Dominating Sets in Planar Graphs[1]

Lesley R. Matheson[2]
Robert E. Tarjan[2,3]

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Abstract

Motivated by an application to unstructured multigrid calculations, we consider the problem of asymptotically minimizing the size of dominating sets in triangulated planar graphs. Specifically, we wish to find the smallest ϵ such that, for n sufficiently large, every n-vertex planar graph contains a dominating set of size at most ϵn. We prove that $\frac{1}{4} \leq \epsilon \leq \frac{1}{3}$, and we conjecture that $\epsilon = \frac{1}{4}$. For triangulated discs we obtain a tight bound of $\epsilon = \frac{1}{3}$. The upper bound proof yields a linear-time algorithm for finding an $n/3$ - size dominating set.

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[2] NEC Research Institute, 4 Independence Way, Princeton, NJ 08540.

[3] Department of Computer Science, Princeton University, Princeton, NJ 08544. Research at Princeton University partially supported by the National Science Foundation, Grant No. CCR-8920505, and the Office of Naval Research, Contract No. N0014-91-J-1463.

1. Introduction

The following problem arises in unstructured multigrid computations in two dimensions. We are given a region of the plane, topologically equivalent to a disc, that is partitioned into triangles. The vertices and edges of the triangles define a planar graph, all of whose faces except the exterior face are triangles. We wish to find a *dominating set* in this graph; namely, a set of vertices such that each vertex is either in the set or adjacent to a vertex in the set. In the multigrid application such a dominating set defines the vertices of a new triangulation that represents the original triangulation but has fewer vertices. For a discussion of the details of this application see [3,6].

One question this approach raises is how small we can make the dominating set, asymptotically in the worst case. More precisely, what is the smallest ϵ such that, for sufficiently large n, every n-vertex triangulated disc has a dominating set containing at most ϵn vertices? Section 2 of this paper is devoted to showing that $\epsilon \leq \frac{1}{3}$, and that a dominating set of size $n/3$ can be found in linear time, for any $n \geq 3$.

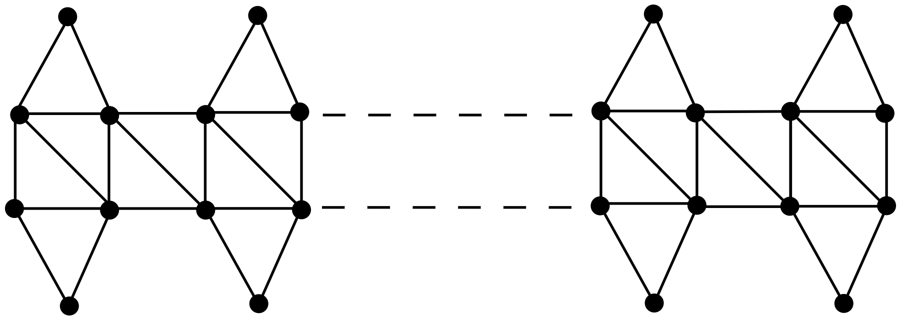

Figure 1: A class of (outer) planar graphs that are triangulated discs and have no dominating sets of size less than $n/3$.

The class of examples illustrated in Figure 1 shows that $\frac{1}{3}$ is the best possible bound. Each graph in the class is outer planer, with $n/3$ vertices of degree two, each adjacent to two distinct vertices. Among each set consisting of a vertex of degree two and its two neighbors, any dominating set must contain at least one vertex.

A related question that we have not been able to answer completely is that of finding the smallest ϵ' such that, for sufficiently large n, every n-vertex triangulated planar graph has a dominating set containing at most $\epsilon' n$ vertices. The construction for triangulated discs shows that $\epsilon' \leq \frac{1}{3}$. The octahedron, with six vertices, has no dominating set of size less than two (see Figure 2). But we know of no way to generalize this example to arbitrarily large n. The class of examples shown in Figure 3, consisting of $n/4$ tetrahedra connected arbitrarily, shows that $\epsilon' \geq \frac{1}{4}$. We conjecture that the actual answer is $\epsilon' = \frac{1}{4}$ and leave as an open problem proving or disproving this conjecture.

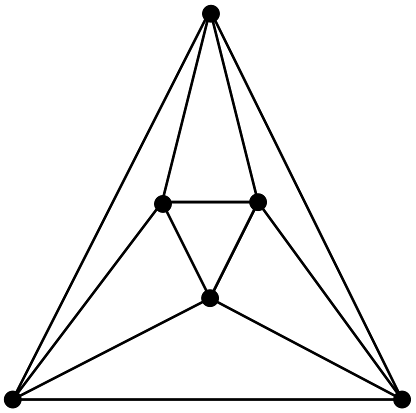

Figure 2: An octahedron. Any dominating set contains at least two vertices.

2. Small Dominating Sets in Triangulated Discs

Our method for finding small dominating sets in triangulated discs is based on a stronger result. The *domatic number* of a graph is defined to be the

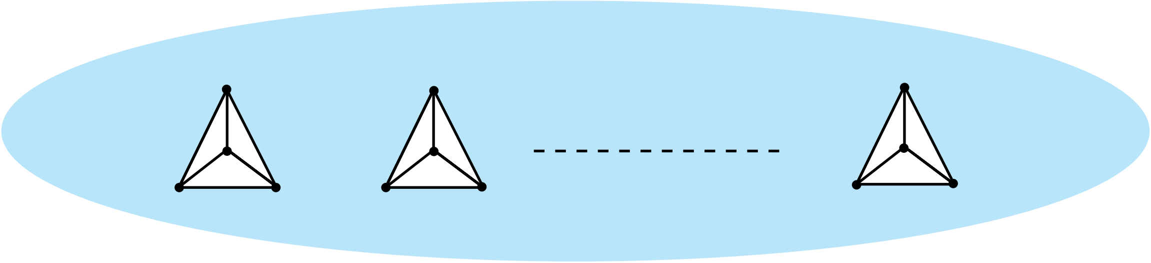

Figure 3: A class of triangulated planar graphs having no dominating sets of size less than $n/4$. The shaded region is triangulated arbitrarily.

maximum number of dominating sets into which the vertices can be partitioned. We show that every triangulated disc has a domatic number of at least three, by exhibiting a linear-time algorithm that partitions the vertices into three dominating sets. The smallest of these three dominating sets must have size at most $n/3$. Thus every triangulated disc has a dominating set of size at most $n/3$.

Our partitioning algorithm assigns the vertices one-by-one to dominating sets, in an order with specific properties, which we now define. An *outward numbering* of a triangulated disc is a numbering of the vertices from 1 to n such that, for any i, the set of vertices numbered 1 through i induces a triangulated disc D_i such that all remaining vertices are outside the disc. The notion of an outward numbering is essentially the restriction to triangulated discs of Mondshein's [**?**] notion of a $(2,1)$-connected sequence of a biconnected graph. Essentially the same notion, restricted to triconnected planar graphs, was independently rediscovered by Kant [**?**]. For completeness, we include here a proof of the existence of outward numberings and a linear-time algorithm for binding them.

Lemma 1 *Given any triangulated disc and any of its triangles, there is an outward numbering that numbers the vertices of the triangle 1, 2, 3 in any desired order.*

Proof: See below. □

Theorem 1 *Any triangulated disc has a domatic number of at least three.*

Proof: By Lemma 1 the disc has an outward numbering. We will label each vertex a, b, or c such that each set of vertices with the same label is a dominating set. We label the vertices one-by-one in the order of the outward numbering. We maintain two invariants:

(i) The labeling constructed so far defines three dominating sets of the subgraph induced by the labeled vertices, and

(ii) the boundary of the disc defined by the vertices labeled so far has no two adjacent vertices with the same label.

We begin by labeling vertices 1, 2, 3 with labels a, b, c, respectively. This makes (i) and (ii) true. In general, let i be the next vertex to be labeled. Because the numbering is outward, the set of vertices to which i is adjacent in disc D_{i-1} is a set $\{x_1, x_2, \ldots, x_k\}$ of vertices occurring consecutively on the boundary of D_{i-1}, with $k \geq 2$. The boundary of disc D_i is formed from the boundary of D_{i-1} by deleting $x_2, x_3, \ldots, x_{k-1}$ (if $k \geq 3$) and inserting i in between x_1 and x_k. To label i, we consider two cases. If x_1 and x_k are labeled differently, we label i with the label not used for x_1 or x_k. This preserves (i) and (ii). If x_1 and x_k are labeled the same, invariant (ii) implies that $k \geq 3$ and that x_2 has a different label than x_1. In this case we label i with the label not used for x_1 or x_2. Again this preserves (i) and (ii).

□

Corollary 1 *Any triangulated disc has a dominating set containing at most* $\frac{1}{3}$ *of the vertices.*

Proof: Apply Theorem 1 and select the smallest of the dominating sets into which the vertices are partitioned. □

The proofs of Theorem 1 and Corollary 1 give simple linear-time constructions, assuming that one is given an outward numbering.

Proof of Lemma 1 : Let D be the triangulated disc. Number the vertices of the initial triangle arbitrarily 1, 2, 3. Given that numbers $1, 2, \ldots, i$ for $i \geq 3$ have been assigned to vertices satisfactorily, we describe how to find a vertex to number $i + 1$.

The disc D properly contains the disc D_i. Consider any triangle T of D that is outside of D_i but shares an edge with the boundary of D_i. There must be at least one such triangle, since otherwise the boundaries of D and D_i would be equal, a contradiction. One vertex of T must be unnumbered, otherwise T would be inside of D_i, also a contradiction. Thus there is at least one unnumbered vertex v adjacent to two or more vertices on the boundary of D_i. Let the set of such vertices be S.

For a vertex $v \epsilon S$, let $x_1, x_2, \ldots, x_k$ be the vertices adjacent to v on the boundary of D_i, indexed clockwise around the boundary, with x_1 chosen so that none of the boundary of D_i from x_1 clockwise to x_k is part of the boundary of D. This choice is possible since the edges of the boundary of D_i are partitioned by $x_1, x_2, \ldots, x_k$ into k parts, at most one of which can intersect the boundary of D. Consider the set of triangles $T(v)$ inside the simple cycle $v, x_k, x_{k-1}, \ldots, x_1, v$. We define the *score* of v to be the number of triangles in $T(v)$.

Choose any vertex $v \epsilon S$ of minimum score. We assert that such a vertex can be numbered $i + 1$. For such a vertex v, the vertices $x_1, x_2, \ldots, x_k$ must occur consecutively around the boundary of D_i, and $T(v)$ must consist exactly of the set $\{x_1, x_2, v\}$, $\{x_2, x_3, v\}, , \ldots, \{x_{k-1}, x_k, v\}$. Otherwise, there would be some vertex of S inside the cycle $v, x_k, x_{k-1}, \ldots, x_1, v$, and such a vertex would have smaller score than v. Thus v can be numbered $i + 1$.

□

We can obtain a linear-time algorithm for finding outward numberings by modifying the construction in the proof of Lemma 1. We assume that we are given an embedding of the disc D in the plane; if not, one can be found in linear time using any of the known linear-time planarity-testing algorithms, such as [**?**, **?**, **?**], appropriately extended [2]. As the numbering proceeds, we maintain the set of vertices $x_1, x_2, \ldots, x_k,\ x_{k+1} = x_1$ on the boundary of D_i, indexed clockwise around the boundary. For each pair $\{x_i, x_{i+1}\}$ we maintain the triple $\{x_i, x_{i+1}, v_i\}$, if any, that defines a triangle of D sharing an edge with the boundary of D_i and outside of D_i. Finally, for each unnumbered vertex v, we maintain a count of the number of triples $\{x_i, x_{i+1}, v\}$ such that $\{x_{i-1}, x_i, v\}$ is not a triple. Any vertex whose count is exactly one can be numbered next. Updating all the required information can easily be done so that the total running time of the algorithm is linear.

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