computer Graphics Annual Conference Series 1993* (August 1993), Siggraph, pp. 213–220.