LAYOUT OF ROOTED TREES

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Abstract. Let $S$ be a set of $n$ points in the plane in general position. The depth of a point $p \in S$ is the minimum number of elements of $S$ in a closed halfplane containing $p$. We prove that, if $p$ is not the deepest point of $S$ or the depth of $p$ is at most $\frac{n}{3} + 1$, then any tree with $n$ vertices and with root $r$ can be straight-line embedded on $S$ so that $r$ is mapped onto $p$. This gives a partial answer to a problem raised by Micha Perles.

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Let \( S \) be a set of \( n \) points in the plane in general position, i.e., no 3 of them are on the same line. We say that a graph \( G = (V, E) \) with \( n \) vertices can be laid down (or can be straight-line embedded) onto \( S \), if there exists a one-to-one mapping \( \phi : V \to S \) that takes the edges of \( G \) into non-crossing straight-line segments, i.e.,

\[
(\phi(u_1), \phi(v_1)) \cap (\phi(u_2), \phi(v_2)) = \emptyset \quad \text{for any } u_1v_1 \neq u_2v_2 \in E.
\]

It is easy to see that any tree \( T \) (and, in fact, any outerplanar graph) can be laid down onto any set \( S \) with the same number of points (cf. [FPP], [GMPP]). Micha Perles [P] raised the question whether one can arbitrarily specify the image of the root under such an embedding. The aim of this note is to give a partial answer to this question.

The depth of an element \( p \in S \) is defined as the minimum number of elements of \( S \) in a closed halfplane containing \( p \). A point \( p \in S \) is a vertex of the convex hull if and only if its depth \( d(p) = 1 \).

**Theorem.** Let \( T \) be a tree with \( n \) vertices and with root \( r \), and let \( S \) be a set with \( n \) points in the plane in general position. Suppose that some point \( p \in S \) satisfies at least one of the following conditions:

(i) \( p \) is not the unique deepest point of \( S \), or
(ii) the depth of \( p \), \( d(p) \leq \frac{n}{3} + 1 \).

Then there is a straight-line embedding \( \phi \) of \( T \) onto \( S \) such that \( \phi(r) = p \).

For any point \( x \) of \( T \), let \( v^0(x) = x, v^1(x), ..., v^k(x) = r \) denote the vertices of the path connecting \( x \) to \( r \) in \( T \). \( v^1(x) \) is called the father of \( x \), and \( x \) is the son of \( v^1(x) \). The set of all vertices \( x \) for which the path connecting \( x \) to \( r \) passes through \( y \) induces a subtree \( T(y) \subseteq T \). The vertex \( y \) is called the root of \( T(y) \).

**Algorithm 1.** The following trivial algorithm finds a straight-line embedding \( \phi \) of \( T \) onto \( S \) with \( \phi(r) = p \) in the special case when \( p \) is a vertex of the convex hull of \( S \).

Enumerate the points of \( S \setminus \{p\} \) by \( p_1, p_2, ..., p_{n-1} \) in clockwise order around \( p \). Let \( r_1, r_2, .. \) denote the sons of \( r \) in \( T \), and let \( |T(r_j)| \) be the number of vertices of the subtree \( T(r_j) \). (See fig. 1.)

Let \( S_i = \{p_k \mid \Sigma_{j<i}|T(r_j)| < k \leq \Sigma_{j\leq i}|T(r_j)|\} \), and find a point \( p_{ki} \in S_i \) nearest to \( p \) (i=1,2,...).

Construct recursively a straight-line embedding \( \phi \) of the subtree \( T(r_i) \) onto \( S_i \) with \( \phi(r_i) = p_{ki} \) (i = 1, 2, ...) and set \( \phi(r) = p \).
Algorithm 2. Let $p$ and $q$ be two consecutive vertices of the convex hull of $S$, and let $x$ be any vertex of $T$ different from the root $r$. The following slightly modified version of Algorithm 1 enables us to construct a straight-line embedding $\phi$ of $T$ onto $S$ with $\phi(r) = p$ and $\phi(x) = q$.

**Step 0.** Let $p_1, p_2, ..., p_{n-1}$ denote the elements of $S - \{q\}$ listed (say) in clockwise order around $q$, and assume by symmetry that $p_{n-1} = p$.
Use Algorithm 1 to find a straight-line embedding $\phi$ of $T(x)$ onto the point set \{p_1, p_2, ..., $p_{|T(x)|-1}, q\}$, such that $\phi(x) = q$. (See fig. 2.)
Let $v_0^i(x) = x, v_1^i(x), ..., v_k^i(x) = r$ denote the vertices of the path connecting $x$ to $r$ in $T$.

**Step i.** (1 $\leq i < k$) Let $S_i = S - \phi(T(v_{i-1}^i(x)))$, and let $q_i$ be the next vertex of the convex hull of $S_i$ that comes after $p$ in the clockwise order. Renumber the points of $S_i - \{q_i\}$ by $p_1, p_2, ..., p_{|S_i| - 1} = p$ in clockwise order around $q_i$.
Use Algorithm 1 to find a straight-line embedding $\phi$ of $T_i = T(v_i^i(x)) - T(v_{i-1}^i(x))$ onto the point set \{p_1, p_2, ..., $p_{|T_i| - 1}, q_i\}$ such that $\phi(v_i^i(x)) = q_i$.

**Step k.** Use Algorithm 1 to find a straight-line embedding $\phi$ of $T_k = T - T(v_{k-1}^k(x))$ onto $S_k$ with $\phi(r) = p$.

Figure 1.

\[\text{Figure 2.}\]
Now we are in the position to prove our main result.

**Proof of Theorem.** Let us build the subtree $T' \subseteq T$ from $T' = r$ by repeating the following step as long as possible.

If $T - T'$ consists of at least two trees, then let $T_{min}$ denote one of them having the smallest number of vertices, and

if $|T'| + |T_{min}| \leq d(p)$, then set $T' = T' + T_{min}$
else stop.

If $T - T'$ consists of one tree, then let $x$ denote its root, and

if $|T'| + 1 \leq d(p)$, then set $T' = T' + x$
else stop.

After the above process has come to an end,

if $T - T'$ consists of at least two trees, then set $T'' = T_{min}$
if $T - T'$ consists of one tree, then set $T'' = \emptyset$.

Furthermore, let $F$ denote the forest $T - |T'| - |T''|$. (See fig. 3.)

![Figure 3.](image)

Note that the decomposition $T = T' \cup T'' \cup F$ is usually not uniquely determined, but it can be fixed arbitrarily in the rest of the argument. It follows from the above construction that $|T'| \leq d(p)$, $|T'| + |T''| \geq d(p)$, $|T''| \leq |F|$, thus $|F| = |T| - (|T'| + |T''|) \leq n - d(p)$. Observe that $T''$ and each component of $F$ are connected to the same vertex $c$ of $T'$, which is called the center of $T$.

**Case 1.** $|F| \geq d(p) - 1.$
Then $d(p) \leq |T'| + |T''| \leq n - d(p) + 1$.

By the definition of $d(p)$, there exists a closed halfplane $H$ containing $p$ on its boundary such that $|H \cap S| = d(p)$. Letting $\overline{H}$ denote the closure of the complement of $H$, we have $|\overline{H} \cap S| = n - d(p) + 1$.
Suppose first that \( d(p) < |T'| + |T''| \). Then by a suitable rotation of \( H \), we obtain a closed halfplane \( H_{pq} \) with boundary line \( pq \) such that \( q \in S \) and \( |H_{pq} \cap S| = |T'| + |T''| \). Cut \( H_{pq} \) into two convex cones \( C', C'' \) whose apices are at \( q \) so that they have no interior points in common, \( C' \cup C'' = H_{pq}, |C' \cap S| = |T'| \) and \( |C'' \cap S| = |T''| + 1 \). By Algorithm 2, we can find a straight-line embedding \( \phi \) of \( T' \) onto \( C' \cap S \) with \( \phi(r) = p \) and \( \phi(c) = q \). Using Algorithm 1, \( T'' \cup c \) and \( F \cup c \) can be laid down onto \( C'' \cap S \) and \( (H_{pq} \cap S) - \{p\} \), respectively, so that \( c \) is mapped onto \( q \). (Fig. 4.)

Suppose next, that \( d(p) = |T'| + |T''| \). Then \( T'' = \emptyset \), and \( F \) consists of a single tree whose root is denoted by \( c' \). Rotating \( H \) around \( p \), now we obtain a closed halfplane \( H_{pq} \) such that \( q \in S \) and \( |H_{pq} \cap S| = d(p) + 1 = |T'| + 1 \). Using Algorithm 2, we can find a straight-line embedding \( \phi \) of \( T' \cup c' \) onto \( H_{pq} \cap S \) with \( \phi(r) = p \) and \( \phi(c') = q \). This can be extended to a straight-line embedding of \( T \) by laying down \( F \) onto \( (H_{pq} \cap S) - \{p\} \).

\[ \overline{H_{pq}} \]
\[ H_{pq} \]
\[ q \]
\[ p \]
\[ C' \]
\[ C'' \]

Figure 4.

**Case 2.** \( |F| < d(p) - 1 \).

Assume first that condition (ii) of the theorem holds, i.e., \( d(p) \leq \frac{n}{3} + 1 \). Then \( |T''| \leq |F| \leq d(p) - 2 \) and \( |T'| \leq d(p) \), therefore \( |T'| + |T''| + |F| \leq 3d(p) - 4 < n \), which is a contradiction.

So we can suppose that (i) is true, i.e., there exists a point \( q \neq p \) in \( S \) with \( d(q) \geq d(p) \). Let \( H_{pq} \) and \( \overline{H_{pq}} \) denote the two closed halfplanes bounded by the line \( pq \). Obviously, \( |H_{pq} \cap S|, |\overline{H_{pq}} \cap S| \geq d(q) \). In view of the fact that \( |T'| \leq d(p) \leq d(q) \) and \( |T''| \leq |F| \leq d(p) - 2 \leq d(q) - 2 \), we can find two convex cones \( C' \subseteq H_{pq}, C'' \subseteq \overline{H_{pq}} \) whose intersection is the ray \( qP \) so that \( |C' \cap S| = |T'| \) and \( |C'' \cap S| = |T''| + 2 \). (See Fig. 5.) Hence, by Algorithms 2 and 1, we can get a straight-line embedding \( \phi \) of \( T' \) and \( T'' \) onto \( C' \cap S \) and \( C'' \cap S \), respectively, with \( \phi(r) = p \) and \( \phi(c) = q \).

On the other hand, \( |(\mathbb{R}^2 - (C \cup C'')) \cap S| = |F| \leq d(p) - 2 \leq d(q) - 2 \), hence \( (\mathbb{R}^2 - (C' \cup C'')) \) is either convex or it contains an open convex cone covering all points of \( (\mathbb{R}^2 - (C' \cup C'')) \cap S \). That is, \( \phi \) can be extended to a straight-line embedding of \( T \) by laying down \( F \) onto \( (\mathbb{R}^2 - (C' \cup C'')) \cap S \). This completes the proof. \( \square \)
An immediate consequence of our theorem is the following.

**Corollary.** Let $T$ be a tree of $n$ vertices with root $r$, let $S$ be a set of $n$ points in the plane in general position, $p_1, p_2 \in S$. Then $T$ can be laid down onto $S$ so that the image of $r$ is either $p_1$ or $p_2$.

□

**Remarks.**

1. In fact, the above proof shows that the Theorem remains true if we replace (i) by the somewhat weaker condition that there exists $q \in S$ with $d(q) \geq d(p) - 1$.

2. Algorithm 2 can be generalized to establish the following statement.

**Proposition.** Let $T$ be a tree on $n$ vertices, and let $v_1, v_2, ..., v_k$ be a simple path in $T$. Let $S$ be a set of $n$ points in the plane in general position, and let $p_1, p_2, ..., p_j$ ($j \leq k$) be consecutive vertices of the convex hull of $S$. Then for any $1 = i_1 < i_2 < ... < i_j = k$, there is a straight-line embedding $\phi$ of $T$ onto $S$ such that $\phi(v_{i_1}) = p_1, ..., \phi(v_{i_j}) = p_j$.

**Time Complexity.**

Algorithm 1 requires $O(n^2)$ time, since for each node $r$ we have to solve a selection problem among the points corresponding to the subtree rooted at the father of $r$.

Algorithm 2 requires $O(n^2)$ time. Observe that in Step i we need to find only $q_i$, \{p_1, p_2, ..., p_{|T_i|-1}\} and a mapping of $T_i$ onto them. So we have to solve a selection problem and we can do the mapping in $O(|T_i|^2)$ time using Algorithm 1.

To decide if condition (ii) of the Theorem holds and if it doesn’t then to find a point $q$ with $d(q) \geq d(p) - 1$ takes $O\left(\frac{n^{3/2}\log^3 n}{\log^* n}\right)$ time, where $\log^* n$ denotes the iterated logarithm function. This follows from the fact that for fixed $k$ the number of $k$-sets in $S$ is $O\left(\frac{n^{3/2}}{\log^* n}\right)$ (see [PSSz], and [E] for terminology), and we can find all of them spending $O(\log^2 n)$ time on each [OL].

To decompose our tree $T$ into $T' \cup T'' \cup F$ takes only linear time, and the suitable rotations can be done in time $O(n \log n)$.

Therefore the overall running time is $O(n^2)$. To improve the running time we should only improve Algorithm 1.
References


[P] M. Perles, Open problem proposed at the DIMACS Workshop on Arrangements, Rutgers University, 1990.