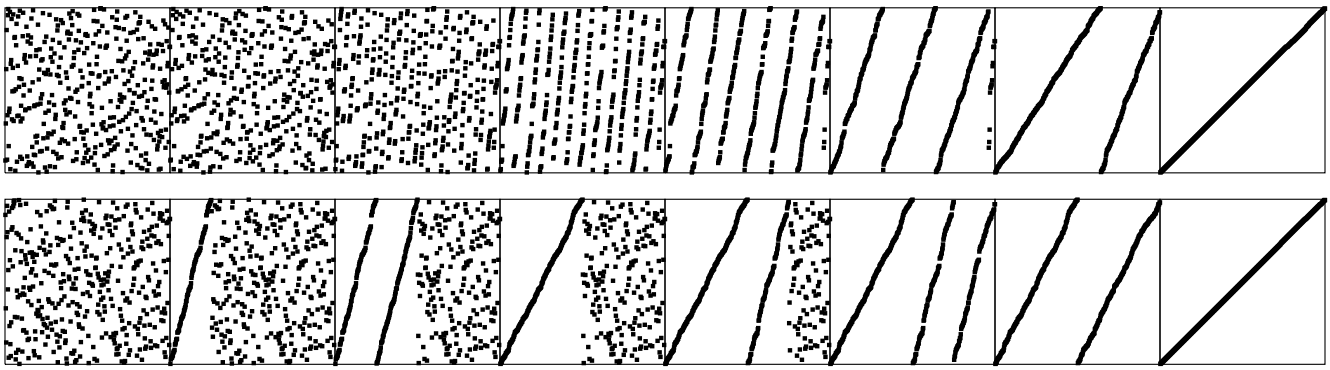
COS 226 Lecture 4: Mergesort

Prototypical divide-and-conquer algorithm



Guaranteed to run in O(N log N) steps

Drawback:

- Linear extra space for the merge
- (can only sort half the memory)

An "optimal" sorting method

Leads us to consider

- recurrence relationships
- computational complexity
- deep hacking
- fractals

Merging two sorted files

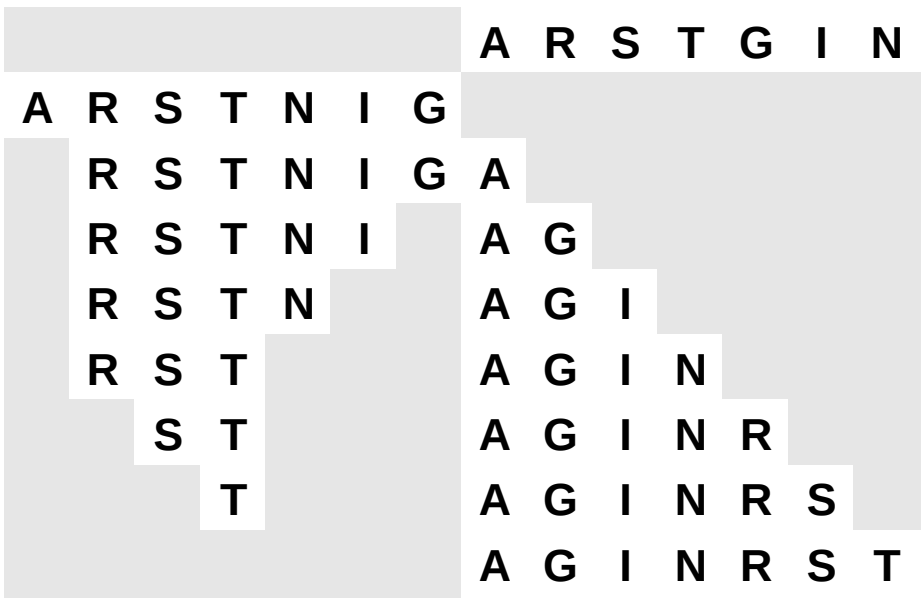
```
#define T Item
merge(T c[], T a[], int N, T b[], int M )
{ int i, j, k;
   for (i = 0, j = 0, k = 0; k < N+M; k++)
      {
         if (i == N) { c[k] = b[j++]; continue; }
         if (j == M) { c[k] = a[i++]; continue; }
         if (less(a[i], b[j]))
            c[k] = a[i++]; else c[k] = b[j++];
      }
}
```

Seems trivial, but try doing it

without using linear extra space

Abstract inplace merge

```
Item aux[maxN];
merge(Item a[], int l, int m, int r)
{ int i, j, k;
   for (i = m+1; i > l; i--) aux[i-1] = a[i-1];
   for (j = m; j < r; j++) aux[r+m-j] = a[j+1];
   for (k = l; k <= r; k++)
            if (less(aux[i], aux[j]))
          a[k] = aux[i++]; else a[k] = aux[j--];
}
```

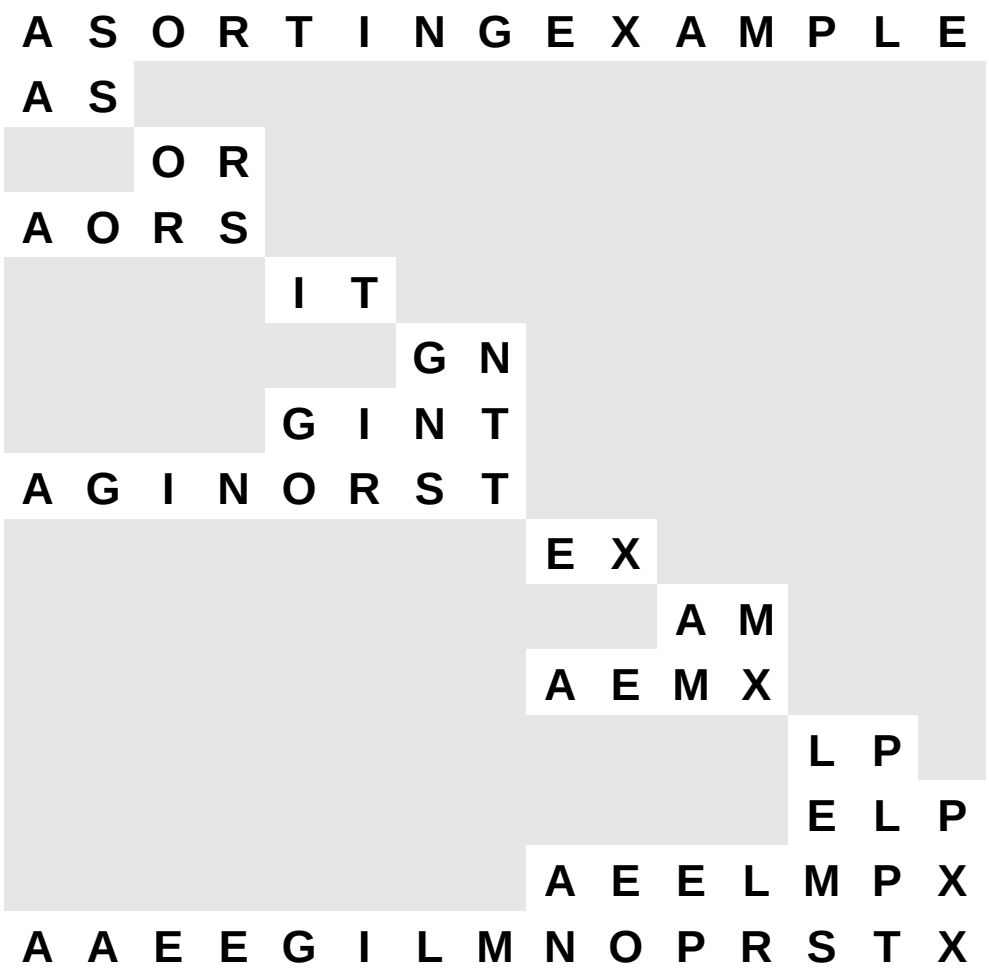


Tricky way to avoid special tests for ends of arrays

Still uses extra space, but easier for calling routine to assume inplace

Mergesort

Sort a file by (recursively) sorting two halves, then merge the results



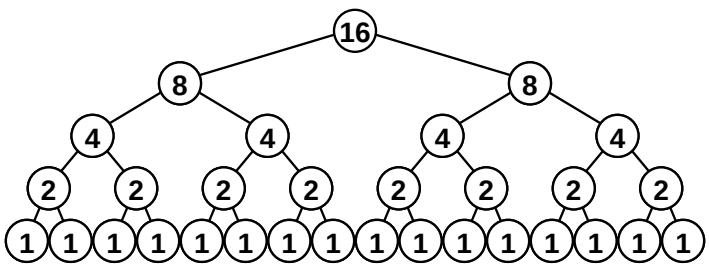
Mergesort implementation

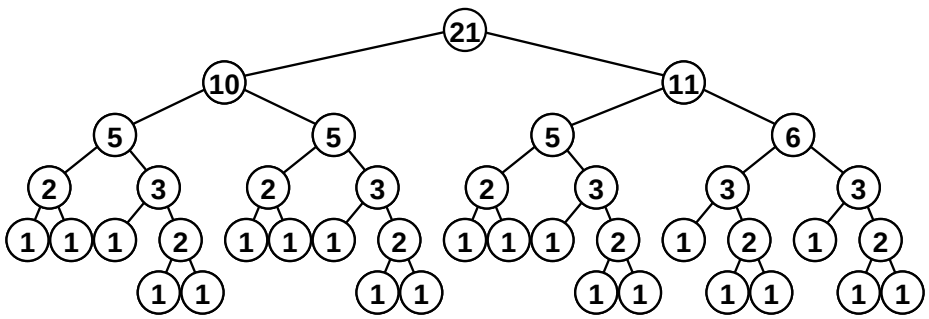
Direct recursive code

Uses abstract inplace merge

```
void mergesort(Item a[], int l, int r)
   {
      int m = (r+l)/2;
      if (r <= l) return;
      mergesort(a, l, m);
      mergesort(a, m+1, r);
      merge(a, l, m, r);
   }
```

Tree structures describe merge file sizes





Recurrences

Direct relationship to recursive programs

(most programs are "recursive")

Easy telescoping recurrences

T(N) = T(N-1) + 1 T(N) = N

T(2^n) = T(2^(n-1)) + 1 T(N) = lg N if N=2^n

Short list of important recurrences

T(N) = T(N/2) + 1 T(N) = lg N

T(N) = T(N/2) + N T(N) = N

T(N) = 2T(N/2) + 1 T(N) = N

T(N) = 2T(N/2) + N T(N) = N lg N

Details in Chapter 2

THM: Mergesort uses N lg N comparisons

(guaranteed worst-case bound)

Divide-and-conquer "master theorem" gives similar results for other algorithms

T(N) = aT(N/b) + N^c(lg N)^d

Interested in learning more?

Stay tuned for a few more in CS 226

Take CS 341, CS 423

Read "Introduction to the Analysis of Algs" by Sedgewick and Flajolet

Computational Complexity

Framework to study efficiency of algorithms

Machine model:

- count fundamental operations

Average case:

- predict performance (need input model)

Worst case:

- guarantee performance (any input)

Upper bound:

- algorithm to solve the problem

Lower bound:

- proof that no algorithm can do better

Caveats:

- worst-case may be unrealistic
- average-case bounds difficult to derive
- costs ignored in analysis may dominate
- machine model may be restrictive

Complexity studies provide

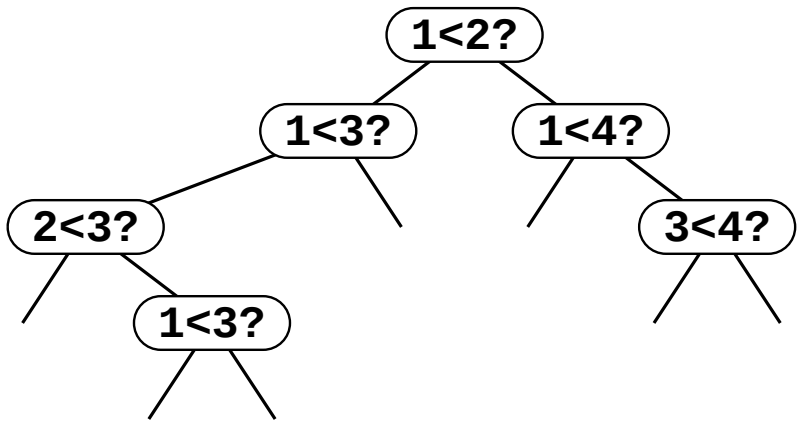
- starting point for practical implementations
- indication of approaches to be avoided

Complexity of sorting

N lg N comparisons necessary and sufficient

Upper bound: N lg N (Mergesort)
Lower bound: N lg N – N/(ln 2) + lg N

THM: All comparison-based sorting methods
     must use at least N lg N comparisons
Proof:
     DECISION TREE (all sequences of comparisons)



Claim 1: at least N! leaves
Claim 2: height at least lg N!
Claim 3: (Stirling's formula for lg N!)
     height at least N lg N – N/(ln 2) + lg N

Caveat: what if we don't use comparisons??
     (stay tuned for radix sort)

Deep hacking on Mergesort inner loop

Avoid move with recursive argument switch on call to straight merge.

```
void mergesort(T a[], T b[], int l, int r)
  { int m = (l+r)/2;
    if (r-l <= 10)
      { insertion(a, l, r); return; }
    mergesort(b, a, l, m);
    mergesort(b, a, m+1, r);
    merge(a+l, b+l, m-l+1, b+m+1, r-m);
  }
void sort(Item a[], int l, int r)
  { int i;
    for (i = l; i <= r; i++) aux[i] = a[i];
    mergesort(a, aux, l, r);
  }
```

Avoid sentinels with "up-down" trick

Combine the two? Doable, but mindbending

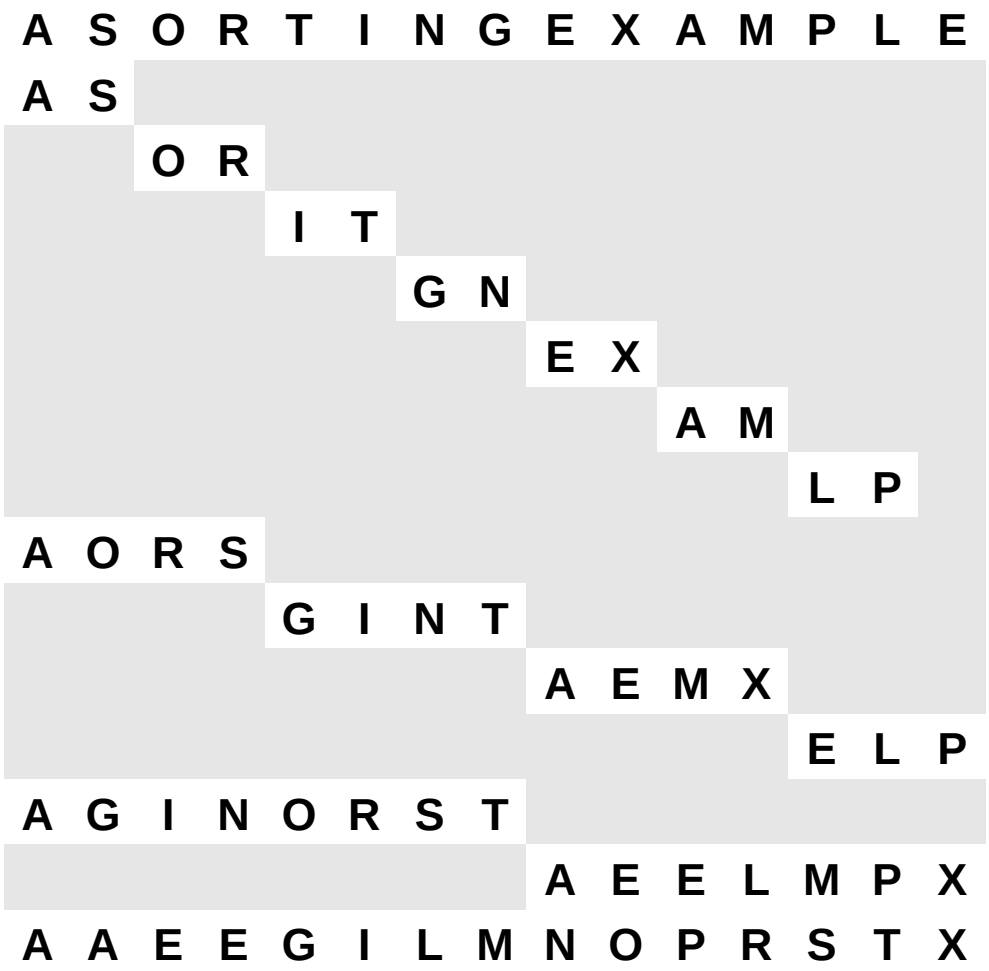
Inner loop: compare, store, two incs
(Quicksort: compare, inc)

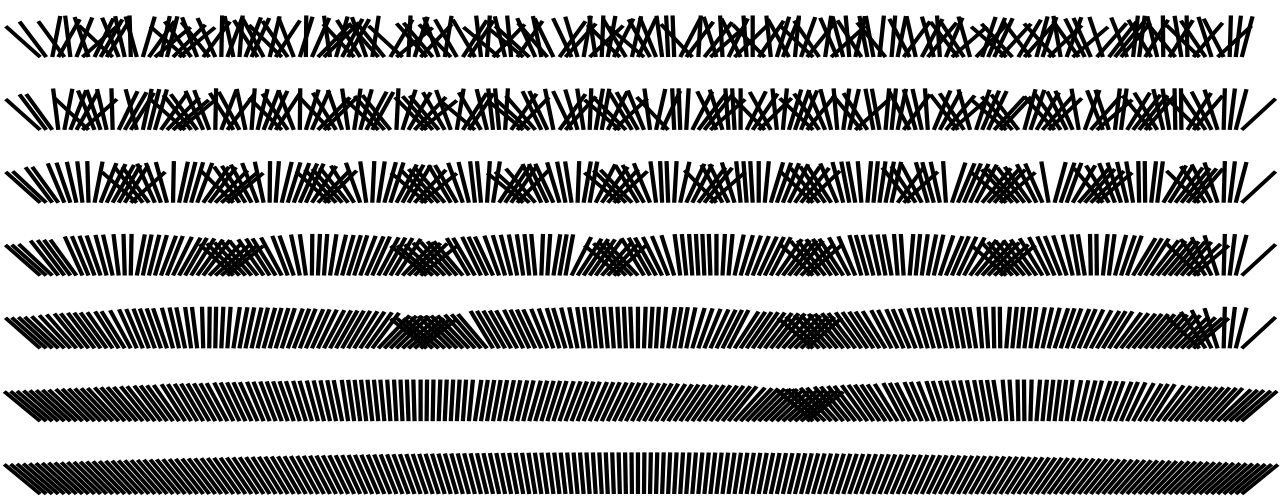
Bottom-up mergesort

Pass through the file

merge adjacent subfiles

size doubles each time through



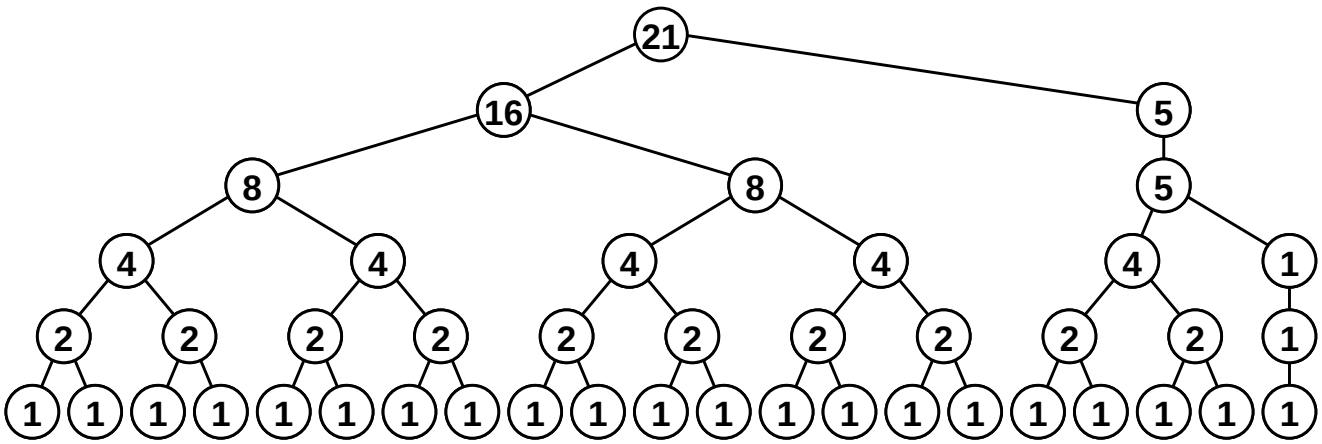


Bottom-up mergesort implementation

```
void mergesort(Item a[], int l, int r)
  { int i, m;
    for (m = 1; m < r-l; m = m+m)
      for (i = l; i <= r-m; i += m+m)
        merge(a, i, i+m-1, min(i+m+m-1, r));
  }
```

Different set of merges than for top-down

unless N is a power of two



Mergesort and numbers

Relationship to binary representation of numbers

THM:
The number of comparisons used by Mergesort
for a file of size N
    is the same as
the number of bits in the binary representations
of all the numbers less than N (plus N-1).

0 0 0 0
0 0 0 1
0 0 1 0
0 0 1 1
0 1 0 0
0 1 0 1
0 1 1 0
0 1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1

Proof: They satisfy the same recurrence

$$C(2N) = C(N) + C(N) + 2N$$

$$C(2N+1) = C(N) + C(N+1) + 2N+1$$

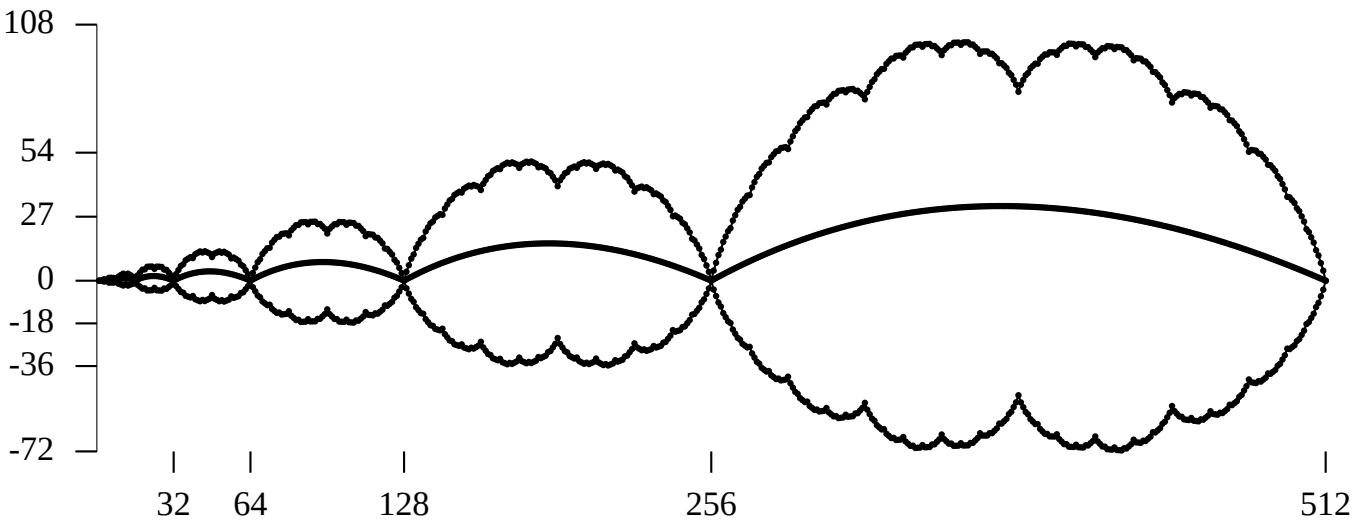
Mergesort and fractals

Divide-and-conquer algorithms
can exhibit erratic periodic behavior

number of bits in numbers less than N
= number of 0 bits
+ number of 1 bits

= (N lg N)/2 + periodic term
+ (N lg N)/2 + periodic term

= N lg N + periodic term



Details in Sedgewick-Flajolet

Merging linked lists

Problem: sort data on a linked list

(rearrange list so items are in order)

```
typedef struct node *link;
struct node { Item item; link next; };
```

First step: merge implementation

```
link merge(link a, link b)
  { struct node head; link c = &head;
    while ((a != NULL) && (b != NULL))
      if (less(a->item, b->item))
        { c->next = a; c = a; a = a->next; }
      else
        { c->next = b; c = b; b = b->next; }
    c->next = (a == NULL) ? b : a;
    return head.next;
  }
```

Next step: sort implementation

top-down and bottom-up methods both apply

List mergesorts

Top-down: split, sort, and merge

```
link mergesort(link c)
   { link a, b;
      if (c->next == NULL) return c;
      a = c; b = c->next;
      while ((b != NULL) && (b->next != NULL))
         { c = c->next; b = b->next->next; }
      b = c->next; c->next = NULL;
      return merge(mergesort(a), mergesort(b));
   }
```

Bottom-up: cycle through a circular list

```
link mergesort(link t)
   { link u;
      for (initQ(); t != NULL; t = u)
         { u = t->next; t->next = NULL; putQ(t); }
      t = getQ();
      while (!emptyQ())
         { putQ(t); t = merge(getQ(), getQ()); }
      return t;
   }
```

Mergesort is the method of choice for lists