CS 423: Theory of Algorithms **Problem Set 1**
Spring, 1999
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Problem Set No. 1 Solutions

1.(a) Answer: $O(n)$

Proof: We prove by induction that $T(n) = 2n - 1$. We have $2(1) - 1 = 1 = T(1)$ so the result is true for $n = 1$. Let $n > 1$, and suppose the result is true for values less than n. Let k be a value that achieves the maximum in the definition of $T(n)$. Since $k < n$ and $n - k < n$, the induction hypothesis implies $T(n) = T(k) + T(n-k) + 1 = 2k - 1 + 2(n-k) - 1 + 1 = 2n - 1$, and the induction hypothesis is true for n. By induction, it is true for all n.

(b) Answer: $O(n \log n)$

Proof: We prove by induction that $T(n) \leq n \lg n + n$, where $\lg n$ is the binary (base-two) logarithm. The proof technique and the words are the same as in (a), so we show only the calculations.

$1 \lg 1 + 1 = 0 + 1 = 1 = T(1)$. For $n > 1$,

$$\begin{aligned} T(n) &= T(k) + T(n-k) + k \leq k \lg k + k + (n-k)\log(n-k) + (n-k) + k \\ &\leq k \lg k + (n-k)\log n + n + k \\ &\leq k \lg(n/2) + (n-k)\log n + n + k \text{ since } k \leq n/2 \\ &= k(\lg n - 1) + (n-k)\log n + n + k = n \log n + n \end{aligned}$$

(c) Answer: $O(n^2)$

Proof: We prove by induction that $T(n) \leq n^2/2 + n/2$, as in (a) and (b).

$$\begin{aligned} 1^2/2 + 1/2 &= 1 + T(1). \\ T(n) &= T(k) + T(n-k) + k^2 \leq k^2/2 + k/2 + (n-k)^2/2 + (n-k)/2 + k^2 \\ &= \tfrac{1}{2}(k^2 + n^2 - 2nk + k^2) + n/2 + k^2 = n^2/2 + n/2 + 2k^2 - n\,k \\ &\leq n^2/2 + n/2 + k(2k - n) \leq n^2/2 + n/2 \text{ since } k \leq n/2. \end{aligned}$$

Remark: For n a power of two, the bounds proved in (b) and (c) are tight, as can be proved by induction.