–Independent Work Report, Spring 2015–

Universal Traversal Sequences for Arbitrarily Labeled Graphs

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Abstract

A universal traversal sequence is a deterministic walk on an edge-labeled graph that is guaranteed to visit all of the graph's vertices. We look at the problem of constructing universal traversal sequences for arbitrarily labeled graphs and give a combinatorial construction of quasi-polynomial length universal traversal sequences for expander graphs. This is an alternative to the quasi-polynomial length construction given by Nisan [5] for general graphs, which makes use of a pseudorandom generator. We also give a guarantee on the number of vertices a universal traversal sequence for a smaller graph will visit when run on a larger graph. This may allow smaller traversal sequences to be combined to give short universal traversal sequences for arbitrarily labeled graphs.

1. Introduction

Constructing universal traversal sequences is an interesting combinatorial problem in its own right, but it also has connections to other important problems, such as the undirected s,t connectivity problem and various problems related to randomness and space-bounded computation. We will survey some of the most notable work to date involving universal traversal sequences of graphs and present a couple of new and interesting results. First, we look at the behavior of universal traversal sequences for small graphs when run on larger graphs. For directed or undirected graphs allowing multiple edges and self-loops, we conclude that a universal traversal sequence for graphs with $n-i$ vertices will always visit at least $n-i$ vertices when run on a graph with n vertices. In addition, we look at the class of expander graphs and give a construction of universal traversal sequences of quasi-polynomial ($n^{O(\log n)}$) length using a different technique than the pseudorandom generator used by Nisan [5] to obtain the same bound. Our construction relies on spectral expansion of graphs and exploits the properties that expander graphs share with complete graphs.

2. History

2.1. Universal traversal sequences

Universal traversal sequences were introduced by Aleliunas et. al. [1] in 1979 in connection with the heavily studied problem of graph reachability. Undirected s,t connectivity (or USTCON) is the problem of determining whether two vertices in an undirected graph are in the same connected component. There are well-known linear time algorithms for solving USTCON, namely depth-first search and breadth-first search, but the space complexity of this problem has proven to be more interesting. Aleliunas used universal traversal sequences to give a randomized log-space algorithm for the problem, showing that a random walk of length $O(n^3 d^2 \log n)$ on a d-regular undirected graph with n vertices will visit every vertex of G with high probability. This result was obtained via the probabilistic method, meaning that such a sequence was never explicitly constructed.

Since then, there has been an ongoing search to explicitly construct universal sequences of polynomial length. Informally, a sequence is universal for d-regular graphs with n vertices if for every graph and every start vertex, the walk defined by the sequence will visit every vertex in the graph. To produce a walk defined by a sequence, we need some notion of graph labeling. For each vertex, label the adjacent edges from 1 to d. Then a sequence is a string of edge labels which determines some walk through the graph. Consistent labeling is a restrictive way to label graphs and provides more flexibility when attempting to construct universal traversal sequences. A labeling is consistent if every edge labeled i (for any i between 1 and d) leads to a different vertex. In other words, there is no pair of vertices that both have an edge labeled i that leads to the same vertex. This is a useful notion because the same sequence started at two different vertices of a graph will end at different vertices (and will be traversing different vertices at every point in the sequence) . With arbitrary labeling, this is certainly not guaranteed.

Istrail [3] constructed polynomial length universal sequences for cycles in 1988, but to this day cycles remain the only class of (arbitrarily labeled) graph for which polynomial length universal traversal sequences have been constructed. Soon after, there was an interesting result given by

Karloff et al [4], who constructed quasi-polynomial universal traversal sequences for complete graphs. There has been considerably more success constructing these sequences for consistently labeled graphs. Hoory and Widgerson [2] constructed polynomial length universal traversal sequences for consistently labeled expander graphs, using a similar technique as Karloff et al [4]. The technique involves picking any vertex v in some graph and building a set of sequences until there must be some sequence in the set that will visit v when started at any vertex in the graph.

This is the technique that we use to give the same bound for arbitrarily labeled expanders that Karloff [4] gave for complete graphs. Karloff uses the fact that in a complete graph, every vertex is directly connected to every other vertex. There are two properties of expanders that allow them to behave like complete graphs in this way. First, they have small diameter (logarithmic in the number of vertices), so there is a short path between any two vertices. Also, random walks on expanders quickly converge to the uniform distribution, meaning that for every vertex v, a short walk from arbitrary vertex u has substantial probability of ending at v. We will use these facts to generalize Karloff's construction to all directed and undirected expander graphs.

Around the same time that Karloff and Hoory published their papers, Nisan [5] used a completely different technique to show quasi-polynomial universal traversal sequences for general graphs. He constructed a pseudorandom generator that is able to take a short string of truly random bits of length O(S log R) and output a pseudorandom string of length R, using space S. If R is poly(n) and S is logarithmic in n, this generator can be used to produce a polynomial length pseudorandom string of bits using only $O(\log^2 n)$ truly random bits and $O(\log n)$ space. Going back to Aleliunas' probabilistic construction, polynomial length walks of a certain size have high probability of being universal. So if we look at all possible outputs of the pseudorandom generator on a random input of length $O(\log^2 n)$, a non-zero percentage of them will be universal. Therefore, the concatenation of all possible outputs of this size will be universal. Since there are $2^{O(\log^2 n)} = n^{O(\log n)}$ possible inputs and each output is poly(n), the concatenation of all possible outputs is $n^{O(\log n)}$. This remains the best construction of universal traversal sequences for general arbitrarily labeled graphs.

2.2. Expander graphs

As mentioned earlier, there has been more success constructing polynomial length universal traversal sequences for consistently labeled graphs. In fact, using techniques from Reingold's [7] 2008 paper which resolved the space complexity of USTCON, polynomial length universal sequences can be constructed for all consistently labeled graphs. The construction makes heavy use of expanders, and in particular the zig-zag product, which is a graph product introduced by Reingold, Vadham, and Widgerson in 2002 [6] and used to preserve expansion. We will now formally define expander graphs in order to talk about their use in Reingold's sequences and in our construction involving arbitrarily labeled expanders.

Loosely speaking, expanders are graphs that are sparse (not too many edges) but well-connected. There are various ways to measure the "well-connectedness" or expansion of a graph. First, we will show two combinatorial ways to measure expansion, and then focus on a third algebraic measure that was used by Reingold in the analysis of the zig-zag product, and used here to give our universal traversal sequences for expanders.

We will consider directed graphs where (u,v) denotes a directed edge from vertex u to vertex v. Let the neighborhood $N(S)$ of a set of vertices S be all vertices of G such that there is an edge (u,v) where $u \in S$ and $v \in N(S)$. From [9], G is a (K,A) vertex expander if for all sets S of at most K vertices, $|N(S)| \geq A|S|$. In other words, there are a substantial number of vertices connected to any set of vertices of G under a certain size. If a graph with n vertices has degree = O(1) and $K = \Omega(n)$, then for large enough A, it is considered a good expander. It can be shown that, in fact, random d-regular graphs are good vertex expanders with high probability [9].

Next we will look at edge expansion, which is a very similar notion. There are various definitions of edge expansion, but we will look at the one given by Hoory and Widgerson [2] as it was used to prove the result mentioned previously about polynomial length universal traversal sequences for consistently labeled expanders. A graph is said to be a c-expander if for every subset S of the vertex set V, the number of edges between S and $V-S$ is at least $\frac{c|S|(n-|S|)}{n}$ [2]. This guarantees a certain number of edges between any set of vertices and the rest of the graph. This was useful in the construction given in [2] since a set of vertices was built from which some sequence *s* would visit a distinguished vertex when started at any vertex in the set. It was shown that since there were many edges between the set and its complement, there must be some edge label j such that the concatenated sequence *sjs* considerably grew the size of the set. However, this also took advantage of the consistent labeling of the graph, as the construction relied on the fact that the sequence s when started at x different vertices would end at x different vertices. For our construction, we did not have this at our disposal, so we had to look at the properties of random walks on expanders. We found that spectral expansion was a much more useful notion for our purposes.

Spectral expansion can be thought of in terms of how quickly a random walk on a graph converges to the uniform distribution. Start with a probability distribution π = (1,0,0....0) over the vertices of d-regular graph G, which represents a walk started at some specific vertex. Define the uniform distribution $u = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$. Let M be the adjacency matrix of G divided by d. Each column of this matrix then represents the probability of ending at each vertex after one random step from some particular vertex. Now we give a definition from [9] that measures the spectral expansion ($\lambda(G)$) of a graph G:

$$\lambda(G) = \max_{\pi} \frac{||\pi M - u||}{||\pi - u||}$$

Where $||x||$ is the l_2 norm of x. Each step on a random walk of G brings the current probability distribution closer to the uniform distribution. The fraction above measures how much closer one step on a random walk brings the current distribution to uniform. Values closer to 0 correspond to a bigger change, so the smaller $\lambda(G)$ is, the quicker the probability distribution approaches the uniform distribution. The spectral gap of G, or $\gamma(G)$, is equal to 1- $\lambda(G)$. Therefore, a bigger spectral gap means better expansion. Now, vertex and edge expansion are clearly related, but the relationship between spectral expansion and these two other combinatorial measures is not as obvious. A discussion by Vadham [9] shows that indeed, spectral and vertex expansion are closely related. Any graph that is a good vertex expander has a large spectral gap and vice versa.

2.3. The zig-zag product

Now we are ready to discuss Reingold's approach to constructing universal traversal sequences for consistently labeled graphs. The construction involves taking any d-regular graph with n vertices, transforming it into an expander, and then using the result given in [2] for consistently labeled expanders to construct the universal sequence. In order to make any graph an expander, the spectral expansion γ(G) must increase while the degree stays bounded. It is easy to increase the expansion of a graph by squaring the graph, which results in a graph that contains an edge between every two vertices of the original graph that were connected by a path of length at most 2. This clearly improves the overall connectivity of the graph. However, this also squares the degree of the graph. The hard part is then is to reduce the degree of the resulting graph while maintaining good expansion. This is accomplished by the zig-zag product, which was originally introduced for use in the explicit combinatorial construction of expanders [6].

The zig-zag product operates on two graphs, ideally one big expander and one smaller 'auxiliary' expander. Call the big graph G and the small graph H. The product essentially takes each vertex of G and replaces it with all vertices of H. Then the edges between these vertices are determined by taking a step in H followed by a step in G and finishing with another step in H. In order to efficiently represent the input and output graphs of this operation, each graph can be thought of as a rotation map. This is a function that takes two integers as input (v,i) where v is some vertex of an undirected labeled graph, and i is an edge label. The output is the pair (w, j) where w is the vertex reached by taking the i'th edge from v and j is the edge that would lead back to v. Given the rotation map for a graph G and an auxiliary expander H, the rotation map function for the zig-zag product of G and H would then consist of a rotation on H followed by a rotation on G and another rotation on H.

The graph that results from the zig-zag product roughly inherits its degree from H and its expansion properties from G. From [9], if G is a d_1-regular graph with n_1 vertices and spectral gap γ_1, and H is a d_2-regular graph with n_2 vertices and spectral gap γ_2, then the zig-zag product of G and H is a D_2^2 - regular graph with N_1D_1 vertices and spectral gap $\gamma_1\gamma_2^2$. Therefore, if H has much smaller degree than G and is a very good expander, the resulting graph will have degree much less than G and only slightly worse expansion. So, by picking a good H and alternating between squaring and taking the zig-zag product with H, any graph G will eventually become a good expander with bounded degree.

The actual construction of polynomial length sequences for consistently labeled graphs using the zig-zag product is described in a 2006 paper by Reingold, Trevisan, and Vadham [8]. It is given for directed graphs but can extend to undirected graphs since any consistently labeled undirected graph can be made a consistently labeled directed graph by replacing each undirected edge with two directed ones. The first step is to convert a d-regular graph G with n vertices into a graph G' with $d * n$ vertices by replacing each vertex with a cycle of d vertices, labeled 1 to d. The cycle is constructed out of edges labeled 1 and 2, and the edge labeled 3 from each vertex leads to a vertex on a different cycle. A number of self-loops are also added to each vertex. This graph, which is consistently labeled, is then turned into an expander G_{exp} by the technique just described. The zig-zag product preserves consistent labeling, so G_{exp} has an explicitly constructible polynomial length universal traversal sequence by [2]. Furthermore, the sequence on G_{exp} can easily be converted to the corresponding sequence on G'. Then, the sequence on G' can be converted to a sequence for any consistently labeled G with n vertices by looking at the label of the current vertex when the edge label 3 is encountered in the sequence (which is the only edge label that corresponds to changing vertices in G). This result hinges on the consistent labeling of the input graph G. It is unclear how to generalize this to arbitrarily labeled graphs. However, this is the most recent significant result given involving universal traversal sequences. An important question remains regarding whether there exist explicitly constructible polynomial length universal traversal sequences for arbitrarily labeled graphs.

3. Definitions

- $G(n,d)$ is a d-regular graph with n vertices. If G is directed, d = indegree(v) = outdegree(v) for all vertices v.
- A multigraph is one which can contain multiple edges and self-loops; a simple graph cannot.
- In a directed graph, edge e is an out-edge of vertex v if it is connected to v and directed away from v, and an in-edge of v it if is connected to v and directed towards v.
- $G(n,d)$ is labeled if for every vertex v, the edges coming out of v are labeled with some permutation of $[1,2,...,d]$. In an undirected graph, each edge has 2 labels (one associated with each vertex it is connected to) and in a directed graph, each edge has 1 label (associated with the vertex of which it is an out-edge).
- $G(n,d)$ is consistently labeled if there is no pair of distinct vertices of G such that the edge labeled i from each vertex leads to the same vertex.

For the purposes of this paper, $G(n,d)$ will always refer to an arbitrarily labeled graph, unless stated otherwise.

- A sequence S is a string of integers between 1 and d that determines a walk on some $G(n,d)$.
- A sequence S is started at vertex v of $G(n,d)$ if v is the first vertex of a walk and the next vertex is determined by taking the i'th edge from v, where i is the next integer in S.
- A sequence S started at vertex v visits vertex u if u is traversed at any point in the walk.
- A (n,d) universal traversal sequence is a sequence which visits all vertices of G when started at any vertex v for any $G(n,d)$.

4. Behavior of universal traversal sequences for smaller graphs on larger graphs

We will consider both undirected and directed multigraphs.

> **Theorem 1**: Given $G(n,d)$ and a $(n-i,d)$ universal traversal sequence, the sequence will visit at least $n-i$ vertices of G when started at any vertex of G.

We will make use of this lemma:

> **Lemma 1**: If a sequence started at any vertex of G does not visit t vertices, then for every $j \leq t$, there are j unvisited vertices such that removing them from G results in a graph G' that is connected (contains only one connected component).

Proof of Lemma 1:

Call the set of visited vertices A and the set of unvisited vertices B. A is connected since every vertex in A was visited by the same sequence. Thus, removing the t unvisited vertices will not disconnect the graph. Now, since G is connected, there must be some $v \in B$ that is connected to a vertex in A. Move v from B to A, so there are now $t-1$ unvisited vertices such that removing them will not disconnect the graph. By induction, this is true for any $j \leq t$. This argument holds for both directed and undirected graphs. ■

Proof of Theorem 1:

Undirected case:

Assume for the sake of contradiction that the universal traversal sequence visits $< n-i$ vertices when started at arbitrary vertex u. Then there are $> i$ unvisited vertices, so by Lemma 1, there are i unvisited vertices that can be removed such that the remaining graph is connected. Remove these vertices. All of the edges between any two of these removed vertices are also removed, but there remain the edges between one removed vertex and one remaining vertex. Each of these edges is now only connected to one vertex, so combining any two of these edges creates one valid edge (even if it is a self-loop or multiple edge). The arbitrary labeling remains valid since each remaining vertex still has d adjacent edges numbered from 1 to d.

Now we show that the number of edges to combine is even. Call the set of removed vertices X. Let E be the set of edges with both endpoints in X and C the set of edges with only one endpoint in X. Each edge can be represented as a set of two vertices, its endpoints. Every vertex is represented d times in the union of all these sets and each edge in E contains 2 vertices in X and each edge in C contains 1 vertex in X. $|X| = i$, so we get the equation:

$$2|E|+|C|=d*i$$

If $d*i$ is even, then $|C|$ must be even. If d is even this holds. If d is odd, then the graph has odd degree, so it must have an even number of vertices since for any d-regular graph with n vertices and e edges, $2e=d*n$. Then, there can only exist universal traversal sequences for d-regular graphs with even number of vertices, so $n-i$ is even and therefore i is even, so $d*i$ is even which means that $|C|$ is even.

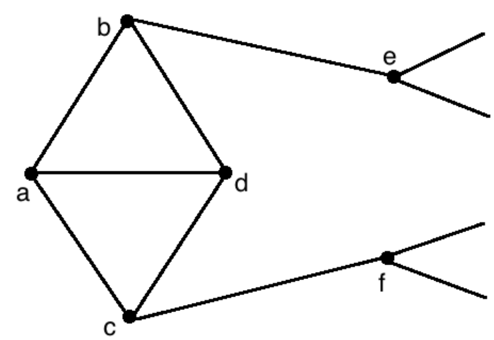
So, we can split C into groups of 2 and combine each group of 2 into one new edge, forming a valid d-regular graph with $n-i$ vertices. There is at least one unvisited vertex in this graph because less than $n-i$ vertices were visited. The universal traversal sequence started at u in this new graph results in the same exact walk as when it was started at u in the original graph G, since only unvisited vertices were removed. Therefore we have a contradiction, since starting the universal traversal sequence at u in this graph must visit all of the vertices. ■

Directed case:

The proof is the same as for undirected graphs, except that when we remove a set of vertices, we are left with a slightly different situation. Keep all edges that are connected to at least one remaining edge. Now we are left with a number of out-edges of remaining vertices that are no longer connected to another vertex and a number of in-edges of remaining vertices that are no longer connected to another vertex. Each remaining vertex still has outdegree d, indegree d, and the valid labeling has not changed. If the number of unconnected in-edges is the same as the number of unconnected out-edges, we can simply combine every unconnected out-edge with an unconnected in-edge and produce a valid d-regular directed graph with $n-i$ vertices, and the rest of the proof follows the undirected case.

Let e be the number of edges removed (connected to two vertices that were removed). There are a total of $d*i$ out-edges and $d*i$ in-edges associated with each removed vertex, and each edge removed is an out-edge of one of these removed vertices and an in-edge of another, so $d*i-e$ is both the number of out-edges connected to a remaining vertex and the number of in-edges connected to a remaining vertex, so the theorem is proved. ■

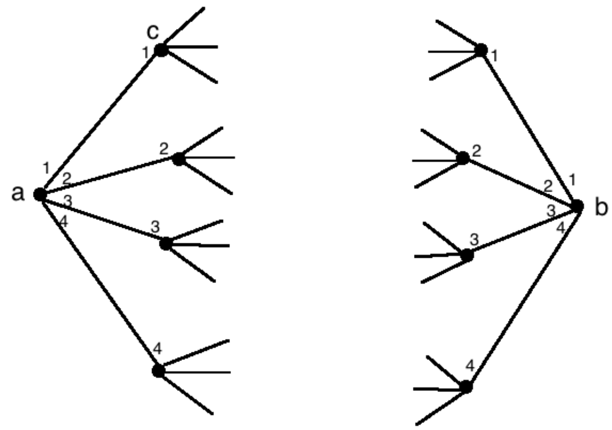
Next we will show by use of an example that this approach does not work for simple graphs. We have a simple graph $G(n,3)$ and a $(n-2,3)$ universal traversal sequence for simple graphs. Say that the sequence fails to visit 4 vertices of G: the vertices $\{a,b,c,d\}$ in the fragment of G below.



There is no way to remove 2 of the unvisited vertices and reconnect edges such that the remaining graph has no multiple edges of self-loops. If we try to remove $\{a,d\}$, there are 2 edges from b and 2 edges from c that need to be connected. This results in either 2 edges between b and c or a self-loop on both b and c. If we try to remove $\{a,b\}$ (which results in the same situation as removing $\{b,d\}, \{a,c\},$ or $\{c,d\}$), there are 2 edges from d, 1 edge from e, and 1 edge from c that need to be connected. If the 2 edges from d do not become a self-loop then one of them has to connect with the edge from c, which would form a multiple edge. The only other possibility is removing $\{b,c\}$, but then a and d each have 2 edges that need to be connected, and e and f each have 1. So, one edge from a or d has to be connected to another edge from a or d, but that would result in either a multiple edge or a self-loop. Therefore, in this example, there is no way to remove 2 vertices from G and form a simple 3-regular graph with $n-2$ vertices.

Finally, we will show by use of another example that this approach does not work for consistently labeled graphs. We have consistently labeled $G(n,4)$ and a $(n-1,4)$ universal traversal sequence for consistently labeled graphs. Say that the sequence fails to visit 2 vertices of G: the vertices

$\{a,b\}$ in the fragment of G below.



Trying to remove either a or b will result in the same situation. If a is removed, vertex c now needs to be connected to a vertex by an edge labeled 1 by the other vertex. However, the only edges that the edge from c can connect with are labeled 2, 3, or 4 by the other vertex so this is impossible. In this example, there is no way to remove 1 vertex from G and form a consistently labeled 4-regular graph with $n-1$ vertices.

5. Construction of universal traversal sequences for arbitrarily labeled expanders

Theorem 2 : There is a universal traversal sequence of length $n^{O(\log n)}$ for all constant degree expander graphs explicitly constructible without the use of pseudorandom generators.

Proof of Theorem 2 :

We follow the basic structure of the constructions given for cliques [4] and for consistently labeled expanders [2]. We will prove this for arbitrarily labeled directed expander graphs and note that any undirected graph can be made directed by thinking of each undirected edge as two directed edges. From Vadham, chapter 2 [9], we have that for any labeled directed graph $G(n,k)$ with spectral expansion $\gamma(\mathrm{G})$ and vertex v, a random walk of length $\frac{\ln(2\mathrm{n})}{\gamma(\mathrm{G})}$ from any start vertex will end at v with probability $\geq \frac{1}{2\mathrm{n}}$. So we immediately have this lemma:

Lemma 2 : Given directed $G(n,k)$ and a set S of its vertices of size A, a random sequence of length $\frac{\ln(2n)}{\gamma(G)}$ started at any vertex of G will end at a vertex in S with probability $\geq \frac{A}{2n}$

We also use make use of a lemma from both [4] and [2] (reworded here to apply to arbitrarily labeled expanders):

Lemma 3 : Given a set T of sequences with the property that for every directed expander $G(n,d)$ and vertex v, there is a sequence $t \in T$ that visits v when started at any vertex of G, the concatenation of all sequences in T (in any order) is universal.

The idea is that any vertex could be the distinguished vertex v, so given any $G(n,d)$, for every vertex in G there is a sequence in T that visits that vertex regardless of its starting point. The concatenation of all those sequences is then universal. Now our goal is to build up a set of sequences that has the property described in Lemma 3.

Consider directed expander $G(n,d)$ with spectral expansion $\gamma(G)$ and distinguished vertex v.
Let $c = \frac{1}{\gamma(G)}$, $L = c\ln(2n)$ and form the set of all possible sequences of length L.
Let S be the set of all such sequences, so $|S| = d^L$

Now recursively define a set of sequences T_i:

$$T_0 = \{\emptyset\}$$

$$T_{i+1} = \{\ tst \mid t \in T_i,\ s \in S\ \}$$

Where tst is a concatenation of sequences.

Let A_i be the maximum set of vertices such that there is some $t \in T_i$ such that t visits v when started at any vertex $w \in A_i$, and let $|A_i| = a_i$

We want A_i to grow until its size is n, and at that point, by Lemma 3, the concatenation of all strings in T_i is a universal traversal sequence. What value of i guarantees this?

We set up a recurrence to measure how much larger a_{i+1} is than a_i. Take $t \in T_i$ such that t visits v when started at any vertex in A_i. If t is not started in A_i, it will end up at some arbitrary node u. By Lemma 2, a random sequence of length L starting from u has probability $\geq \frac{a_i}{2n}$ of ending at a vertex in A_i. Since S is the set of all sequences of length L, $\geq \frac{a_i}{2\text{n}}|S|$ sequences in S will end at a vertex in A_i when started at u. So for each vertex outside of A_i, there are $\geq \frac{a_i}{2\text{n}}|S|$ sequences s such that ts ends at a vertex in A_i.

There are $n - a_i$ vertices outside of A_i, so by averaging, there must exist some sequence s such that ts ends at a vertex in A_i when started at

$$\geq \frac{(n-a_i)\frac{a_i}{2\text{n}}|S|}{|S|} = \frac{a_i(n-a_i)}{2\text{n}}$$

vertices outside of A_i.

Vertex v will be visited by the sequence tst if either the first or second t starts at a vertex in A_i, so we have established the recurrence

$$a_{i+1} \geq a_i + \frac{a_i(n-a_i)}{2\text{n}}\ ,\ a_0 = 1$$

$$a_{i+1} \geq \frac{2a_in + a_in - a_i^2}{2n} = \frac{3a_in - a_i^2}{2n} = a_i(\frac{3}{2} - \frac{a_i}{2n})$$

Now we split the evaluation of the recurrence into two cases.

If $a_i < \frac{n}{2}$:

$$a_{i+1} \geq a_i(\frac{3}{2} - \frac{1}{4}) = \frac{5}{4}a_i$$

So when $i = \lceil \log_{\frac{5}{4}}(\frac{n}{2}) \rceil$, $a_i \geq \frac{n}{2}$

Now let $b_i = n - a_i$

$$b_{i+1} \leq n - a_i(\frac{3}{2} - \frac{a_i}{2n}) = n - (n-b_i)(\frac{3}{2} - \frac{n-b_i}{2n}) = n - (n-b_i)(1 + \frac{b_i}{2n})$$

$$= n - (n - \frac{b_i}{2} - \frac{b_i^2}{2n}) = \frac{b_i}{2} + \frac{b_i^2}{2n} = b_i(\frac{b_i}{2n} + \frac{1}{2})$$

So when $a_i \geq \frac{n}{2}$ (meaning $b_i < \frac{n}{2}$),

$$b_{i+1} \leq b_i(\frac{1}{4}+\frac{1}{2}) = \frac{3}{4}b_i$$

So now when $i = \lceil \log_{\frac{5}{4}}(\frac{n}{2}) \rceil + \lceil \log_{\frac{4}{3}}(\frac{n}{2}) \rceil$, $b_i \leq 1$

And since $\frac{4}{3} > \frac{5}{4}$, when $i = 2\lceil \log_{\frac{5}{4}}(\frac{n}{2}) \rceil$, $b_i < 1$, meaning that $a_i = n$ since there are no fractional vertices.

So now that we have established a sufficient value for i, we must look at the length of the concatenation of all sequences in T_i to get a bound on the universal traversal sequence. The size of each sequence in T_i is the same, so let $z(i)$ be the size of a sequence in T_i. Looking at the definition of T_{i+1}, we get this recurrence:

$$z(i+1) = 2z(i) + L,\ z(1) = L$$

So we get that $z(i) \leq L2^i$.

The number of sequences in the set T_i is represented by the recurrence:

$$|T_{i+1}| = |S||T_i|\ , |T_0| = 1$$

So $|T_i| = |S|^i$.

Putting this all together, the length of the concatenation of all strings in $T_i \leq L2^i|S|^i = L(2|S|)^i$

So our universal traversal sequence is of length at most

$$L(2|S|)^{2\lceil \log_{\frac{5}{4}}(\frac{n}{2}) \rceil} = c\ln(2n)(2d^{c\ln(2n)})^{2\lceil \log_{\frac{5}{4}}(\frac{n}{2}) \rceil}$$

$$= c\ln(2n)(d^{\mathrm{O}(\log^2 n)}) = n^{\mathrm{O}(\log n)}$$

Since d = O(1). ■

6. Discussion

The first result we give involving the behavior of $(n-i,d)$ universal traversal sequences on (n,d) graphs is only a start. The hope is that this idea can be used in combination with other ideas to give better constructions of universal traversal sequences. There are a few ways that we can see this result being used. First of all, we can think about building up universal sequences incrementally. We know that (for even-degree graphs), a $(n-1,d)$ universal traversal sequence will visit at least all but one vertex of any (n,d) graph. So the question becomes how much we need to add to a sequence in order to guarantee that it visits just one more vertex. However, to end up with polynomial length universal traversal sequences, the length of the sequence that we add can not be proportional to our current sequence, it must be polynomial in n. So we would need to set up a recurrence like this:

$$|(n,d)\text{UTS}| = |(n-1,d)\text{UTS}| + \text{poly}(n)$$

However, this seems difficult because the unvisited vertex could be any vertex. If we simply add a polynomial length sequence at the beginning or end of our current sequence, it would have to be guaranteed to visit the unvisited vertex. Since that vertex could be any vertex, it seems like the sequence we add would have to be a universal traversal sequence itself. So this is not an option because it is polynomial length universal traversal sequences that we are looking for in the first place.

Perhaps there are more promising alternatives. If we look at universal sequences for graphs with $\frac{n}{2}$ vertices, we have more flexibility in how much we can add to them in the hope of producing polynomial length universal sequences for n vertices. In particular, if we can combine a constant number of universal sequences for graphs with $\frac{n}{2}$ vertices to get one for graphs with n vertices, then we have what we are looking for. This is seen by solving this recurrence:

$$|(n,d)\text{UTS}| = c|(\frac{n}{2},d)\text{UTS}|$$

We get that

$$|(n,d)\text{UTS}| = c^{\log_2 n} = c^{\frac{\log_c n}{\log_c 2}} = n^{\frac{1}{\log_c 2}} = \text{poly}(n)$$

Also note that any constant could replace 2 and the result would be the same. So if there were a way to combine a constant number of ($\frac{n}{a}$, d) universal traversal sequences (where a is a constant > 1 and independent of n) to get a (n,d) universal traversal sequence, we would have polynomial length universal traversal sequences for arbitrarily labeled graphs.

Our second result about arbitrarily labeled expanders is interesting mainly because it provides a new, more combinatorial, way to get quasi-polynomial universal traversal sequences for expanders. It also exploits a connection between complete graphs and expanders and suggests that if some new technique were discovered to construct polynomial length universal traversal sequences for complete graphs, it might be easily generalized to expanders. It seems that other than cycles, complete graphs would be the easiest class of graph for which to construct short sequences. Generalizing a technique from complete graphs to expanders is a big step because, as mentioned earlier, random regular graphs are expanders with high probability.

7. Further research

Our first result suggests many opportunities for further research. First, it would be valuable to get a sense of the effect of combining a number of small universal sequences in various ways. For example, can we guarantee that more than $n-i$ vertices are covered if we start a concatenation of two $(n-i,d)$ universal traversal sequences on some $G(n,d)$? With arbitrary labeling, this question is difficult to answer, but perhaps there are more clever ways of combining sequences than simple concatenation. Whereas Reingold used operations on graphs to construct his universal sequences, our result suggests that it could be interesting and informative to look at operations on sequences. We now have a guarantee about how various individual universal sequences will operate on larger graphs, but we need a way to combine them and get guarantees about their combination.

Also, the counterexamples given for simple graphs and consistently labeled graphs only show that our particular method for proving the result about arbitrarily labeled multigraphs does not extend to these other classes. They do not show that the same result does not hold for these classes. It very well may be the case that $(n-i,d)$ universal traversal sequences for simple or consistently labeled graphs do in fact cover $n-i$ vertices on a graph of size n. It would be interesting to either find an alternate method of proving this, or show a concrete counterexample that demonstrates a $(n-i,d)$ universal sequence for one of these classes that visits less than $n-i$ vertices on some graph with n vertices. It may be easier to reason about combining universal traversal sequences on consistently labeled graphs, so proving this result for consistently labeled graphs might quickly lead to polynomial length universal sequences for consistently labeled graphs. Even though Reingold already showed this using his zig-zag product, it would be informative to obtain a similar result using completely different techniques.

Our second result suggests further research into the properties of expanders and how they can be used to construct short universal sequences. In particular, it suggests that spectral expansion and the properties of random walks on expanders might be used to construct even shorter universal sequences. Our construction which gives quasi-polynomial length sequences certainly leaves room for improvement. For example, the bound given in Lemma 2 should be able to be improved for sets of vertices of a certain size. If the set is of size n - 1, the lemma states that a random sequence of logarithmic length started at any vertex will end at a vertex in this set with probability at least one half, but clearly many more than half of all possible sequences of this length will. It remains to be seen whether various improvements of this type could bring the length of the sequence constructed down from quasi-polynomial to truly polynomial.

8. Acknowledgements

I would like to thank my advisor Zeev Dvir for introducing me to this topic and providing valuable ideas and insights throughout the semester. I would also like to thank Ava Chen for help with editing this paper.

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