SOME PROBLEMS ON DOUBLY PERIODIC INFINITE GRAPHS

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Some problems on doubly periodic infinite graphs[†]

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Abstract

We show that finding weak components, finding an Eulerian path, and testing 2-colorability of two-dimensional doubly periodic graphs can be done in polynomial time with respect to the size of the static graph.

1. Introduction

A k-dimensional *dynamic graph* is obtained by repeating a basic cell in a k-dimensional orthogonal grid. The nodes in each cell are connected to a finite number of nodes in other cells, and, furthermore, the pattern of the inter-cell connections is the same for each cell. Thus, a dynamic graph is a finitely described infinite graph, with a periodic structure. In this paper we study the following problems for two-dimensional dynamic graphs: finding weakly connected components, deciding whether there is an (undirected or directed) Eulerian path, and testing 2-colorability.

A two-dimensional dynamic graph can be represented by a finite graph with two-dimensional labels on each edge, which is called a *static graph*. From the definition, every two-dimensional dynamic graph is *doubly periodic*, *locally finite*, and infinite. Note that a graph is said to be *doubly periodic* if its automorphism group has two non-parallel translations, and an infinite graph is said to be *locally finite* if all vertices have finite valencies. We will show that the class of two-dimensional dynamic graphs is the same as the class of doubly periodic graphs.

Two-dimensional dynamic graphs arise naturally in the study of regular VLSI circuits, such as systolic arrays and VLSI signal processing arrays

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(Cappello and Steiglitz 1981; Iwano and Steiglitz 1986b). In these applications, the graphs associated with the circuits can be regarded as subgraphs of two-dimensional dynamic graphs. Orlin (1984) pointed out that many problems in transportation planning, communications, and operations management can be modeled by one-dimensional dynamic graphs. Doubly-weighted digraphs, which can be regarded as static graphs of two-dimensional dynamic graphs, have also been well studied. For example, Dantzig, Blatter, and Rao (1967) and Lawler (1967) studied optimal cycles with minimum ratio of two labels; Reiter (1968) studied these graphs for scheduling parallel computation. Orlin (1984) investigated various problems for one-dimensional dynamic graphs, such as finding weak or strong components, finding an Eulerian path, and testing 2-colorability. The authors studied the acyclicity problem (Iwano and Steiglitz 1986a, 1987) and planarity testing for two-dimensional dynamic graphs (Iwano and Steiglitz 1986c).

The regularity of dynamic graphs may lead us to efficient solutions of certain problems because we may be able to restrict problems to finite graphs which adequately represent them. We will show that finding weak components, finding an Eulerian path, and testing 2-colorability of two-dimensional dynamic graphs can be solved efficiently using this idea. Our algorithms also solve the problems of finding weak components and testing 2-colorability for one-dimensional dynamic graphs which are discussed by Orlin (1984).

2. Graph terminology

In this section, we review basic graph terminology (See Harary 1969; Christofides 1975) and define a dynamic graph as an infinite graph induced by a finite graph.

Definition 2.1: Given a digraph $G = (V, E)$, a *path* P in G is a sequence of vertices

$$P = v_0, v_1, \ldots, v_l$$

where

$$e_i = (v_{i-1}, v_i) \in E \text{ for } 1 \leq \forall i \leq l$$

and

$$v_i \in V \text{ for } 0 \leq \forall i \leq l.$$

If all vertices $v_0, v_1, \ldots, v_{l-1}$ are distinct, a path P is *simple*. A path P such that $v_0 = v_l$ is called a *cycle*. A path P is a *trail* if all edges in P are distinct. A *chain* P in G is a sequence of vertices

$$P = v_0, v_1, \ldots, v_l$$

where

$$v_i \in V \text{ for } 0 \leq \forall i \leq l$$

and either

$$e_i = (v_{i-1}, v_i) \in E \text{ or } \overline{e_i} = (v_i, v_{i-1}) \in E \text{ for } 1 \leq \forall i \leq l.$$

A digraph G is said to be *weakly connected* or *weak* if there is at least one chain joining every pair of distinct vertices. □

Definition 2.2: A *countable graph* is one in which both the vertex set and the edge set are finite or countably infinite. A graph is *locally finite* if the valence of every vertex is finite. □

Definition 2.3: Let $G^0 = (V^0, E^0)$ be a finite directed graph with

$$V^0 = \{v_1, v_2, \ldots, v_n\}.$$

Let

$$T^k : E^0 \to Z^k$$

be a k-dimensional labeling of E^0 such that

$$T^k(e) = \{e_{(1)}, e_{(2)}, \ldots, e_{(k)}\} \in Z^k$$

for every $e \in E^0$. For each $\mathbf{x} \in Z^k$, we call $v_{i,\mathbf{x}}$ *the* $\mathbf{x}$*-th copy* of $v_i \in V^0$, and

$$V_\mathbf{x} = \{v_{1,\mathbf{x}}, v_{2,\mathbf{x}}, \ldots, v_{n,\mathbf{x}}\}$$

the $\mathbf{x}$*-th copy of* V^0. Then we can define the *k-dimensional dynamic graph* $G^k = (V^k, E^k, T^k)$ induced by G^0 as follows:

$$\begin{cases} V^k = \bigcup_{\mathbf{x} \in Z^k} V_\mathbf{x} \\ E^k = \{ (v_{i,\mathbf{x}}, v_{j,\mathbf{y}}) \mid (v_i, v_j) \in E^0, \mathbf{y} - \mathbf{x} = T^k((v_i, v_j)) \}. \end{cases} \tag{2.1}$$

We call G^0 the *static graph* of G^k. The edge with the $T^k(e)$-label is called the *$T^k(e)$-edge*. □

Note that G^k is an infinite graph and is locally finite. Moreover, we have the following theorem, which is easy to prove.

Theorem 2.1: The class of two-dimensional dynamic graphs is the same as the class of doubly periodic graphs. □

We use $\mathbf{0}$ to represent the origin in Z^k; that is, $\mathbf{0} = (0, 0, . . . , 0)$. We now define *the basic cell of G^k* as follows:

Definition 2.4: For $\mathbf{x}, \mathbf{y} \in Z^k$, let

$$E_{\mathbf{x},\mathbf{y}} = \{ (v_{i,\mathbf{x}}, v_{j,\mathbf{y}}) \in E^k \}.$$

When $\mathbf{x} \neq \mathbf{y}$, we call $E_{\mathbf{x},\mathbf{y}}$ *the connecting edges*. We call

$$C_{\mathbf{x}} = (V_{\mathbf{x}}, E_{\mathbf{x},\mathbf{x}})$$

the $\mathbf{x}$*-th cell of G^k*. In particular, we call $C_{\mathbf{0}}$ the *basic cell of G^k*. □

From this definition, we can regard G^k as the union of cells and connecting edges. When we regard each cell of G^k as a point, we have another dynamic graph that we call the *cell-dynamic graph* G_c^k. We call the static graph of G_c^k the *cell-static graph* G_c^0. Fig. 1c shows our notation: the superscript k of G indicates a k-dimensional dynamic graph, while the superscript 0 indicates a static graph. The subscript c of G^0 (resp. G^k) indicates a cell-static (resp. cell-dynamic) graph. We now define G_c^0 and G_c^k formally as follows:

Definition 2.5: Let G^k be a dynamic graph defined in Eq. (2.1). Then the *cell-static graph* G_c^0 is the following multidigraph:

$$G_c^0 = (V_c^0, E_c^0, T_c^k)$$

where

$$\begin{cases} V_c^0 = \{ v \} \\ E_c^0 = \{ e' = (v, v) \mid e \in E^0, T(e) \neq \mathbf{0} \} \\ T_c^k(e') = T^k(e) \text{ for } \forall e' \in E_c^0. \end{cases} \tag{2.2}$$

The *cell-dynamic graph*

$$G_c^k = (V_c^k, E_c^k, T_c^k)$$

is the dynamic graph induced by G_c^0 and described as follows:

$$\begin{cases} V_c^k = Z^k \\ E_c^k = \{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in V_c^k, \exists (v_{i,\mathbf{x}}, v_{j,\mathbf{y}}) \in E_{\mathbf{x},\mathbf{y}}, \mathbf{x} \neq \mathbf{y} \}. \end{cases}$$

□

From now on, we assume that every vertex of the cell-dynamic graph G_c^k is located at an integer lattice point of the Euclidean plane Z^k, and we sometimes use $\mathbf{x} \in Z^k$ to represent the vertex of G_c^k which is located at $\mathbf{x}$.

In Fig. 1a, the two-dimensional dynamic graph G^2 is induced by a static graph G^0, while in Fig. 1b, the cell-dynamic graph G_c^2 is induced by the cell-static graph G_c^0. The cell-dynamic graph G_c^2 represents the interconnection between cells in the dynamic graph G^2, and the cell-static graph G_c^0 consists of edges with non-$\mathbf{0}$ labels in G^0.

3. Weak connectivity

In this section, we will investigate the problem of finding the number of weakly connected components in a two-dimensional dynamic graph.

If the basic cell is not connected, the associated dynamic graph is also not connected. Thus without loss of generality, we can assume the following:

1) The basic cell C_0 is connected.

Since we are concerned with weak connectivity, each connected cell can be regarded as a point. Therefore, we can assume the following 1') instead of 1).

1') The basic cell C_0 consists of one point; that is, $G^2 = G_c^2$. In other words, the dynamic graph is the same graph as its cell-dynamic graph.

Hence we can assume the following static graph $G^0 = (V^0, E^0, T^2)$ which induces the two-dimensional dynamic graph $G^2 = (V^2, E^2, T^2)$:

$$\begin{cases} V^0 = \{ v \} \\ E^0 = \{ e_1, e_2, \ldots, e_m \} \text{ where } e_i = (v, v) \\ T^2(e_i) = \mathbf{e_i} = (x_i, y_i) \in Z \times Z \text{ for } i = 1, 2, \ldots, m. \end{cases} \tag{3.1}$$

Definition 3.1: Let I_2 be the set of two-dimensional integer vectors. Let $\mathbf{f_1}, \mathbf{f_2} \in I_2$. Then we denote the parallelogram formed by $\mathbf{0}, \mathbf{f_1}, \mathbf{f_2}$, and $\mathbf{f_1} + \mathbf{f_2}$ by $P(\mathbf{f_1}, \mathbf{f_2})$. □

Definition 3.2: Let $\mathbf{e_i} = (x_i, y_i) \in I_2$ for $i = 1, 2, \ldots, m$. Define $\mathbf{e_1} \bigcirc \mathbf{e_2}$ and $\bigcirc gcd$ in the following way:

$$\mathbf{e_1} \bigcirc \mathbf{e_2} = x_1 y_2 - y_1 x_2$$

and

$$\bigcirc gcd \{ \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \} = gcd \{ | \mathbf{e_i} \bigcirc \mathbf{e_j} |, 1 \leq i < j \leq m \}.$$

□

Note that the area of parallelogram $P(\mathbf{e_1}, \mathbf{e_2})$ is $| \mathbf{e_1} \bigcirc \mathbf{e_2} |$.

Definition 3.3: Let $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \in I_2$. Then we denote the set of linear combinations of $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}$ by $(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m})$. □

We sometimes use $\mathbf{e}$ to represent the point (x, y) in the plane when $\mathbf{e} = (x, y)$.

Theorem 3.1: Let $G^2 = (V^2, E^2, T^2)$ be a two-dimensional dynamic graph. Then G^2 is weakly connected if and only if

$$\bigcirc gcd \{ \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \} = 1 \tag{3.2}$$

where the $\mathbf{e}_i$ are defined in (3.1).

Proof: Suppose G^2 is weakly connected. Then $(1, 0)$ and $(0, 1)$ can be expressed by linear combinations of the $\mathbf{e_i}$. Thus there exist some $a_i, b_j \in Z$

such that

$$(1, 0) = \sum_{i=1}^{m} a_i \mathbf{e_i},\ (0, 1) = \sum_{j=1}^{m} b_j \mathbf{e_j}. \tag{3.3}$$

Note that $(1, 0) \bigcirc (0, 1) = 1$. Therefore, from (3.3),

$$\begin{aligned} & \left(\sum_{i=1}^{m} a_i \mathbf{e_i} \right) \bigcirc \left(\sum_{j=1}^{m} b_j \mathbf{e_j} \right) \\ = & \left(\sum_{i=1}^{m} a_i x_i \right)\left(\sum_{j=1}^{m} b_j y_j \right) - \left(\sum_{i=1}^{m} a_i y_i \right)\left(\sum_{j=1}^{m} b_j x_j \right) \\ = & \sum_{1 \le i,j \le m} a_i b_j (x_i y_j - y_i x_j) = \sum_{1 \le i,j \le m} a_i b_j (\mathbf{e_i} \bigcirc \mathbf{e_j}) = 1. \end{aligned}$$

Therefore, (3.2) holds.

Conversely, suppose (3.2) holds. Then there exist $c_{ij} \in Z$ for $1 \le i,j \le m$ such that

$$\sum_{1 \le i,j \le m} c_{ij} (\mathbf{e_i} \bigcirc \mathbf{e_j}) = 1. \tag{3.4}$$

Therefore, we have

$$\sum_{1 \le i,j \le m} (c_{ij} - c_{ji}) x_i y_j = 1. \tag{3.5}$$

Let

$$X_i = \sum_{j=1}^{m} (c_{ij} - c_{ji}) y_j,\ Y_j = \sum_{i=1}^{m} (c_{ij} - c_{ji}) x_i, \tag{3.6}$$

and

$$\mathbf{f_x} = \sum_{i=1}^{m} X_i \mathbf{e_i},\ \mathbf{f_y} = \sum_{j=1}^{m} Y_j \mathbf{e_j}. \tag{3.7}$$

Then we have $\mathbf{f_x} = (1, 0)$ and $\mathbf{f_y} = (0, 1)$ as follows. From (3.5), we have

$$\sum_{i=1}^{m} X_i x_i = 1,\ \sum_{j=1}^{m} Y_j y_j = 1. \tag{3.8}$$

We also have

$$\begin{cases} \displaystyle\sum_{i=1}^{m} X_i y_i = \sum_{1 \le i,j \le m} (c_{ij} - c_{ji}) y_i y_j = 0 \\ \displaystyle\sum_{j=1}^{m} Y_j x_j = \sum_{1 \le i,j \le m} (c_{ij} - c_{ji}) x_i x_j = 0. \end{cases} \tag{3.9}$$

Therefore, we have $\mathbf{f_x} = (1, 0)$ and $\mathbf{f_y} = (0, 1)$, and thus G^2 is weakly connected. □

Corollary 3.1: The weak connectivity of G^2 can be tested in

$$O(m^2 \log e_{\max})$$

steps where m is the number of edges in the static graph and

$$e_{\max} = \max \{ |x_i|, |y_i| \mid \mathbf{e_i} = (x_i, y_i) \in E^0 \text{ for } 1 \le \forall i \le m \}.$$

Proof: Euclid's algorithm computes $gcd(a, b)$ in $< 2\log N = O(\log N)$ iterations for $0 \le a \le b \le N$ (Lipson 1981, 208). We need m^2 gcd computations. □

We have a stronger result than Theorem 3.1 as follows:

Theorem 3.2: The number of weakly connected components of G^2 is

$$Ogcd\{ \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \}.$$

Before proving Theorem 3.2, we need the following lemmas:

Lemma 3.1: Let $\mathbf{g}, \mathbf{f_1}, \mathbf{f_2} \in I_2$ be such that $\mathbf{g}$ is properly contained in $P(\mathbf{f_1}, \mathbf{f_2})$. Then

$$0 < \max \{ |\mathbf{g} \bigcirc \mathbf{f_1}|, |\mathbf{g} \bigcirc \mathbf{f_2}| \} < \mathbf{f_1} \bigcirc \mathbf{f_2}.$$

Proof: Trivial. □

Lemma 3.2: Let

$$\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}, \mathbf{f} \in I_2$$

be such that

$$\mathbf{f} \in (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}).$$

Then

$$Ogcd\{ \mathbf{f}, \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \} = Ogcd\{ \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \}. \tag{3.10}$$

Proof: Let p be the right-hand-side of (3.10). Let $\mathbf{f} = \sum_{i=1}^{m} a_i \mathbf{e_i}$. Then for any $1 \le j \le m$,

$$\mathbf{f} \bigcirc \mathbf{e_j} = \sum_{i=1}^{m} a_i (\mathbf{e_i} \bigcirc \mathbf{e_j}).$$

Thus, p divides $\mathbf{f} \bigcirc \mathbf{e_j}$. Therefore,

$$p = \bigcirc gcd\{ \mathbf{f}, \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \}.$$

□

Lemma 3.3: Let $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \in I_2$ be such that

$$p = \bigcirc gcd\{ \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \}.$$

Suppose there exist

$$\mathbf{f_1}, \mathbf{f_2} \in (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m})$$

such that

$$\mathbf{f_1} \bigcirc \mathbf{f_2} = p.$$

If $\mathbf{g} \in I_2$ is properly contained in $P(\mathbf{f_1}, \mathbf{f_2})$, then

$$\mathbf{g} \notin (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}).$$

Proof: Suppose

$$\mathbf{g} \in (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}).$$

Then from Lemma 3.2,

$$p = \bigcirc gcd\{ \mathbf{g}, \mathbf{f_1}, \mathbf{f_2}, \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \}.$$

Therefore, p divides $\mathbf{g} \bigcirc \mathbf{f_1}$. However, from Lemma 3.1,

$$0 < | \mathbf{g} \bigcirc \mathbf{f_1} | < \mathbf{f_1} \bigcirc \mathbf{f_2} = p.$$

This is a contradiction. □

Corollary 3.2: Let p, $\mathbf{f_1}$, $\mathbf{f_2}$ be defined as above in Lemma 3.3. If two different vectors $\mathbf{g_1}, \mathbf{g_2} \in I_2$ are properly contained in $P(\mathbf{f_1}, \mathbf{f_2})$, then

$$\mathbf{g_2} - \mathbf{g_1} \notin (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}).$$

Proof: If $\mathbf{g}_2 - \mathbf{g}_1$ is properly contained in $P(\mathbf{f}_1, \mathbf{f}_2)$, Corollary 3.2 holds by Lemma 3.3. If not, there exists a rational number $0 < r < 1$ such that

$$\mathbf{g}_2 - \mathbf{g}_1 = r\mathbf{f}_1 \text{ or } r\mathbf{f}_2.$$

Suppose

$$r\mathbf{f}_1 \in (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m).$$

Then, p divides $r\mathbf{f}_1 \bigcirc \mathbf{e}_1 = rp < p$, which is a contradiction. Thus,

$$\mathbf{g}_2 - \mathbf{g}_1 \notin (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m).$$

□

Lemma 3.4: Let p, $\mathbf{f}_1$, $\mathbf{f}_2$ be defined as above in Lemma 3.3. Then for any $\mathbf{g} \in I_2$, there exists a vector $\mathbf{g}' \in P(\mathbf{f}_1, \mathbf{f}_2)$ such that

$$\mathbf{g} - \mathbf{g}' \in (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m).$$

Proof: Let R_{ab} be the translated parallelogram formed by

$$a\mathbf{f}_1 + b\mathbf{f}_2, \ (a+1)\mathbf{f}_1 + b\mathbf{f}_2,$$
$$(a+1)\mathbf{f}_1 + (b+1)\mathbf{f}_2, \text{ and } a\mathbf{f}_1 + (b+1)\mathbf{f}_2.$$

Then there exists some R_{ab} which contains $\mathbf{g}$. Let

$$\mathbf{g}' = \mathbf{g} - (a\mathbf{f}_1 + b\mathbf{f}_2).$$

Then

$$\mathbf{g}' \in P(\mathbf{f}_1, \mathbf{f}_2)$$

and

$$\mathbf{g} - \mathbf{g}' = a\mathbf{f}_1 + b\mathbf{f}_2 \in (\mathbf{f}_1, \mathbf{f}_2) \subset (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m).$$

□

Lemma 3.5: Let

$$p = \bigcirc gcd\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}.$$

Then there exist two vectors

$$\mathbf{f}_1, \mathbf{f}_2 \in (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m)$$

such that

$$(\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathbf{m}).$$

Therefore,

$$|\ \mathbf{f}_1 \bigcirc \mathbf{f}_2\ | = p.$$

Proof: Suppose not. Let

$$\mathbf{f}_1, \mathbf{f}_2 \in (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathbf{m})$$

be such that

$$\mathbf{f}_1 \bigcirc \mathbf{f}_2 = \min \{ \mathbf{g}_1 \bigcirc \mathbf{g}_2 \mid \mathbf{g}_1, \mathbf{g}_2 \in (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathbf{m}), \mathbf{g}_1 \bigcirc \mathbf{g}_2 > 0 \}. \tag{3.11}$$

Since we assumed $\mathbf{f}_1 \bigcirc \mathbf{f}_2 \neq p$, there exists an integer $k > 1$ such that

$$\mathbf{f}_1 \bigcirc \mathbf{f}_2 = kp.$$

Note that

$$\exists\, \mathbf{g} \in \{ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathbf{m} \} \ \ s.t.\ \ \mathbf{g} \notin (\mathbf{f}_1, \mathbf{f}_2), \tag{3.12}$$

because otherwise,

$$(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathbf{m}) \subset (\mathbf{f}_1, \mathbf{f}_2),$$

and then

$$kp = \bigcirc gcd \{ \mathbf{f}_1, \mathbf{f}_2 \} = \mathbf{f}_1 \bigcirc \mathbf{f}_2$$

can divide

$$\bigcirc gcd \{ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathbf{m} \} = p.$$

This contradicts $k > 1$.

From (3.12) and Lemma 3.4,

$$\exists\, \mathbf{g}' \in P(\mathbf{f}_1, \mathbf{f}_2) \ \ s.t.\ \ \mathbf{g} - \mathbf{g}' \in (\mathbf{f}_1, \mathbf{f}_2).$$

From Lemma 3.1,

$$0 < \max\{ \mid \mathbf{g}' \bigcirc \mathbf{f}_1 \mid, \mid \mathbf{g}' \bigcirc \mathbf{f}_2 \mid \} < \mathbf{f}_1 \bigcirc \mathbf{f}_2. \tag{3.13}$$

Note that since

$$\mathbf{g}, \mathbf{f}_1, \mathbf{f}_2 \in \{ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathbf{m} \}, \mathbf{g}' \in (\mathbf{g}, \mathbf{f}_1, \mathbf{f}_2),$$

we have

$$\mathbf{g}' \in (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathbf{m}).$$

Therefore, (3.13) contradicts the definition of $\mathbf{f_1} \bigcirc \mathbf{f_2}$ in (3.11). □

We now prove Theorem 3.2.

Proof of Theorem 3.2: From Lemma 3.5,

$$\exists \mathbf{f_1}, \mathbf{f_2} \in \{ \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \}$$

such that

$$\mathbf{f_1} \bigcirc \mathbf{f_2} = \bigcirc gcd\{ \mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m} \} = p.$$

Corollary 3.2 implies that any two distinct vectors $\mathbf{g_1}, \mathbf{g_2} \in P(\mathbf{f_1}, \mathbf{f_2})$ cannot be connected by linear combinations of $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}$. Thus $\mathbf{g_1}$ and $\mathbf{g_2}$ are in two different weakly connected components. Therefore, there exist at least p components. Lemma 3.4 implies that there exist at most p components. □

Note that each weakly connected component of G^2 corresponds to an element of the quotient ring

$$(Z \times Z) \,/\, (\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_m}).$$

Orlin (1984) computed the number of weakly connected components in a one-dimensional dynamic graph. We can obtain the same result from Theorem 3.2 as follows:

Theorem 3.3: Let $G^1 = (V^1, E^1, T^1)$ be a connected one-dimensional dynamic graph induced by a static graph

$$G^0 = (V^0, E^0, T^1)$$

with

$$\begin{cases} E^0 = \{ e_1, e_2, \ldots, e_m \} \\ T^1(e_i) = x_i \in Z \quad \text{for } 1 \le \forall i \le m. \end{cases}$$

Then the number of weakly connected components of G^1 is

$$gcd(x_1, x_2, \ldots, x_m).$$

Proof: We can assume that the basic cell of G^0 is connected, as we assumed for G^2. As we discussed in the beginning of this section, we can also assume that V^0 consists of only one vertex. We now define a two dimensional dynamic graph $G^2 = (V^2, E^2, T^2)$ induced by the following static graph $S^0 = (V, E, T^2)$:

$$\begin{cases} V = V^0 = \{ v \} \\ E = E^0 \cup \{ e_0 = (v, v) \} \\ T^2(e_i) = \begin{cases} \mathbf{e_i} = (T^1(e_i), 0) & \text{if } 1 \le \forall i \le m \\ \mathbf{e_0} = (0, 1) & \text{if } i = 0. \end{cases} \end{cases}$$

Fig. 2 illustrates examples of G^1 and G^2.

Let $V^1 = \{ v_x \mid x \in Z \}$ and $V^2 = \{ v_{xy} \mid x, y \in Z \}$ where v_x denotes the x-th node in G^1 and v_{xy} denotes the (x, y)-th node in G^2. Let $v_a \longleftrightarrow v_b$ denote that v_a and v_b are connected by a chain; that is, v_a and v_b are in the same weakly connected component. We use the same notation for G^2, namely $v_{ab} \longleftrightarrow v_{cd}$.

If $v_{ab} \longleftrightarrow v_{cd}$, then $v_{ad} \longleftrightarrow v_{ab} \longleftrightarrow v_{cd}$, because v_{ad} and v_{ab} are connected by $(d - b)$ copies of e_0 edges. Therefore, $v_a \longleftrightarrow v_c$.

Conversely, if $v_a \longleftrightarrow v_c$, then

$$v_{ab} \longleftrightarrow v_{cd} \text{ for } \forall b, d \in Z.$$

This is because $v_{ab} \longleftrightarrow v_{ad}$ by $(d - b)$ copies of e_0 edge and $v_{ad} \longleftrightarrow v_{cd}$. Suppose G^1 is located horizontally along with the x-axis as shown in Fig. 2. Then

$$v_{ab} \longleftrightarrow v_{ad} \longleftrightarrow v_{cd}.$$

Therefore, the number of weakly connected components in G^1 is the same as in G^2. Thus, from Theorem 3.2, the number of weakly connected components of G^2 is

$$Ogcd\{ \mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_m \} = gcd(x_1, x_2, \ldots, x_m).$$

□

4. Eulerian path

In this section, we will show that a two-dimensional connected cell-dynamic graph is Eulerian and that two-dimensional dynamic graph is

Eulerian if and only if its static graph is Eulerian. Except when otherwise stated, all graphs discussed are undirected.

The following are well known facts about finite Eulerian graphs:

Theorem 4.1: (See Harary 1969) A connected finite graph G is Eulerian if and only if every vertex has even valency. □

Corollary 4.1: (See Harary 1969) Let G be a connected finite graph with exactly $2n$ odd vertices, $n > 0$. Then the set of edges of G can be partitioned into n open trails. □

Before introducing an extension of Theorem 4.1 to infinite graphs, we need the following definition:

Definition 4.1: Let S be a finite set of edges in an infinite graph G. Denote by $\mid G \mid_\infty$ the number of infinite components in G, and by G/S the graph obtained by deleting S from G. A connected infinite graph G is said to be *k-separable* for a positive integer k, if there is a finite set S of edges in G such that $\mid G/S \mid_\infty \geq k$. □

Erdős, Grünwald and Vázsonyi (1938) extended the above theorem to infinite graphs as follows: (See Thomassen 1983)

Theorem 4.2: (Erdős, Grünwald and Vázsonyi 1938) A connected multi-graph has a 2-way infinite Eulerian trail if and only if

1) $E(G)$ is countably infinite;
2) all vertices have even or infinite valency;
3) G is not 3-separable;
4) there is no finite Eulerian subgraph whose edge-deletion leaves more than one infinite component. □

We have the following theorem about the separability of connected two-dimensional cell-dynamic graphs.

Theorem 4.3: Let G_c^2 be a connected cell-dynamic graph. Then G_c^2 is not 2-separable.

Proof: We can assume that every vertex of G_c^2 is located at an integer lattice point in the Euclidean plane. Let S_0 be an arbitrary finite set of edges in G_c^2. Without loss of generality, we can assume that

$$S_0 \subset [\,0,\ R\,] \times [\,0,\ R\,].$$

Then let

$$S_1 = \bigcup_{\mathbf{x},\mathbf{y} \in [0,R]\times[0,R]} E_{\mathbf{x},\mathbf{y}}.$$

Note that $E_{\mathbf{x},\mathbf{y}}$ is the set of connecting edges between the **x**-th and **y**-th cells as defined in Definition 2.4. Then

$$|\ G/S_1\ |_\infty \geq |\ G/S_0\ |_\infty.$$

Therefore, without loss of generality, we can assume that $S_0 = S_1$. See Fig. 3 for the following discussion. Suppose $|\ G/S_0\ |_\infty \geq 2$. Since G_c^2 is weakly connected, from Section 3, there exist $a_i,\ b_i \in Z$ for $i = 1, 2, \ldots, m$ such that

$$\sum_{i=1}^{m} a_i \mathbf{e}_i = (\,1,\ 0\,),\ \sum_{i=1}^{m} b_i \mathbf{e}_i = (\,0,\ 1\,) \tag{4.1}$$

where

$$E_c^0 = \{\, e_1,\ e_2,\ \ldots,\ e_m \,\},\ T_c^2(\, e_i \,) = \mathbf{e}_i = (\, x_i,\ y_i \,).$$

Let

$$M = \max \{\ \sum_{i=1}^{m} |\ a_i\ ||\ x_i\ |,\ \sum_{i=1}^{m} |\ a_i\ ||\ y_i\ |, \tag{4.2}$$

$$\sum_{i=1}^{m} |\ b_i\ ||\ x_i\ |,\ \sum_{i=1}^{m} |\ b_i\ ||\ y_i\ |\ \}.$$

Since we assume that $|\ G/S_0\ |_\infty \geq 2$, there exist two points $s = (\, s_x,\ s_y \,)$ and $t = (\, t_x,\ t_y \,)$ that lie in two different infinite components and

$$s,\ t \notin [\,-M,\ R + M\,] \times [\,-M,\ R + M\,]. \tag{4.3}$$

Let $v \longleftrightarrow w$ denote that two points v and w in G are connected by a chain that uses no edges in S_0. Then from (4.1), (4.2), and (4.3),

$$s \longleftrightarrow s + (\,1,\ 0\,)$$

by a chain consisting of a_i copies of $\mathbf{e}_i$ for each $1 \le i \le m$. In the same way, we have

$$s = (s_x, s_y) \longleftrightarrow s + (1,0) \longleftrightarrow \cdots \longleftrightarrow (R+2M, s_y)$$

and

$$(R+2M, s_y) \longleftrightarrow (R+2M, s_y) + (0,1) \longleftrightarrow \cdots \longleftrightarrow (R+2M, R+2M).$$

Therefore, we have

$$s \longleftrightarrow (R+2M, R+2M). \qquad (4.4)$$

In the same way, we have

$$t \longleftrightarrow (R+2M, R+2M). \qquad (4.5)$$

Therefore, from (4.4) and (4.5), we have $s \longleftrightarrow t$, which is a contradiction. Thus, $| G/S_0 |_\infty = 1$. □

Corollary 4.2: A connected two dimensional dynamic graph G^2 is not 2-separable.

Proof: Suppose we delete a finite set of edges S_1 in G^2. Instead of deleting S_1, we delete all edges in cells which have endpoints of edges in S_1. Let S_0 be the set of all edges in deleted cells. Then the number of infinite components, $| G^2/S_0 |_\infty$, is the same as $| G_c^2/S_0 |_\infty$. Since the cell-dynamic graph G_c^2 is connected, from Theorem 4.3,

$$| G^2/S_1 |_\infty = | G_c^2/S_0 |_\infty = 1.$$

□

Theorem 4.4: A connected cell-dynamic graph is Eulerian.

Proof: The four properties in Theorem 4.2 are satisfied as follows: By definition, dynamic graphs are countable. Since there are the same number of outgoing and incoming edges in every cell, all vertices have even valency. From Theorem 4.3, 3) and 4) are immediate. □

Theorem 4.5: A connected two-dimensional dynamic graph G^2 is Eulerian if and only if its static graph G^0 is Eulerian.

Proof: The "only if" part is immediate. From Theorem 4.2, all vertices in G^2 have even valency, and this implies that every vertex in G^0 also has even valency. Therefore, a connected finite graph G^0 is Eulerian from Theorem 4.1.

We now prove the "if" part. If the static graph G^0 is Eulerian, then every vertex in G^0 has even valency. Let $v_{\mathbf{x}}$ be the **x**-th copy of $v \in V^0$ for $\mathbf{x} \in Z \times Z$. Then $v_{\mathbf{x}}$ has the same valency as v. Therefore, every vertex in G^2 also has even valency. From Corollary 4.2, properties 3) and 4) in Theorem 4.2) are satisfied. □

We can show that an Eulerian path P in G^2 can be obtained from an Eulerian path P_c in G_c^2 as follows: Every cell of G^2 has $2k$ edges connecting to other cells. Therefore, from Corollary 4.1, the set of edges in each cell can be partitioned into k open trails. Thus, P is obtained by attaching these k open trails to P_c at each cell.

From now on, we will study *directed Eulerian paths* in two-dimensional dynamic graphs. A *directed Eulerian path* is an Eulerian path which is directed. Nash-Williams (1966) showed the following necessary and sufficient conditions for the existence of directed Eulerian paths in a directed infinite graph. We use $\rho^+(v)$ (resp. $\rho^-(v)$) to represent the out-valency (resp. in-valency) of vertex v.

Theorem 4.6: (Nash-Williams 1966) A connected multigraph G has a 2-way infinite directed Eulerian path if and only if

1) $E(G)$ is countable;
2) the valencies satisfy $\rho^+(v) = \rho^-(v)$ for all $v \in V(G)$;
3) any set of vertices with infinitely many out-going edges must have infinitely many in-coming edges;
4) G is not 3-separable;
5) If G is 2-separable, G possesses a set of vertices X such that X has a finite number (say n_{out}) of out-going edges and a finite number (say n_{in}) of in-coming edges and $n_{out} = n_{in} + 1$. □

Then we have the following theorem which corresponds to Theorem 4.5.

Theorem 4.7: A connected two-dimensional dynamic graph G^2 is directed Eulerian if and only if its static graph is directed Eulerian.

Proof: The proof is similar to that of Theorem 4.5. Note that a directed Eulerian static graph G^0 implies 2). □

Corollary 4.3: The existence of an (undirected or directed) Eulerian path in a two-dimensional dynamic graph G^2 can be tested in linear time with respect to the number of edges in the static graph.

Proof: We have to test only whether there exists an (undirected or directed) Eulerian path in the static graph. □

5. 2-colorability

We now deal with the 2-colorability of dynamic graphs. We have two basic theorems about k-colorability as follows:

Theorem 5.1: (De Bruijn and Erdős 1951) A (finite or infinite) graph is k-colorable if and only if every finite subgraph is. □

Theorem 5.2: (König 1936) A graph G is 2-colorable if and only if there is no odd cycle. □

We assume, without loss of generality, that two-dimensional dynamic graphs G^2 are connected. For, otherwise, we can consider the components separately. We can also assume that if G^0 is the static graph corresponding to a two-dimensional dynamic graph G^2, then G^0 has an arborescence whose labels are all $(0,0)$, because the following two-dimensional labeling T_d^2 induces the same dynamic graph and the desired arborescence (Orlin 1984). Let S be a spanning tree of G^0 with root v_0. Let $\Delta(u)$ for $u \in V^0$ be the distance in S from v_0 to u. Then we can define a two-dimensional labeling T_d^2 by

$$T_d^2(e) = T^2(e) + \Delta(u) - \Delta(v) \text{ for } \forall e = (u, v) \in E^0.$$

Without loss of generality, we can also assume that the basic cell is 2-colorable. For, otherwise, a dynamic graph which includes a basic cell as a subgraph cannot be 2-colorable. Orlin (1984) solved the problem of determining the 2-colorability of one-dimensional dynamic graphs. We, however, will give another 2-colorability test and then extend our approach to two-dimensional dynamic graphs.

Before describing our 2-colorability test, we need some definitions. Let A_{G^0} be an arborescence of G^0 with root v_0 such that all edge labels are $(0, 0)$. Let

$$V_{even} = \{ v \in V^0 \mid \textit{the distance from } v_0 \textit{ to } v \textit{ is even.} \}$$

and let V_{odd} be defined similarly. Let v_0 be the root of an arborescence in the static graph and use $v_{0,\mathbf{x}}$ to indicate the $\mathbf{x}$-th copy of v_0. We can now define what we call the *constraint graph*

$$H_{G^0} = (V(H_{G^0}), E(H_{G^0}))$$

of G^0 and a two-dimensional edge labeling T_H as follows:

$$\begin{cases} V(H_{G^0}) = \{ a, b \} \\ E(H_{G^0}) = \{ e_i' \mid e_i \in E^0, T^2(e_i) \neq (0, 0) \} \end{cases}$$

where e_i' and their labels are defined as follows:

$$e_i' = \begin{cases} (a, a) \text{ if } e_i \in V_{even} \times V_{even} \cup V_{odd} \times V_{odd} \\ (a, b) \text{ if } e_i \in V_{even} \times V_{odd} \cup V_{odd} \times V_{even} \end{cases} \tag{5.1}$$

and

$$T_H(e_i') \in \{ 0, 1 \} \times \{ 0, 1 \}$$

is defined by

$$T_H(e_i') \equiv T^2(e_i) \; (mod\ 2).$$

For example, in Fig. 4a, the static graph G^0 induces a constraint graph H_{G^0}.

Suppose we use two colors red and black. If $v_{0,\mathbf{x}}$ is colored by red (resp. black), we call the cell $C_{\mathbf{x}}$ the *red type* (resp. *black type*). Let R (resp. B) represent a red (resp. black) type cell. Then we have the following lemma:

Lemma 5.1: Let G^2 be a bipartite dynamic graph and let G^0 be its static graph. Let $f \in E(H_{G^0})$ and $C_{\mathbf{x}}$, $C_{\mathbf{y}}$ be two cells in G^2 such that

$$T_H(f) = \mathbf{f} \equiv \mathbf{y} - \mathbf{x} \ (mod\ 2).$$

If $f = (a, a)$, then the two cells $C_{\mathbf{x}}$ and $C_{\mathbf{y}}$ are different types. On the other hand, if $f = (a, b)$, then the two cells $C_{\mathbf{x}}$ and $C_{\mathbf{y}}$ are the same type.

Proof: Suppose $f = (a, a)$. Then there exists an edge $e \in E^0$ with

$$T^2(e) = \mathbf{e} \neq (0, 0)$$

which induces an edge $f = (a, a)$ in H_{G^0} and

$$\mathbf{e} \equiv \mathbf{f} \ (mod\ 2).$$

Since G^2 is connected, there are two closed chains $P_{0,1}$ and $P_{1,0}$ in G^0 such that $v_0 \in P_{0,1}, P_{1,0}$ and

$$T^2(P_{0,1}) = (0, 1),\ T^2(P_{1,0}) = (1, 0).$$

Since

$$\mathbf{y} - \mathbf{x} \equiv \mathbf{f} \equiv \mathbf{e} \ (mod\ 2),$$

there are some $p, q \in Z$ such that

$$2pT^2(P_{0,1}) + 2qT^2(P_{1,0}) + \mathbf{e} = \mathbf{y} - \mathbf{x}.$$

This means that there is an odd length chain from $v_{0,\mathbf{x}}$ to $v_{0,\mathbf{y}}$ which consists of $2|\ p\ |$ copies of $P_{0,1}$ and $2|\ q\ |$ copies of $P_{1,0}$ and a copy of the edge e. Therefore, $v_{0,\mathbf{x}}$ and $v_{0,\mathbf{y}}$ are colored differently, which implies that the two cells $C_{\mathbf{x}}$ and $C_{\mathbf{y}}$ are different types. The other case is handled in the same way. □

Lemma 5.2: Let G^1 be a connected bipartite one-dimensional dynamic graph. Then, as illustrated in Fig. 4b, the pattern of cell types of G^0 is either one of the following:

(RR): Every cell has the same cell type (say R).

(RB): Two different cell types appear alternately.

Proof: Since G^1 is connected, there is a path P between $v_{0,x}$ and $v_{0,x+1}$ for $\forall x \in Z$ where $v_{0,x}$ is the root of an arborescence of the x-th cell. If P is an even length path, $v_{0,x}$ and $v_{0,x+1}$ should be colored the same. Therefore, the two cells C_x and C_{x+1} are the same type (say R) for $\forall x \in Z$. If P is an odd length path, the two cells C_x and C_{x+1} are different types. Therefore, two different types appear alternately. □

We have a similar lemma for two-dimensional dynamic graphs.

Lemma 5.3: Let G^2 be a connected bipartite two-dimensional dynamic graph. Then the pattern of cell types in G^2 is one of the following four patterns as illustrated in Fig. 4c.

$\begin{bmatrix} RR \\ RR \end{bmatrix}$: Every cell has the same cell type (say R).

$\begin{bmatrix} RR \\ BB \end{bmatrix}$: Two one-dimensional patterns (RR) and (BB) appear alternately in the y-axis direction.

$\begin{bmatrix} RB \\ RB \end{bmatrix}$: Two one-dimensional patterns (RR) and (BB) appear alternately in the x-axis direction.

$\begin{bmatrix} RB \\ BR \end{bmatrix}$: Two one-dimensional patterns (RB) and (BR) appear alternately in both the x-axis and y-axis directions.

Proof: Since G^2 is connected, there exist some $a_i,\ b_i \in Z^+ \cup \{ 0 \}$ such that

$$(0, 1) = \sum_{i=1}^{m} a_i \mathbf{e}_i,\ (1, 0) = \sum_{i=1}^{m} b_i \mathbf{e}_i.$$

If $\sum_{i=1}^{m} a_i \equiv 0 \ (mod\ 2)$, all cells with the same x-coordinates are the same type; that is, RR- or BB-type. On the other hand, if $\sum_{i=1}^{m} a_i \equiv 1 \ (mod\ 2)$, all cells with the same x-coordinates are colored alternately; that is, RB-type. In the same way, all cells with the same y-coordinates have RR-, BB-, or RB-type. Therefore, we have four possible two-dimensional patterns. Note that there cannot be a pattern of type $\begin{bmatrix} RR \\ RB \end{bmatrix}$, because an (RR)- (resp. (RB)-) pattern in the x-axis direction implies the existence of an even (resp. odd) length path from $v_{0,\mathbf{x}}$ to $v_{0,\mathbf{x}+(0,1)}$ for $\forall \mathbf{x} \in Z \times Z$. Then, from Theorem 5.2, G^2 could not be 2-colorable, which is a contradiction. □

We now have the following necessary and sufficient conditions for 2-colorability of dynamic graphs.

Theorem 5.3: Let G^1 be a connected one-dimensional dynamic graph. Then G^1 is 2-colorable if and only if the two-dimensional labeling of the constraint graph H_{G^0} satisfies the following table.

cell pattern	$T_H((a, a))$	$T_H((a, b))$
(RR)	none(*)	0, 1
(RB)	1	0

* means that there is no (a, a) edge.

Proof: Suppose G^1 is 2-colorable. Then from Lemma 5.2, the cell pattern is either (RR) or (RB). From Lemma 5.1, an edge with the label (a, a) (resp. (a, b)) connects the two cells which are different (resp. the same) types. Therefore, the above table is correct.

Conversely, let us suppose the two-dimensional labeling T_H satisfies the above table.

First, suppose also that there are only (a, b) edges in $E(H_{G^0})$. Then there exists a 2-coloring which colors all $v_{0,x}$ by the same color as follows: Let $e_x = (v_{i,x}, v_{j,y})$ in G^1 be an arbitrary connecting edge which is the x-th copy of the edge $e = (v_i, v_j) \in E^0$ with $T^2(e) = y - x$. Since there are only (a, b) edges in H_{G^0}, from (5.1), v_i and v_j are colored differently in the static graph. Since we color all roots of the arborescence of cells by the same color, $v_{i,x}$ and $v_{j,y}$ are colored differently. Therefore, the edge e_x does not violate 2-colorability. Thus, this results the (RR)-type bipartite graph.

Secondly, suppose we only have (a, a) edges of label 1 and (a, b) edges of label 0. The following shows an (RB)-type 2-coloring which colors the roots of arborescences of cells alternately by different colors. Let $e_x = (v_{i,x}, v_{j,y})$ in G^1 be an arbitrary connecting edge which is the x-th copy of the edge $e = (v_i, v_j) \in E^0$ with $T^2(e) = y - x$. If

$y - x \equiv 0 \ (mod\ 2)$, then e induces an (a, b) edge in H_{G^0} from the assumption. This implies that v_i and v_j are colored differently in the static graph. Note that we color $v_{0,x}$ and $v_{0,y}$ by the same color, since $y - x$ is even. Therefore, $v_{i,x}$ and $v_{j,y}$ are colored differently. Thus the connecting edge e_x does not violate 2-colorability. If $y - x \equiv 1 \ (mod\ 2)$, then e induces an (a, a) edge in H_{G^0} from the assumption. This implies that v_i and v_j are colored by the same color in the static graph. Note that we color $v_{0,x}$ and $v_{0,y}$ by different colors, since $y - x$ is odd. Therefore, $v_{i,x}$ and $v_{j,y}$ are colored differently. Thus the connecting edge e_x does not violate 2-colorability. Hence we have the (RB)-type bipartite graph. □

Theorem 5.4: Let G^2 be a connected two-dimensional dynamic graph. Then G^2 is 2-colorable if and only if the constraint graph H_{G^0} satisfies the following table.

cell pattern	$T_H((a, a)) = (x, y)$	$T_H((a, b)) = (x, y)$
$\begin{bmatrix} RR \\ RR \end{bmatrix}$	none(*)	anything(**)
$\begin{bmatrix} RR \\ BB \end{bmatrix}$	$y = 1$	$y = 0$
$\begin{bmatrix} RB \\ RB \end{bmatrix}$	$x = 1$	$x = 0$
$\begin{bmatrix} RB \\ BR \end{bmatrix}$	$x + y = 1$	$x + y = 0$

* means that there is no (a, a) edge.

** means that any edge of this type is allowable.

Proof: This can be proved in the same way as was Theorem 5.3. □

For an example, the constraint graph H_{G^0} in Fig. 4a satisfies the conditions above, and thus the static graph G^0 induces an $\begin{bmatrix} RR \\ BB \end{bmatrix}$-type bipartite dynamic graph as illustrated in Fig. 4d.

Corollary 5.1: The 2-colorability of a two-dimensional dynamic graph G^2 can be tested in linear time with respect to the number of edges in the static graph G^0.

Proof: It takes O($| E^0 |$) time to construct the constraint graph H_{G^0}. □

Note that our approach uses the fact that in a connected bipartite graph a coloring of one vertex determines the coloring of the whole graph. This fact does not hold for k-coloring, and suggest that our approach cannot be extended to $k(\geq 3)$-coloring in a straightforward way.

6. Conclusions

We investigated the problems of finding weak components, finding an Eulerian path, and testing 2-colorability for two-dimensional dynamic graphs and showed that they are done in polynomial time with respect to the size of the associated finite static graphs. We also showed that our algorithms for the problems of finding weak components and testing 2-colorability can be applied to one-dimensional dynamic graphs. The acyclicity problem and planarity testing for two-dimensional dynamic graphs are treated in (Iwano and Steiglitz 1986a, 1986c, 1987).

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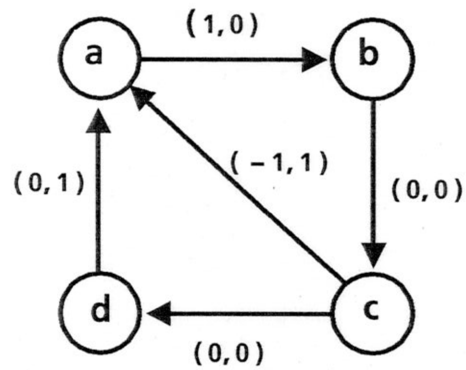
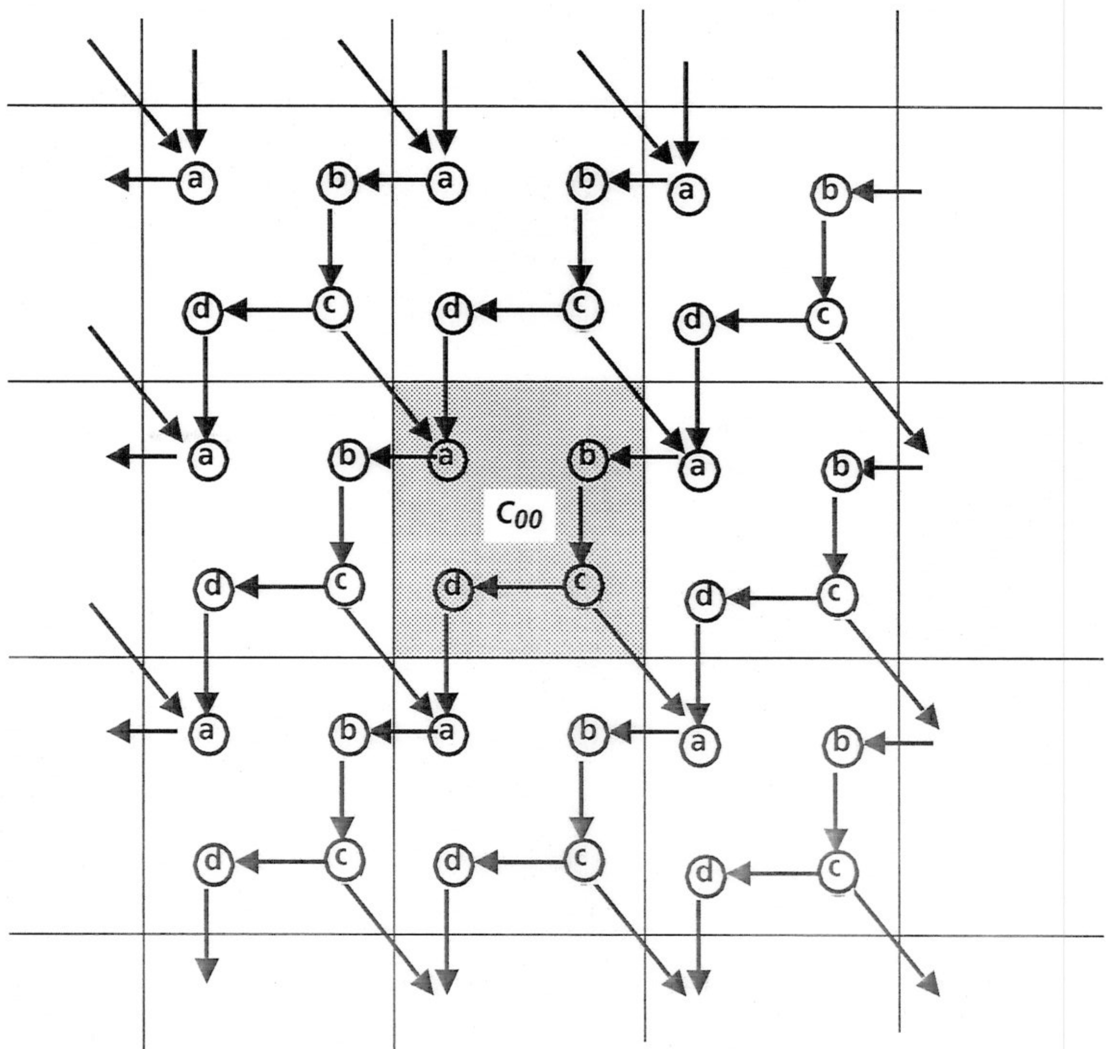
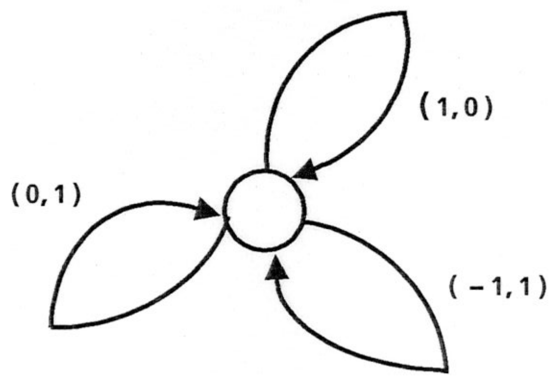


A static graph G^0

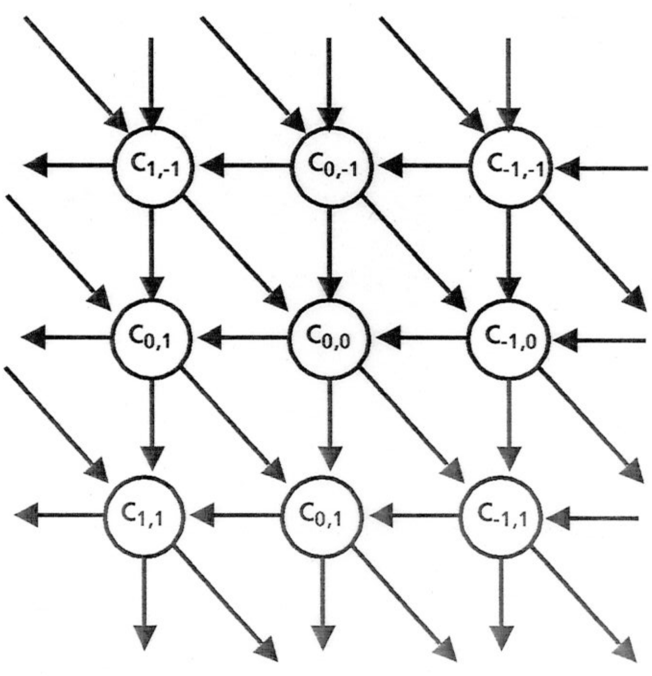


The dynamic graph G^2

Figure 1a. A static graph G^0 shows how to connect the nodes in G^2. The shaded area shows the basic cell C_{00}.



The cell-static graph G_c^0



The cell-dynamic graph G_c^2

Figure 1b. The cell-dynamic graph G_c^2 indicates the interconnection of cells in the dynamic graph G^2 in Fig. 1a.

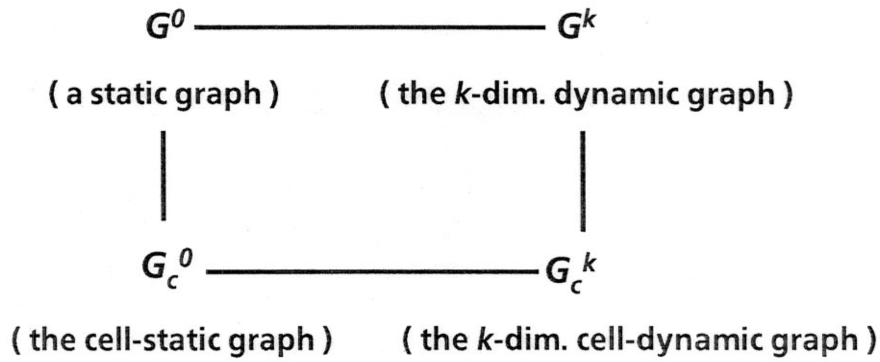
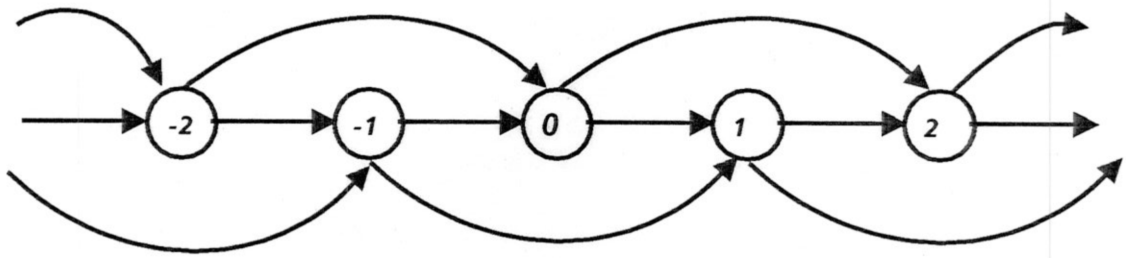
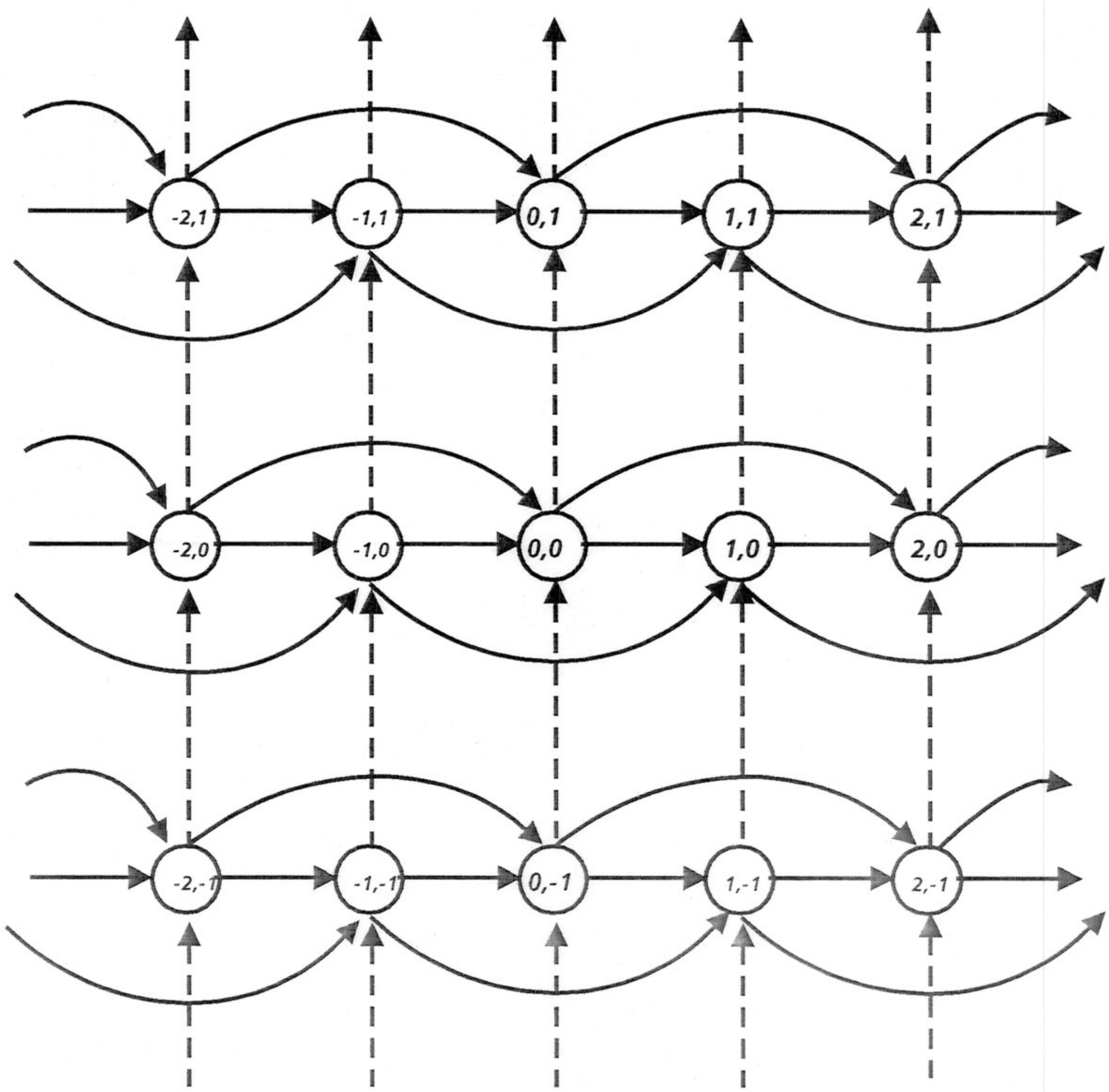


Figure 1c. The superscript *0* indicates a static graph, while the superscript *k* indicates a *k*-dimensional dynamic graph. The subscript *c* indicates a cell graph.



G^1



G^2

Figure 2. The 2-dimensional dynamic graph G^2 is created by repeating G^1 in the direction of the y-axis. Note that the number of weakly connected components of G^1 is the same as one of G^2.

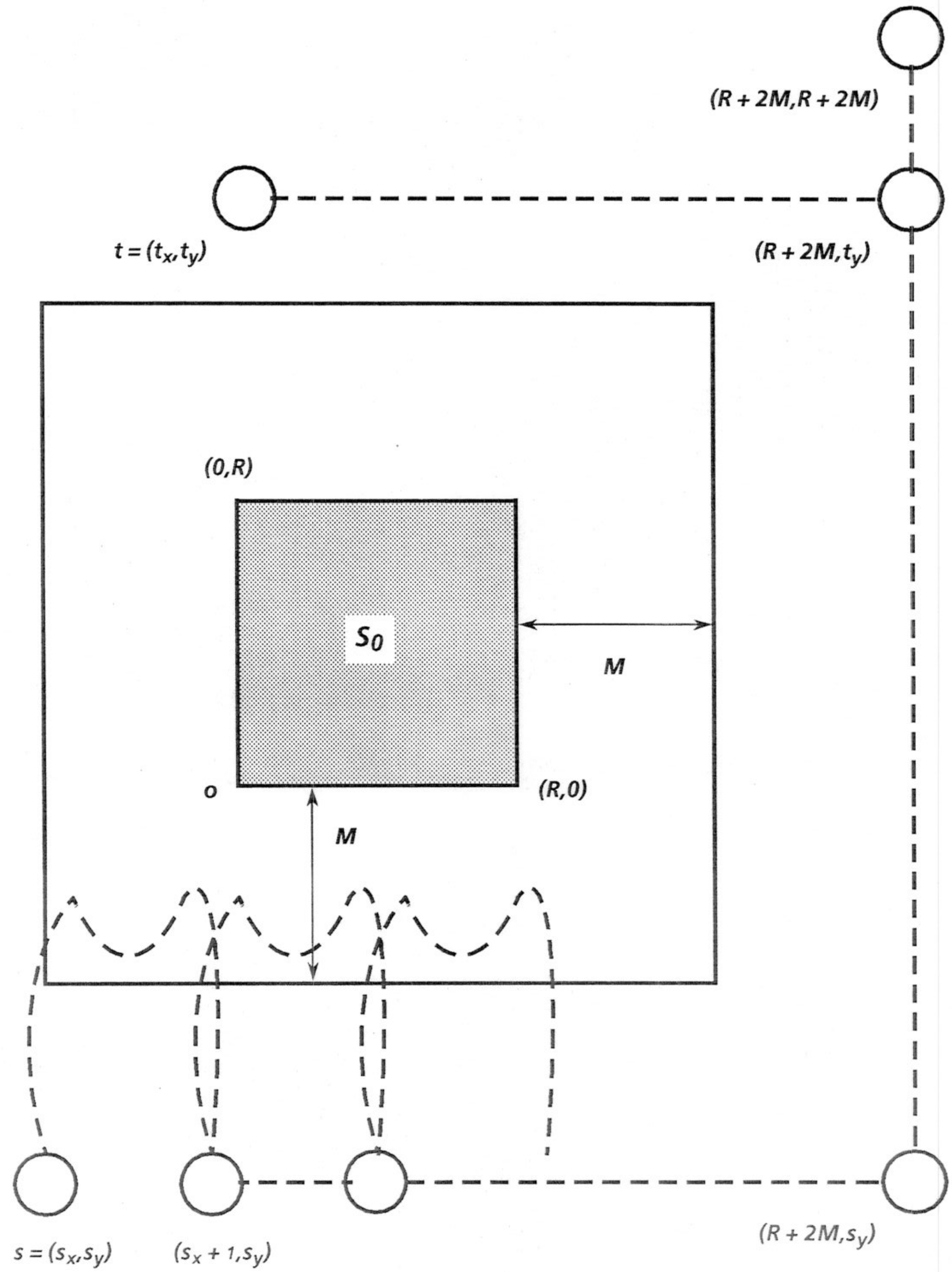
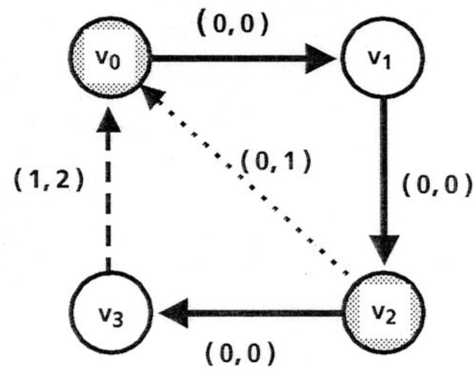
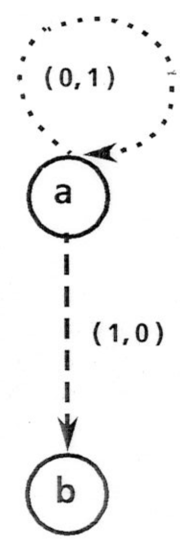


Figure 3. G/S_0 is not 2-separable. Any vertex $s \notin [-R, R+M] \times [-R, R+M]$ is connected to $(R+2M, R+2M)$ by a chain in G/S_0. Thus, any two distinct vertices $s, t \notin [-R, R+M] \times [-R, R+M]$ is in the same weak components.



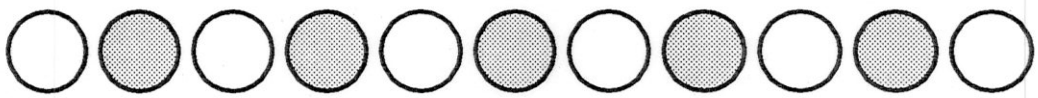
(a) A static graph G^0. The arborescence A_{G^0} with labels $(0,0)$ is illustrated by wide solid lines. $V_{even} = \{v_0, v_2\}$ and $V_{odd} = \{v_1, v_3\}$.



(b) A constraint graph H_{G^0} is created from G^0. The associated lines are indicated by the same line type; For example, the edge (v_3, v_0) with the label $(1,2)$ in G^0 induces the edge (a,b) with the label $(1,0)$ in H_{G^0}, since $(v_3, v_0) \in V_{odd} \times V_{even}$.

Figure 4a.

(RR) type



(RB) type

Figure 4b. Two types appear in bipartite one-dimensional dynamic graphs.

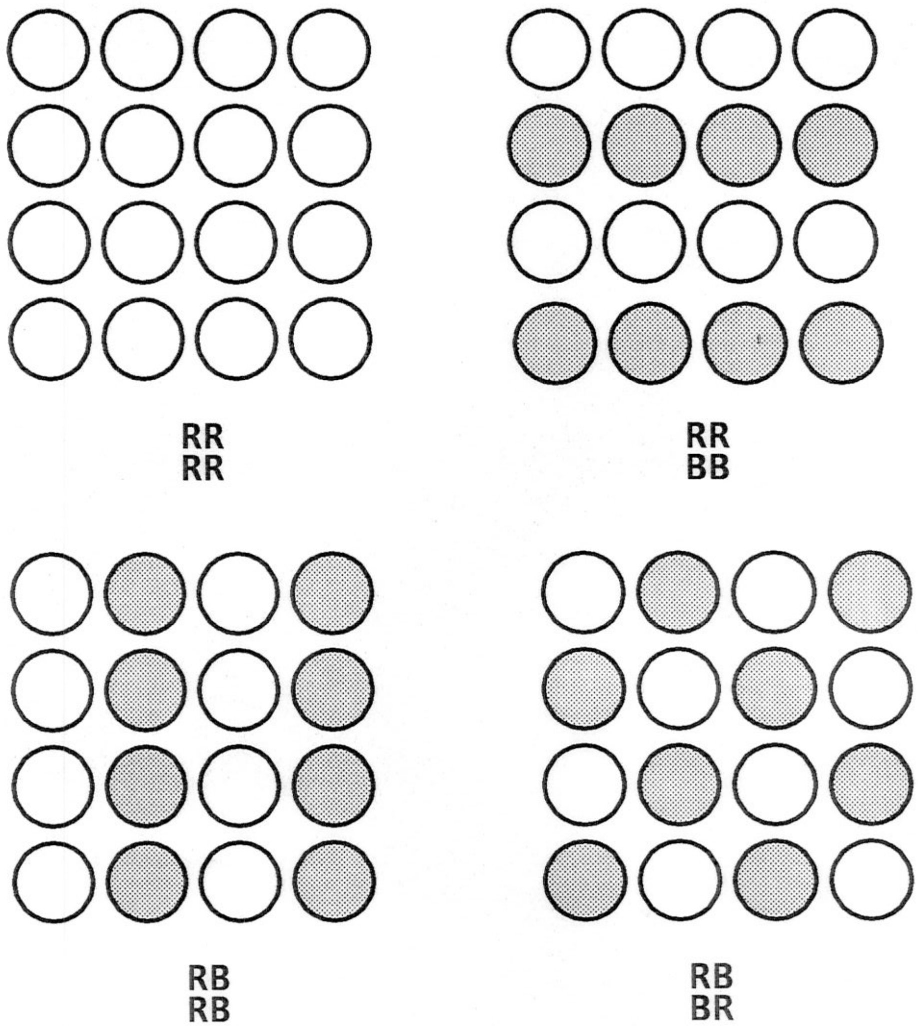


Figure 4c. Four types appear in bipartite two-dimensional dynamic graphs.

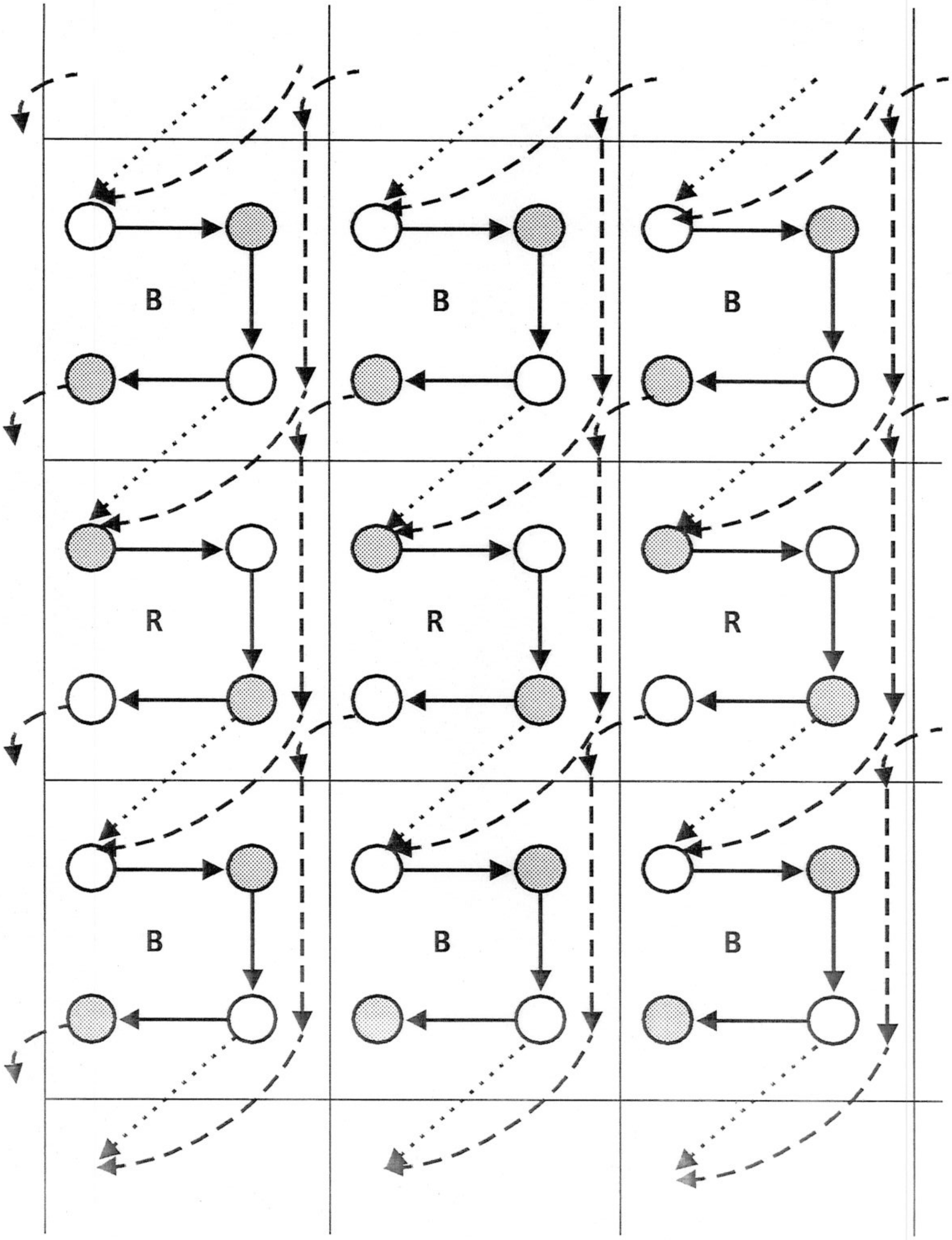


Figure 4d. A static graph G^0 in Fig. 4a induces a $\frac{RR}{BB}$ type bipartite dynamic graph as illustrated above.