Homework #8: Classes P vs. NP, and Reductions
Due Date: Tuesday, April 21, 1998 (5:00pm)

Collaboration Policy

You may collaborate with others on this entire homework. However, please re-read the policy guidelines from Handout #1.

Reading

Read Chapter 36 of CLR, although you may gloss over the parts on the circuit satisfiability problem, including the book's version of the Cook-Levin theorem on pages 933-938.

Practice

Recall, these exercises are purely for your own practice. You need only turn in the official problems.

- Do CLR 36.1-1.
- Do CLR 36.3-1.
- Do CLR 36.4-4.
- Do CLR 36.5-1.

Problems

1. (30 points) Show that if `HAM-CYCLE` $\in$ `P`, then the problem of listing the vertices of a Hamiltonian cycle, in order, is polynomial-time solvable.

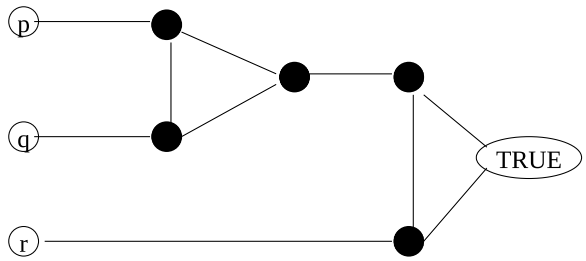
   More specifically, assume that there exists a polynomial-time procedure `TEST-HAM-CYCLE`(G'), which takes any graph G' and returns `True/False` depending on whether or not G' has a Hamiltonian cycle or not. We want you to design a procedure `PRINT-HAM-CYCLE`(G) which will actually print the order of a legitimate Hamiltonian cycle of G, assuming one exists. Of course, your algorithm is allowed to call `TEST-HAM-CYCLE` as needed. Please give high-level pseudo-code for your new procedure, and argue (briefly) why your algorithm is correct, and why it runs in polynomial time.

2. (30 points) The *set-partition problem* takes as input a set S of numbers. The question is whether the numbers can be partitioned into two sets A and $\overline{A} = S - A$, such that $\sum_{x \in A} = \sum_{x \in \overline{A}}$. Show that the set-partition problem is `NP`-Complete. (*Hint:* looks a bit like `Subset-Sum`, doesn't it?!)

3. (40 points) A k-**coloring** of an undirected graph $G = (V, E)$ is a function $c : V \leftarrow \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for every edge $(u, v) \in E$. In other words, the numbers $1, 2, \ldots, k$ represent the k colors, and adjacent vertices must have different colors. We define the k-`COLOR` problem to decide whether or not a given graph can be colored using k colors.

   a. Prove that 2-`COLOR` $\in$ `P`.
   b. Prove that 3-`COLOR` $\in$ `NP`.

   To prove that 3-`COLOR` is `NP`-Hard (and thus `NP`-Complete by part 3b), we use a reduction from 3-`CNF-SAT`. Given a formula ϕ of m clauses on n variables $x_1, x_2, \ldots, x_n$, we construct a graph $G = (V, E)$ as follows. To start with, we create a vertex for each variable, a vertex for the negation of each variable, and three special vertices: **`true`**, **`false`**, **`red`**. We add edges to form a triangle on the three special vertices, and we also add a triangle on $x_i, \neg x_i$, and **`red`** for each $i = 1, 2, \ldots, n$. Now, for each clause $(p \vee q \vee r)$, we create a widget, as shown in the figure below, where the five black vertices are newly created for each separate clause, but the other four vertices are among those we have already introduced.



   e. Argue that for our graph G to be 3-colorable, it must be that $c($**`true`**$)$, $c($**`false`**$)$ and $c($**`red`**$)$ use three distinct colors.
   f. Argue that for our graph G to be 3-colorable, it must either be the case that $c(x_i) = c($**`true`**$)$ and $c(\neg x_i) = c($**`false`**$)$ or else that $c(x_i) = c($**`false`**$)$ and $c(\neg x_i) = c($**`true`**$)$ for a given variable x_i.
   g. Argue that for our graph G to be 3-colorable, it must be that one of $c(p), c(q), c(r)$ is equal to $c($**`true`**$)$, for a given widget. Conversely, argue that so long as any one of $c(p), c(q), c(r)$ is equal to $c($**`true`**$)$, then that particular widget can be successfully 3-colored (at least, locally).
   h. Prove that if graph G is 3-colorable, then ϕ is satisfiable.
   i. Prove that if graph G is not 3-colorable, then ϕ is not satisfiable.
   j. Complete the overall proof that 3-`CNF-SAT` $\leq_p$ 3-`COLOR`, and thus that 3-`COLOR` is `NP`-Hard.

Extra Credit

Let the `Set-Splitting` problem be defined as follows. The input is a finite set S, and a collection of subsets $C_1, C_2, \ldots, C_k$ of the set S. An algorithm should return `true` if it is possible to color the elements of S `red` or `blue` in a way so that no C_i has all its elements colored with the same color.

Show that `Set-Splitting` is `NP`-Complete. (*sorry, no hints on the extra credit problem*)