LAYOUT OF ROOTED TREES

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Abstract. Let S be a set of n points in the plane in general position. The depth of a point $p \in S$ is the minimum number of elements of S in a closed halfplane containing p. We prove that, if p is not the deepest point of S or the depth of p is at most $\frac{n}{3} + 1$, then any tree with n vertices and with root r can be straight-line embedded on S so that r is mapped onto p. This gives a partial answer to a problem raised by Micha Perles.

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Let S be a set of n points in the plane in general position, i.e., no 3 of them are on the same line. We say that a graph $G = (V, E)$ with n vertices can be *laid down* (or can be *straight-line embedded*) onto S, if there exists a one-to-one mapping $\phi : V \to S$ that takes the edges of G into non-crossing straight-line segments, i.e.,

$$(\phi(u_1), \phi(v_1)) \cap (\phi(u_2), \phi(v_2)) = \emptyset \quad \text{for any} \quad u_1v_1 \neq u_2v_2 \in E.$$

It is easy to see that any tree T (and, in fact, any outerplanar graph) can be laid down onto any set S with the same number of points (cf. [FPP], [GMPP]). Micha Perles [P] raised the question whether one can arbitrarily specify the image of the root under such an embedding. The aim of this note is to give a partial answer to this question.

The *depth* of an element $p \in S$ is defined as the minimum number of elements of S in a closed halfplane containing p. A point $p \in S$ is a vertex of the convex hull if and only if its depth $d(p) = 1$.

Theorem. *Let T be a tree with n vertices and with root r, and let S be a set with n points in the plane in general position. Suppose that some point $p \in S$ satisfies at least one of the following conditions:*

(i) p is not the unique deepest point of S, or

(ii) the depth of p, $d(p) \leq \frac{n}{3} + 1$.

Then there is a straight-line embedding ϕ of T onto S such that $\phi(r) = p$.

For any point x of T, let $v^0(x) = x, v^1(x), ..., v^k(x) = r$ denote the vertices of the path connecting x to r in T. $v^1(x)$ is called the *father* of x, and x is the *son* of $v^1(x)$. The set of all vertices x for which the path connecting x to r passes through y induces a subtree $T(y) \subseteq T$. The vertex y is called the *root* of $T(y)$.

Algorithm 1. *The following trivial algorithm finds a straight-line embedding ϕ of T onto S with $\phi(r) = p$ in the special case when p is a vertex of the convex hull of S.*

Enumerate the points of $S - \{p\}$ by $p_1, p_2, ..., p_{n-1}$ in clockwise order around p. Let $r_1, r_2, ...$ denote the sons of r in T, and let $|T(r_j)|$ be the number of vertices of the subtree $T(r_j)$. (See fig. 1.)

Let $S_i = \{p_k \mid \Sigma_{j<i}|T(r_j)| < k \leq \Sigma_{j\leq i}|T(r_j)|\}$, and find a point $p_{k_i} \in S_i$ nearest to p (i=1,2,...).

Construct recursively a straight-line embedding ϕ of the subtree $T(r_i)$ onto S_i with $\phi(r_i) = p_{k_i}$ $(i = 1, 2, ...)$ and set $\phi(r) = p$.

□

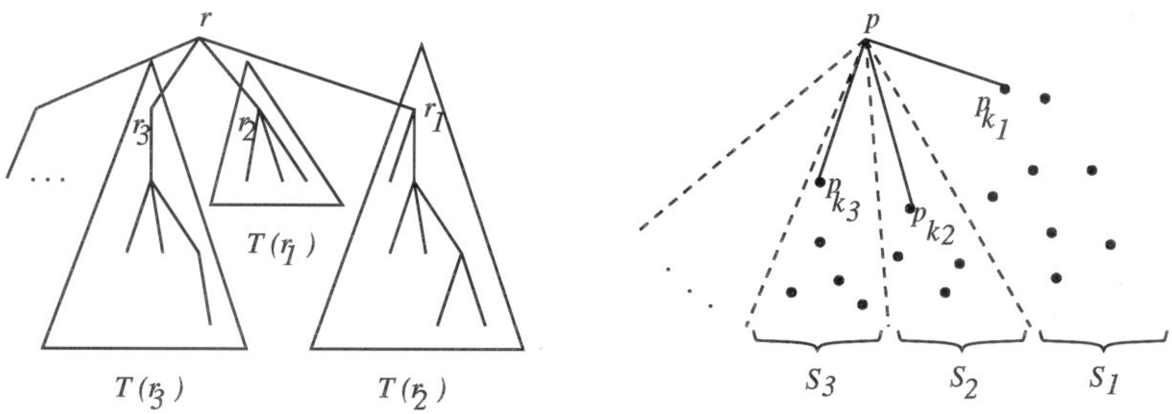

Figure 1.

Algorithm 2. *Let p and q be two consecutive vertices of the convex hull of S, and let x be any vertex of T different from the root r. The following slightly modified version of Algorithm 1 enables us to construct a straight-line embedding ϕ of T onto S with $\phi(r) = p$ and $\phi(x) = q$.*

Step 0. Let $p_1, p_2, ..., p_{n-1}$ denote the elements of $S - \{q\}$ listed (say) in clockwise order around q, and assume by symmetry that $p_{n-1} = p$.
Use Algorithm 1 to find a straight-line embedding ϕ of $T(x)$ onto the point set $\{p_1, p_2, ..., p_{|T(x)|-1}, q\}$, such that $\phi(x) = q$. (See fig. 2.)
Let $v^0(x) = x, v^1(x), ..., v^k(x) = r$ denote the vertices of the path connecting x to r in T.

Step i. $(1 \leq i < k)$ Let $S_i = S - \phi(T(v^{i-1}(x)))$, and let q_i be the next vertex of the convex hull of S_i that comes after p in the clockwise order. Renumber the points of $S_i - \{q_i\}$ by $p_1, p_2, ..., p_{|S_i|-1} = p$ in clockwise order around q_i.
Use Algorithm 1 to find a straight-line embedding ϕ of $T_i = T(v^i(x)) - T(v^{i-1}(x))$ onto the point set $\{p_1, p_2, ..., p_{|T_i|-1}, q_i\}$ such that $\phi(v^i(x)) = q_i$.

Step k. Use Algorithm 1 to find a straight-line embedding ϕ of $T_k = T - T(v^{k-1}(x))$ onto S_k with $\phi(r) = p$.

□

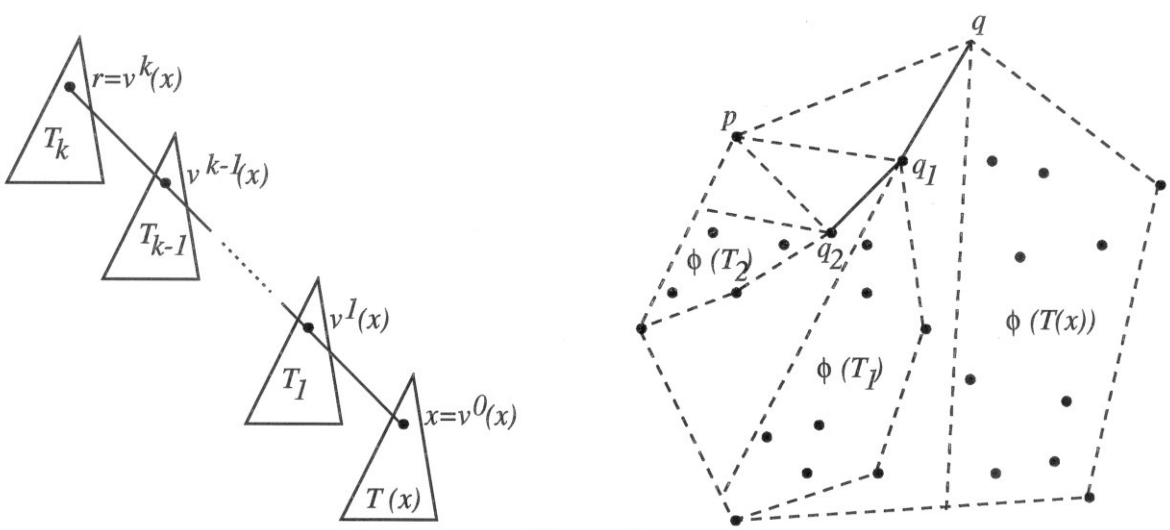

Figure 2.

Now we are in the position to prove our main result.

Proof of Theorem. Let us build the subtree $T' \subseteq T$ from $T' = r$ by repeating the following step as long as possible.

If $T - T'$ consists of at least two trees, **then** let T_{min} denote one of them having the smallest number of vertices, and
if $|T'| + |T_{min}| \leq d(p)$, **then** set $T' = T' + T_{min}$
else stop.

If $T - T'$ consists of one tree, **then** let x denote its root, and
if $|T'| + 1 \leq d(p)$, **then** set $T' = T' + x$
else stop.

After the above process has come to an end,
if $T - T'$ consists of at least two trees, **then** set $T'' = T_{min}$
if $T - T'$ consists of one tree, **then** set $T'' = \emptyset$.
Furthermore, let F denote the forest $T - |T'| - |T''|$. (See fig. 3.)

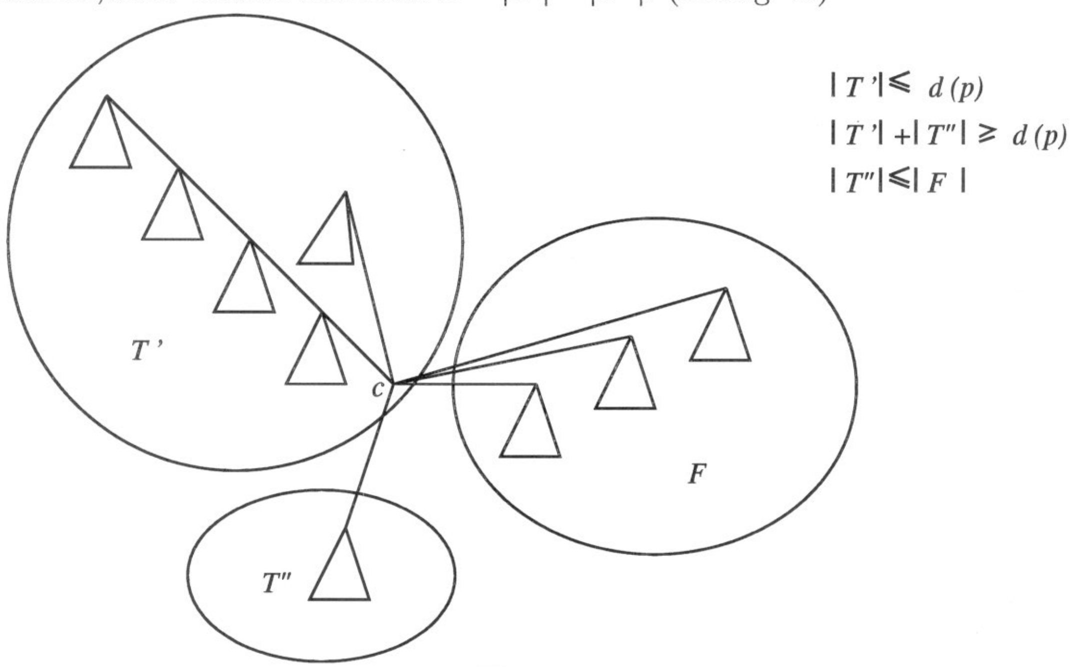

Figure 3.

Note that the decomposition $T = T' \cup T'' \cup F$ is usually not uniquely determined, but it can be fixed arbitrarily in the rest of the argument. It follows from the above construction that $|T'| \leq d(p)$, $|T'| + |T''| \geq d(p)$, $|T''| \leq |F|$, thus $|F| = |T| - (|T'| + |T''|) \leq n - d(p)$. Observe that T'' and each component of F are connected to the same vertex c of T', which is called the *center* of T.

Case 1. $|F| \geq d(p) - 1$.
Then $d(p) \leq |T'| + |T''| \leq n - d(p) + 1$.

By the definition of $d(p)$, there exists a closed halfplane H containing p on its boundary such that $|H \cap S| = d(p)$. Letting $\overline{H}$ denote the closure of the complement of H, we have $|\overline{H} \cap S| = n - d(p) + 1$.

Suppose first that $d(p) < |T'| + |T''|$. Then by a suitable rotation of H, we obtain a closed halfplane H_{pq} with boundary line pq such that $q \in S$ and $|H_{pq} \cap S| = |T'| + |T''|$. Cut H_{pq} into two convex cones C', C'' whose apices are at q so that they have no interior points in common, $C' \cup C'' = H_{pq}$, $|C' \cap S| = |T'|$ and $|C'' \cap S| = |T''| + 1$. By Algorithm 2, we can find a straight-line embedding ϕ of T' onto $C' \cap S$ with $\phi(r) = p$ and $\phi(c) = q$. Using Algorithm 1, $T'' \cup c$ and $F \cup c$ can be laid down onto $C'' \cap S$ and $(\overline{H}_{pq} \cap S) - \{p\}$, respectively, so that c is mapped onto q. (Fig. 4.)

Suppose next, that $d(p) = |T'| + |T''|$. Then $T'' = \emptyset$, and F consists of a single tree whose root is denoted by c'. Rotating H around p, now we obtain a closed halfplane H_{pq} such that $q \in S$ and $|H_{pq} \cap S| = d(p) + 1 = |T'| + 1$. Using Algorithm 2, we can find a straight-line embedding ϕ of $T' \cup c'$ onto $H_{pq} \cap S$ with $\phi(r) = p$ and $\phi(c') = q$. This can be extended to a straight-line embedding of T by laying down F onto $(\overline{H}_{pq} \cap S) - \{p\}$.

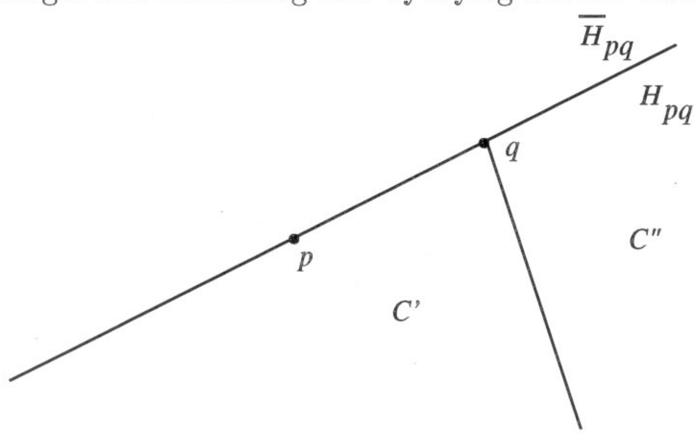

Figure 4.

Case2. $|F| < d(p) - 1$.

Assume first that condition (ii) of the theorem holds, i.e., $d(p) \leq \frac{n}{3} + 1$. Then $|T''| \leq |F| \leq d(p) - 2$ and $|T'| \leq d(p)$, therefore $|T'| + |T''| + |F| \leq 3d(p) - 4 < n$, which is a contradiction.

So we can suppose that (i) is true, i.e., there exists a point $q \neq p$ in S with $d(q) \geq d(p)$. Let H_{pq} and $\overline{H}_{pq}$ denote the two closed halfplanes bounded by the line pq. Obviously, $|H_{pq} \cap S|$, $|\overline{H}_{pq} \cap S| \geq d(q)$. In view of the fact that $|T'| \leq d(p) \leq d(q)$ and $|T''| \leq |F| \leq d(p) - 2 \leq d(q) - 2$, we can find two convex cones $C' \subseteq H_{pq}$, $C'' \subseteq \overline{H}_{pq}$ whose intersection is the ray qp so that $|C' \cap S| = |T'|$ and $|C'' \cap S| = |T''| + 2$. (See Fig. 5.). Hence, by Algorithms 2 and 1, we can get a straight-line embedding ϕ of T' and T'' onto $C' \cap S$ and $C'' \cap S$, respectively, with $\phi(r) = p$ and $\phi(c) = q$.

On the other hand, $|(\mathbf{R}^2 - (C \cup C'')) \cap S| = |F| \leq d(p) - 2 \leq d(q) - 2$, hence $(\mathbf{R}^2 - (C' \cup C''))$ is either convex or it contains an open convex cone covering all points of $(\mathbf{R}^2 - (C' \cup C'')) \cap S$. That is, ϕ can be extended to a straight-line embedding of T by laying down F onto $(\mathbf{R}^2 - (C' \cup C'')) \cap S$. This completes the proof.

□

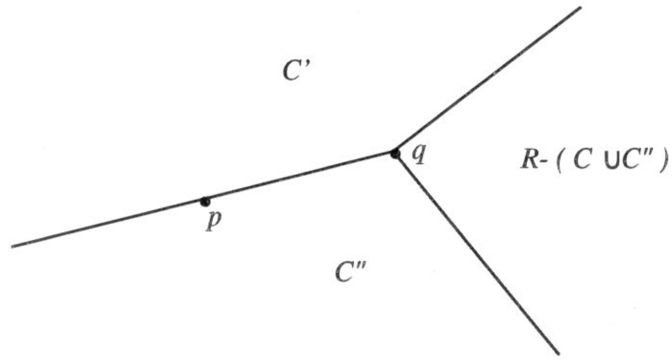

Figure 5.

An immediate consequence of our theorem is the following.

Corollary. Let T be a tree of n vertices with root r, let S be a set of n points in the plane in general position, $p_1,\ p_2 \in S$. Then T can be laid down onto S so that the image of r is either p_1 or p_2.

□

Remarks.

1. In fact, the above proof shows that the Theorem remains true if we replace (i) by the somewhat weaker condition that there exists $q \in S$ with $d(q) \geq d(p) - 1$.

2. Algorithm 2 can be generalized to establish the following statement.
Proposition. *Let T be a tree on n vertices, and let $v_1, v_2, ..., v_k$ be a simple path in T. Let S be a set of n points in the plane in general position, and let $p_1, p_2, ..., p_j$ $(j \leq k)$ be consecutive vertices of the convex hull of S. Then for any $1 = i_1 < i_2 < ... < i_j = k$, there is a straight-line embedding ϕ of T onto S such that $\phi(v_{i_1}) = p_1, ..., \phi(v_{i_j}) = p_j$.*

Time Complexity.

Algorithm 1 requires $O(n^2)$ time, since for each node r we have to solve a selection problem among the points corresponding to the subtree rooted at the father of r.

Algorithm 2 requires $O(n^2)$ time. Observe that in *Step i* we need to find only q_i, $\{p_1, p_2, ..., p_{|T_i|-1}\}$ and a mapping of T_i onto them. So we have to solve a selection problem and we can do the mapping in $O(|T_i|^2)$ time using Algorithm 1.

To decide if condition (ii) of the Theorem holds and if it doesn't then to find a point q with $d(q) \geq d(p) - 1$ takes $O(\frac{n^{3/2} \log^2 n}{\log^* n})$ time, where $\log^* n$ denotes the iterated logarithm function. This follows from the fact that for fized k the number of k-sets in S is $O(\frac{n^{3/2}}{\log^* n})$ (see [PSSz], and [E] for terminology), and we can find all of them spending $O(\log^2 n)$ time on each [OL].

To decompose our tree T into $T' \cup T'' \cup F$ takes only linear time, and the suitable rotations can be done in time $O(nlogn)$.

Therefore the overall running time is $O(n^2)$. To improve the running time we should only improve Algorithm 1.

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