Problem Set No. 1

1. Consider recurrence relations of the following form, where $f(k)$ is a non-decreasing function of integer k:

$$T(1) = 1;$$
$$T(n) = \max\left\{T(k) + T(n-k) + f(k) \;\middle|\; 1 \leq k \leq n/2\right\} \text{ for } n \geq 1.$$

   Such recurrences occur when analyzing the running time of a divide-and-conquer algorithm in which the original problem is split into two disjoint parts whose sizes cannot be controlled, but where the cost to combine subproblem solutions depends on the size k of the smaller subproblem. For each of the recurrences below of this form, give a tight upper bound on $T(n)$, in "O" notation. Prove that your answer is correct.

   (a) $f(k) = 1$

   (b) $f(k) = k$

   (c) $f(k) = k^2$

   Partial credit for the correct answer. Full credit for a complete and careful proof.

2. The MIN algorithm for page replacement chooses the page to move to secondary storage that will be referenced farthest in the future. Prove that, for any reference sequence, the MIN algorithm minimizes the number of page faults.

3. (extra credit)

   Theorem 3 in the paper "Amortized Efficiency of List Update and Paging Rules" is in fact false. Provide a counterexample. Explain where the "proof" in the paper goes wrong.
   Hint. Consider the special case in which the cost to access the i^{th} item is i, the cost of a paid exchange is 1, the list consists of 3 items, and only accesses are performed. Construct a sequence of accesses that are cheaper using one or more paid exchanges than using no paid exchanges.