# COS 226 Lecture 23: Dynamic Programming

DYNAMIC PROGRAMMING

* old term from operations research
* recursion revisited

Modern definition of dynamic programming:

"Bottom-up implementation of recursive programs with overlapping subproblems"

Top-down implementation sometimes easier

Typically improves running time of an algorithm from exponential time to polynomial time (!)

Many applications

# Fibonacci numbers

F(i) = F(i-1) + F(i-2)

| | | | | | |
|---|---|---|---|---|---|
| **0** | **0** | **10** | **55** | **20** | **6765** |
| **1** | **1** | **11** | **89** | **21** | **10946** |
| **2** | **1** | **12** | **144** | **22** | **17711** |
| **3** | **2** | **13** | **233** | **23** | **28657** |
| **4** | **3** | **14** | **377** | **24** | **46368** |
| **5** | **5** | **15** | **610** | **25** | **75025** |
| **6** | **8** | **16** | **987** | **26** | **121393** |
| **7** | **13** | **17** | **1597** | **27** | **196418** |
| **8** | **21** | **18** | **2584** | **28** | **317811** |
| **9** | **34** | **19** | **4181** | **29** | **514229** |

Numbers grow exponentially

F(i)/F(i-1) approaches 1.618... (golden ratio)

F(45) more than one billion

Recursive program to compute F(i)

```
int F(int i)
  {
    if (i < 1) return 0;
    if (i == 1) return 1;
    return F(i-1) + F(i-2);
  }
```

Problem: program is slow!

Easy proof:

running time satistfies the recurrence

F(i) = F(i-1) + F(i-2)

## Overlapping subproblems

Fibonacci computation is slow because subproblems in recursive computation overlap

To compute F(4)

```
     compute F(3)
          compute F(2)
               compute F(1) = 1
               compute F(0) = 0
               return 1
          compute F(1) = 1
          return 2
     compute F(2)
          compute F(1) = 1
          compute F(0) = 0
          return 1
     return 3
return 5
```

Fix by remembering solutions already computed

## Avoid recomputing known results

Maintain array (indexed by parameter value)
* zero if recursive routine not yet called
  for that value yet
* result to be returned otherwise

First call on procedure for a given value:
    compute as before, save value
Second and subsequent calls:
    use value from first call

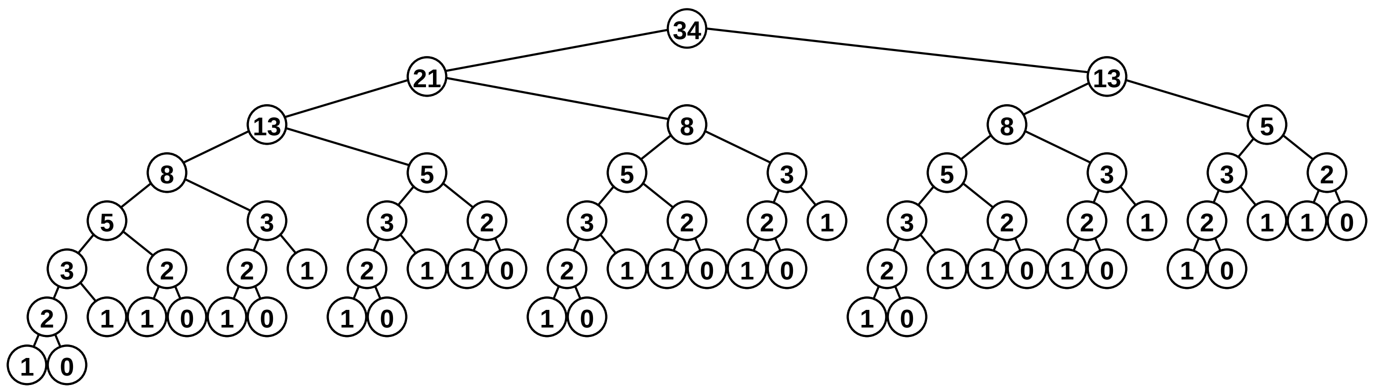
Ex: Fibonacci numbers less than 10^9

```
for (i = 0; i < 45; i++) known[i] = 0;
int F(int i)
  { int t;
    if (known[i]) return known[i];
    if (i == 0) t = 0;
    if (i == 1) t = 1;
    if (i > 1) t = F(i-1) + F(i-2);
    known[i] = t;
    return t;
  }
```

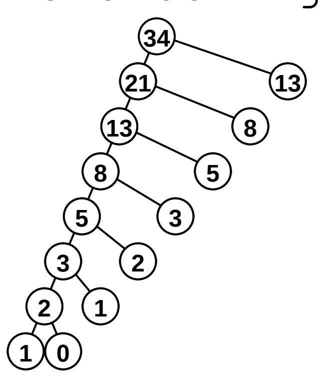
LINEAR running time
LOGARITHMIC amount of extra space

## Recursive call structures

Straight recursive algorithm



Remembering known results



```
To compute F(5)
    compute F(4)
        compute F(3)
            compute F(2)
                compute F(1) = 1
                compute F(0) = 0
                return 1
            use known F(1) = 1
            return 2
        use known F(2) = 1
        return 3
    use known F(3) = 2
    return 5
```

## Bottom-up approach

DYNAMIC PROGRAMMING

* Tabulate answers to subproblems
* Build table in increasing order of problem size
* Use early entries to compute later ones (no recomputation needed)

Ex: Fibonacci numbers

```
F[0] = 0; F[1] = 1;
for (j = 2; j < i; j++)
  F[j] = F[j-1] + F[j-2];
```

"Dynamic programming solution to the problem of computing Fibonacci numbers"

Perhaps the simplest nontrivial example of dynamic programming

Reduces running time from exponential to linear

Similar savings are available for a wide variety of important practical problems

Applicable to *any* pure recursive algorithm with

* no effects on global variables
* enough space to save solutions to subproblems

No overlap: bottom-up divide-and-conquer

# Knapsack problem

Given N types of food items, where each item
* takes a certain amount of space
* provides a certain amount of nutrients

KNAPSACK PROBLEM
Fill a knapsack of capacity M with items
such that total amount of nutrients is maximized

Ex:

| | size | value |
|---|---|---|
| **apple** | **3** | **4** |
| **banana** | **4** | **5** |
| **chocolate** | **7** | **10** |
| **donut** | **8** | **11** |
| **egg** | **9** | **13** |

In a knapsack of capacity 17
    two apples + banana + chocolate gives 23
    donut + egg gives 24 (maximum)

Applications
* shipping cargos
* job scheduling
* ...

## Recursive solution of knapsack problem

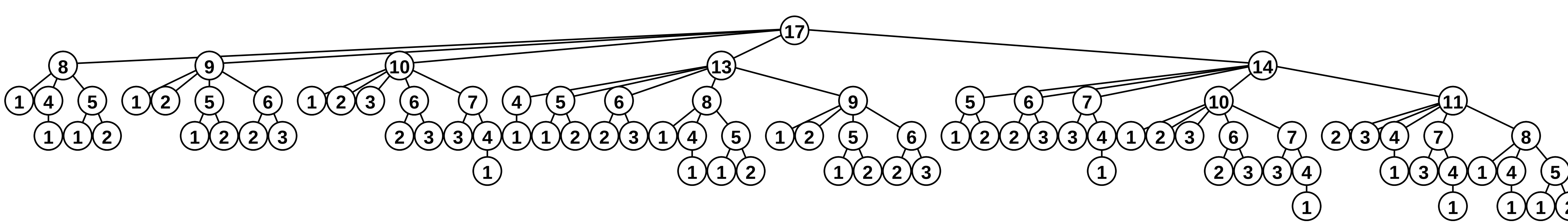
For each item type, compute the value obtained
if the last item added is of that type

Pick the maximum
(also could return ID of last item added)

```
int knap(int capacity)
  { int i, j, max, t;
    for (i = 0, max = 0; i < N; i++)
      if ((j = capacity-size[i]) >= 0)
        if ((t = knap(j) + val[i]) > max)
          max = t;
    return max;
  }
```

DON'T USE THIS PROGRAM!

overlapping subproblems
excessive recomputation
exponential time



Use dynamic programming (TD or BU) to avoid
recomputation for overlapping subproblems

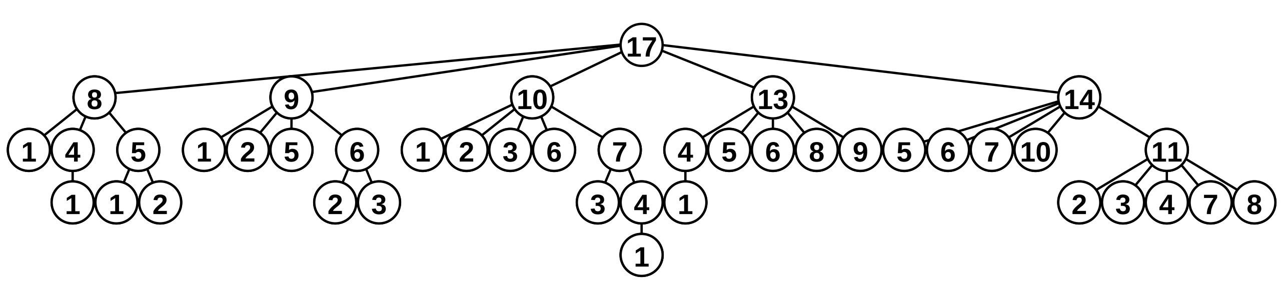
## "Known answers" (TD) approach

To avoid recomputation in knapsack algorithm
remember, for every knapsack capacity

* total cost for best way to fill knapsack
* last item used in filling it

```
int knap(int C)
  { int i, j, max, maxj, t;
    if (costKnown[C]) return costKnown[C];
    for (i = 0, max = 0, j = 0; i < N; i++)
      if ((j = cap-size[i]) >= 0)
        if ((t = knap(j) + val[i]) > max)
          { max = t; maxj = j; }
    costKnown[C] = max;
    bestKnown[C] = maxj;
    return max;
  }
```

Use bestKnown to recover set of items




Maintain elegance of recursive formulation
without sacrificing efficiency

## Bottom-up approach to knapsack problem

Generalized version of Fibonacci numbers program
Use previous solutions to compute next one

```
for (i = 0; i < C; i++) cost[i] = 0;
for (i = 0; i < C; i++)
  for (j = 1; j <= N; j++)
    if (i >= size[j])
      if (cost[i] < cost[i-size[j]]+val[j])
        {
          cost[i] = cost[i-size[j]]+val[j];
          best[i] = j;
        }
```

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 4 | 5 | 5 | 5 | 10 | 11 | 13 | 14 | 15 | | | | | | |
| a | b | b | b | c | d | e | a | a | | | | | | |

Total time proportional to N*C

Another approach

N passes through the cost array

[just interchange "for" loops]

## Dynamic programming summary

Many problem solutions are elegantly
described as recursive programs

If subproblems overlap, programs may
require exponential time

TOP-DOWN (known answers) approach

Mechanically modify recursive routine to

* test if called yet for the given parameter value
* if so, return previously computed value
* if not, compute value and save it

BOTTOM-UP (standard DP) approach

* Tabulate answers to subproblems
* Build table in order of problem size
* Use early entries to compute later ones

Further improvements often possible

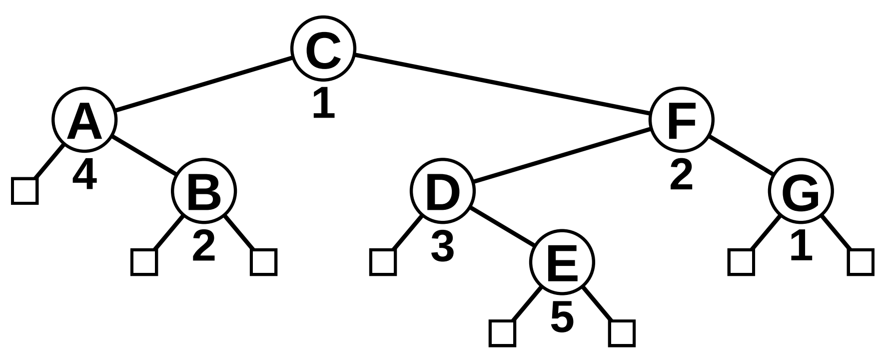
* do computations in particular order
* use specialized knowledge
  to eliminate more computations

## Dynamic programming with two variables

OPTIMAL BST problem

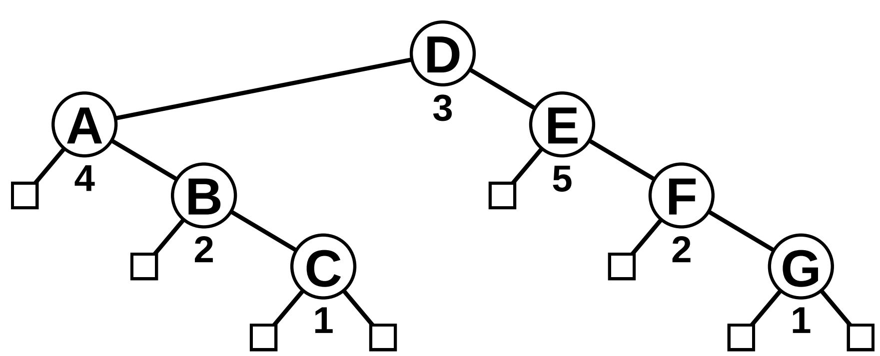
Given access frequencies, minimize average search cost (weighted path length) in BST

Ex:




weighted path length

1*1 + 2*(4+2) + 3*(2+3+1) + 4*5 = 51




weighted path length

1*3 + 2*(4+5) + 3*(2+2) + 4(1+1) = 41

## Top-down solution to optimal BST problem

Recursive program with two arguments

best subtree for ith through jth keys

```
int BST(int i, int j)
  { int k, min, t;
     for (k = i, min = 0; k <= j; k++)
        if (t = BST(i, k-1) + BST(k+1, j) < min)
     return min;
  }
```

DON'T USE THIS PROGRAM

overlapping subproblems, exponential time

Known values (top-down) approach

* precisely the same as for one-variable
* eliminate recomputation for overlapping
  subproblems by saving and recalling
  known values in 2D array

## Bottom-up solution to optimal BST problem

Two-dimensional array
* diagonal has subproblems of size 1
* one up from diagonal has subproblems of size 2
* two up from diagonal has subproblems of size 3

    ...

```
for (i = 1; i <= N; i++)
   for (j = i+1; j <= N+1; j++) cost[i][j] = MAX;
for (i = 1; i <= N; i++) cost[i][i] = f[i];
for (i = 1; i <= N+1; i++) cost[i][i-1] = 0;

for (j = 1; j <= N-1; j++)
   for (i = 1; i <= N-j; i++)
      {
         for (k = i; k <= i+j; k++)
            {
               t = cost[i][k-1]+cost[k+1][i+j];
               if (t < cost[i][i+j])
                 { cost[i][i+j] = t;
                   best[i][i+j] = k; }
            }
         for (k = i; k <= i+j; )
            cost[i][i+j] += f[k++];
      }
```

# Optimal BST example

"cost" array

```
.  4  8 11 19 31 37 41
.     2  4 10 20 25 28
.        1  5 14 18 21
.           3 11 15 18
.              5  9 12
.                 2  4
.                    1
```

"best" array

```
.     A  A  B  D  D  D
.        B  D  D  E  E
.           D  E  E  E
.              E  E  E
.                 E  E
.                    F
```

# Optimal recursive decomposition problems

Trees model many other computation problems

"Optimal divide-and-conquer"

MATRIX CHAIN PRODUCT problem

    Given a list of matrices, find optimal
    prenthiesization for pairwise multiplication

TRIANGULATION

    Divide a convex polygon into triangles
    using minimal lines of total length

Correspondence between trees and

* parenthesizations
* triangulated polygons

...

## String processing DP example

LONGEST COMMON SUBSEQUENCE problem
Find the longest common subsequence in 2 strings
Applications: molecular biology, text editors, ....

subsequence: subset of string chars, in order
    2^N different subseqs in string of length N

Ex: LCS in ABCBDAB and BDCAGA is BCBA

Recursive program is a description of a
    recursively defined solution

```
int LCSlen(int i, int j)
  {
    if ((i == -1) || (j == -1)) return 0;
    if (a[i] == b[j]) return 1+LCSlen(i-1,j-1);
    t = LCSlen(i, j-1); u = LCSlen(i-1, j);
    if (t > u) return t; else return u;
  }
```

As usual, DON'T USE THIS PROGRAM
It takes exponential time as is, but we can
- change to use known values OR
- compute values bottom-up

## "Far too many" subproblems

Dynamic programming cannot be used unless space
    is available to hold answers to subproblems

Three important cases illustrated by different
    versions of knapsack problem

EXAMPLE 1:

Sizes are reals, not integers

| | **size** | **value** |
|---|---|---|
| **apple** | **30000** | **.0004** |
| **banana** | **.0001** | **12345** |
| **chocolate** | **.0002** | **21244** |
| **donut** | **.1234** | **.2125** |
| **egg** | **.0012** | **123** |

    Table size depends on precision, relative sizes
    typically much more precision than space

EXAMPLE 2:

Allow at most one item of each type in knapsack
    Different subproblem for each SUBSET
        of item types
    2^N subsets: EXPONENTIAL space

EXAMPLE 3:

Bananas and eggs can’t go on the bottom, etc.
    Different subproblem for each PERMUTATION
        of item types
    N! permutations: superEXPONENTIAL space
        ( N! grows like 2^(N lgN) )

## NP-complete problems

For a large class of important problems
    no algorithms are known
    that are guaranteed to be fast

EFFICIENT, TRACTABLE
    running time bounded by polynomial
    on input size, no matter what the input
INEFFICIENT, INTRACTABLE
    running time exponential in input size
    on some input

Dynamic programming no help because
    exponential SPACE required for DP solution

Can't guarantee solution to intractable problems
    even with a supercomputer!!
* tough to explain to the boss...
* source of frustration for early programmers
* theory provides some consolation

Includes a very large class of natural problems
    (all equivalent wrt difficulty)
Fundamental issue in the theory of computation

## Examples of NP-complete problems

* TRAVELING SALESMAN
  A salesman needs to visit a set of N cities.
  Find a route that minimizes travel distance.

* SCHEDULING
  A set of jobs of varying length need to be done on two identical machines before a certain deadline. Can the jobs be arranged so that the deadline is met?

* SEQUENCING
  A set of four-character fragments have been obtained by breaking up a long string into overlapping pieces. Can the fragments be reconstituted into the long string?

* TRIPARTITE MATCHING
  Given three sets of N individuals, and a set of triples with one from each set, is there a subset of the triples containing each individual just once?

* SATISFIABILITY
  Is there a way to assign truth values to a given logical formula that makes it true?