MONOTONE BIPARTITE GRAPH PROPERTIES ARE EVASIVE

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Abstract

A Boolean function P from $\{0,1\}^t$ into $\{0,1\}$ is said to be *evasive*, if every decision tree algorithm for evaluating P must examine all t arguments in the worst case. It was known that any nontrivial monotone bipartite graph property on vertex set $V \times W$ must be evasive, when $|V| \cdot |W|$ is a power of a prime number. In this paper, we prove that every nontrivial monotone bipartite graph property is evasive.

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1 Introduction

In [RV2], Rivest and Vuillemin proved the Aanderra-Rosenberg Conjecture [R] which states that, to evaluate any nontrivial monotone graph property on n vertices, every decision tree algorithm must examine $\Omega(n^2)$ entries of the adjacency matrix in the worst case. A stronger conjecture, suggested by Karp (see [R]), that all such graph properties are *evasive*, i.e., *all* entries must be examined in the worst case, was left unresolved. Recently, Kahn, Saks, and Sturtevant [KSS] gave a partial solution by showing that, when n is a power of a prime, all such graph properties are evasive; their proof employed an ingenious topological approach to this complexity problem.

The method used in Rivest and Vuillemin [RV1] [RV2] (also discovered in Best, *et al.* [BBL]) yields immediately that any nontrivial monotone bipartite graph property on vertex set $V \times W$ must be evasive, when $|V| \cdot |W|$ is a power of a prime number. The purpose of this paper is to show that, in fact, every nontrivial monotone bipartite graph property is evasive. We will adopt the topological view for this problem as espoused in [KSS].

2 Main Theorem

Let $V = \{1,2,\ldots,m\}$, $W = \{1,2,\ldots,n\}$, and $\mathcal{G}_{m,n}$ be the set of all bipartite graphs $G = (V \times W, E)$ where $E \subseteq V \times W$. For any two $G = (V \times W, E)$, $G' = (V \times W, E')$, we write $G \le G'$ if $E \subseteq E'$; we say that G and G' are *isomorphic* if there exist permutations p_1, p_2 of V, W such that $(i,j) \in E$ if and only if $(p_1(i), p_2(j)) \in E'$. A *bipartite graph property* on $V \times W$ is a function $P : \mathcal{G}_{m,n} \rightarrow \{0,1\}$ satisfying the constraint that $P(G) = P(G')$ if G and G' are isomorphic. A bipartite graph property P on $V \times W$ is *monotone* if $P(G) \le P(G')$ for all $G \le G'$; P is *nontrivial* if it is not a constant function.

Let P be any bipartite graph property on $V \times W$. We are interested in evaluating $P(G)$, where the input graph $G = (V \times W, E)$ is given as an $m \times n$ adjacency matrix (a_{ij}) with $a_{ij} = 1$ for $(i,j) \in E$ and 0 otherwise. A *decision tree algorithm* T proceeds by asking a sequence of queries: $a_{i_1 j_1} = \ ?$, $a_{i_2 j_2} = \ ?$, ... , until the value of $P(G)$ can be determined; the choice of the $(k+1) - st$ query can depend on the results of all the preceding k values $a_{i_1 j_1}, \ldots, a_{i_k j_k}$. The *cost* of T, *cost*(T), is the maximum number of queries asked for any input $G \in \mathcal{G}_{m,n}$. The *complexity* of property P is defined as $min\{cost(T) | T \in \mathcal{T}(P)\}$, when $\mathcal{T}(P)$ is the set of all decision tree algorithms for property P. We say that P is *evasive* if $C(P) = |V| \cdot |W|$. Our main result is the following theorem; the remainder of this paper is devoted to its proof.

Theorem 1 *Every nontrivial monotone bipartite graph property is evasive.*

3 Preliminaries

We review some needed terminology and facts from standard topology and from [KSS].

3.1 Abstract Complex

An *abstract complex* on a finite set $X = \{x_1, x_2, \ldots, x_t\}$ is a collection Δ of subsets of X with the property that $A \subseteq B \in \Delta$ implies $A \in \Delta$. Each $A \in \Delta$ is a *face*; the *dimension* of A is $|A| - 1$. We call x_i the *vertices*. The *Euler characteristic* of Δ is $\chi(\Delta) = \sum_{i \geq 0}(-1)^i f_i$, where f_i is the number of $i -$ *dimensional* faces of Δ. We say that Δ is *rationally acyclic* if the homology groups of Δ are $H_0(\Delta) = \mathcal{Z}$ and $H_i(\Delta) = 0$ for all $i > 0$.

Let Γ be any permutation group of X. Assume that Δ is *invariant under* Γ, i.e. for all $\sigma \in \Gamma$, $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \in \Delta$ implies[2] $\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_k)}\} \in \Delta$. A face $F = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ is said to be *minimally invariant under* Γ if, for all $\sigma \in \Gamma$, $\{\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_k)\} = \{i_1, i_2, \ldots, i_k\}$, and if in addition, no proper nonempty subset of F has this property. Let $\mathcal{A}(\Delta, \Gamma)$ be the set of all nonempty faces of Δ that are minimally invariant under Γ.

Definition 1. Suppose Δ is invariant under Γ. If $\mathcal{A}(\Delta, \Gamma) = \emptyset$, let $\Delta_\Gamma = \emptyset$. If $\mathcal{A}(\Delta, \Gamma) = \{A_1, A_2, \ldots, A_s\} \neq \emptyset$, let Δ_Γ be the abstract complex on $\mathcal{A}(\Delta, \Gamma)$ defined by $\Delta_\Gamma = \{\{A_i | i \in D\} | D \subseteq \{1, 2, \ldots, s\}, \bigcup_{i \in D} A_i \in \Delta\}$.

3.2 Geometric Complex

Let $\{v_1, v_2, \ldots, v_k\}$ be a set of k independent points in R^q where $q > 0$ is an integer. Denote by $\langle v_1, v_2, \ldots, v_k \rangle$ their *convex hull* , i.e. the set

$$\left\{ \sum_{1 \leq i \leq k} \lambda_i v_i | \lambda_i \geq 0 \text{ for all } i, \text{ and } \sum_{1 \leq i \leq k} \lambda_i = 1 \right\}.$$

A set $M \subseteq R^q$ is called a *geometric realization* of an abstract complex Δ on $X = \{x_1, x_2, \ldots, x_t\}$ if there exists a set of independent points $\{v_1, v_2, \ldots v_t\}$, called the *base*, such that $M = \bigcup_{A \in \Delta} Y_A$, where $Y_A = \langle v_{i_1}, v_{i_2}, \ldots, v_{i_k} \rangle$ for $A = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$. Clearly, any abstract complex Δ on $X = \{x_1, x_2, \ldots, x_t\}$ has a geometric realization in R^q if $q \geq t$.

We will call $M \subseteq R^q$ a *geometric complex* if M is a geometric realization of some abstract complex Δ. It is a well-known fact in Topology (See, *e.g.* [M]) that if M is a geometric realization of two abstract complexes Δ and Δ', then $\chi(\Delta) = \chi(\Delta')$. Thus, we can define $\chi(M)$ as $\chi(\Delta)$ unambiguously.

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[2]In the paper all i_j's are distinct whenever they appear in the notation $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$.

3.3 Fixed Points

Let Δ be an abstract complex on $X = \{x_1, x_2, \ldots, x_t\}$, invariant under a permutation group Γ of $\{1, 2, \ldots, t\}$. Let M be a geometric realization of Δ with base $\{v_1, v_2, \ldots, v_t\}$. Then Γ induces a natural automorphism group on M. Precisely, for each $\sigma \in \Gamma$, let f_σ be the automorphism on M defined by

$$f_\sigma \left(\sum_{1 \leq i \leq k} \lambda_i v_i \right) = \sum_{1 \leq i \leq k} \lambda_i \; v_{\sigma(i)}$$

for $\lambda_i \geq 0$, $\sum_{1 \leq i \leq k} \lambda_i = 1$. Let M^Γ denote the set of *fixed points* of this automorphism group, *i.e.* $M^\Gamma \equiv \{v | v \in M, f_\sigma(v) = v \; \forall \; \sigma \in \Gamma\}$.

Theorem 2 *([KSS]) M^Γ is a geometric realization of Δ_Γ.*

For any two groups F and L , we say that L is a *homomorphic image* of F if there exists a homomorphism from F onto L. Let $\mathcal{Z}_\ell$ be the cyclic group of order ℓ.

Theorem 3 *(Oliver [O]) If Δ is rationally acyclic and Γ is a homomorphic image of $\mathcal{Z}_\ell$, then* $\chi(M^\Gamma) = 1$.

3.4 General String Properties and Topology

In the study of the complexity of evaluating graph properties, it has been found useful ([BBL] [RV2]) to consider the complexity of evaluating a more general class of functions, the *string properties.* A *string property* P is a function from $\{0,1\}^t$ into $\{0,1\}$. As done for graph properties in Section 1, we consider decision tree algorithms T for evaluating $P(a_1, a_2, \ldots, a_t)$ by asking an adaptive sequence of queries $a_{i_1} = \ ?$, $a_{i_2} = \ ?$, ...; we define $cost(T)$ and $C(P)$ in the same way. The property P is said to be *evasive* if $C(P) = t$. We say that P is *nontrivial* if P is not a constant; P is *monotone* if $P(a_1, a_2, \ldots, a_t) \leq P(a'_1, a'_2, \ldots, a'_t)$ when $a_i \leq a'_i$ for all i.

In [KSS], the approach to study a string property P is to associate with P the abstract complex Δ on $X = \{x_1, x_2, \ldots, x_t\}$ defined as follows: $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \in \Delta$ if $P(a_1, a_2, \ldots, a_t) = 0$ where $a_{i_1} = a_{i_2} = \ldots = a_{i_k} = 1$ and $a_j = 0$ for $j \neq i_\ell$. The following fundamental observation was made.

Theorem 4 (KSS) *If P is not evasive, then the associated Δ is rationally acyclic.*

We need one more concept. Let Γ be a permutation group of $\{1, 2, \ldots, t\}$. We say that P is *invariant under* Γ if $P(a_1, a_2, \ldots, a_t) = P(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(t)})$ for all $\sigma \in \Gamma$. It is clear that if P is invariant under Γ, so is the associated abstract complex Δ.

4 Proof of Theorem 1

First we rephrase the problem in the terminology of string property (Section 3.4). Let $\Sigma_{m,n} = S_m \oplus S_n$ where S_m, S_n are the symmetric groups on $V = \{1, 2, \ldots, m\}$, $W = \{1, 2, \ldots, n\}$. Each $\sigma \in \Sigma_{m,n}$ is a permutation of $\{(i,j)|1 \leq i \leq m,\ 1 \leq j \leq n\}$, that is, if $\sigma = (p_1, p_2)$ where $p_1 \in S_m$, $p_2 \in S_n$, then $\sigma(i,j) = (p_1(i), p_2(j))$ for all i, j. Let us regard a bipartite graph property P on $V \times W$ as a string property in the following way: Any input graph $G \in \mathcal{G}_{m,n}$ is identified with $a \equiv (a_{11}, a_{12}, \ldots, a_{ij}, \ldots, a_{mn}) \in \{0,1\}^{mn}$ where a is obtained from the adjacency matrix (a_{ij}) of G by concatenating the entries row by row; this naturally induces a string property $P' : \{0,1\}^{mn} \rightarrow \{0,1\}$. It is easy to see that if P is nontrivial and monotone, so is P' as a string property; also P is evasive if and only if P' is. In addition, P' is invariant under $\Sigma_{m,n}$.

To prove Theorem 1, let P be a nontrivial monotone bipartite graph property on $V \times W$. Denote by P' the corresponding string property on $\{0,1\}^{mn}$. Assume that P is not evasive, implying that P' is not evasive; we will derive a contradiction.

Let $D \subseteq \{1, 2, \ldots, m\}$. Denote by b_D the vector $(a_{11}, a_{12}, \ldots, a_{ij}, \ldots, a_{mn})$ where $a_{i\ell} = 1$ for $i \in D$, $1 \leq \ell \leq n$ and 0 otherwise.

Lemma 1 *There exists an integer* $0 \leq r(P') < m$ *such that* $P'(b_D) = 0$ *if* $|D| \leq r(P)$ *and* 1 *otherwise.*

Proof. As P' is invariant under $\Sigma_{m,n}$, $P'(b_D) = P'(b_{D'})$ if $|D| = |D'|$. It then follows from the monotonicity of P' that there exists an integer $-1 \leq r(P') \leq m$ such that $P'(b_D) = 0$ if and only if $|D| \leq r(P')$. Finally, $r(P') \neq -1, m$, since P' is nontrivial. ∎

Let Δ be the abstract complex associated with P'. Then Δ is rationally acyclic by Theorem 4. Let Γ be the subgroup $1 \oplus \mathcal{Z}_n$ of $\Sigma_{m,n}$, i.e. $\Gamma = \{\sigma_0, \sigma_1, \ldots, \sigma_{n-1}\}$ with $\sigma_\ell(i,j) = (i, (j+\ell) mod\ n)$. Then Δ is invariant under Γ since P' is invariant under Γ.

Now let M be a geometric realization of Δ. As Γ is clearly a homomorphic image of $\mathcal{Z}_n$, we have by Theorem 3 that $\chi(M^\Gamma) = 1$. Thus, $\chi(\Delta_\Gamma) = \chi(M^\Gamma) = 1$, since by Theorem 2 M^Γ is a geometric realization of Δ_Γ.

On the other hand, we have from the definition of Δ_Γ that
$\Delta_\Gamma = \{\{A_i | i \in D\} | D \subseteq \{1, 2, \ldots, m\},\ P'(b_D) = 0\}$ whenever $\Delta_\Gamma \neq \emptyset$. Thus, either $\Delta_\Gamma = \emptyset$ in which case $\chi(\Delta_\Gamma) = 0 \neq 1$, or we have by Lemma 1 that

$$\begin{aligned}
\chi(\Delta_\Gamma) &= \sum_{0 \leq j < r(P')} (-1)^j \binom{m}{j+1} \\
&= \sum_{0 \leq j < r(P')} (-1)^j \left[\binom{m-1}{j+1} + \binom{m-1}{j} \right]
\end{aligned}$$

$$
\begin{aligned}
&= 1+(-1)^{r(P')-1}\binom{m-1}{r(P')} \\
&\neq 1.
\end{aligned}
$$

This contradicts the conclusion of the last paragraph.

We have proved Theorem 1.

5 Remarks

The most tantalizing open question in this subject is whether all nontrivial monotone graph properties are evasive. As mentioned in [KSS], their topological approach cannot resolve this question when only the transitive nature of the underlying group for graph properties is exploited. The proof of our result on bipartite graphs, as well as the proof of evasiveness for graph properties on six vertices in [KSS], suggests that further progress might be possible if one examines in detail the structures of the geometric complexes associated with graph properties.

Another interesting direction for further work is to prove evasiveness for other classes of string properties. For example, any nontrivial monotone string properties that are transitively invariant under cyclic group C_m must be evasive (as can be seen from Theorem 2 in [KSS], or from Theorems 3 and 4 in this paper). Is the analogous result true for string properties invariant under $C_m \oplus C_n$?

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