COS 226 Lecture 3: Quicksort

To sort an array, first divide it so that

* some element a[i] is in its final position
* no larger element left of i
* no smaller element right of i

Then sort the left and right parts recursively

A	S	O	R	T	I	N	G	E	X	A	M	P	L	E
A	A	E	E	T	I	N	G	O	X	S	M	P	L	R
A	A	E												
A	A													
A														
				L	I	N	G	O	P	M	R	X	T	S
				L	I	G	M	O	P	N				
				G	I	L								
					I	L								
					I									
								N	P	O				
									O	P				
										P				
												S	T	X
													T	X
													T	
A	A	E	E	G	I	L	M	N	O	P	R	S	T	X

Partitioning

To partition an array, pick a partitioning element

* scan from right for smaller element
* scan from left for larger element
* exchange
* repeat until pointers cross

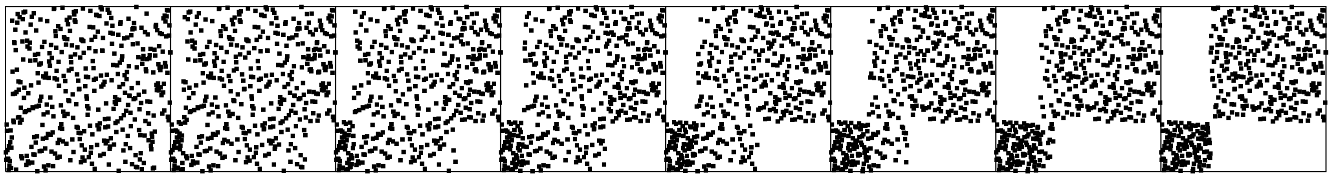
```
A S O R T I N G E X A M P L E

A S
                    A M P L
A A                 S M P L E

    O
                E X
A A E           O X S M P L E

      R
    E R T I N G

A A E E T I N G O X S M P L R
```



Partitioning implementation

v: partitioning element

i: left-to-right pointer

j: right-to-left pointer

```
int partition(Item a[], int l, int r)
  {
    int i, j; Item v;
    v = a[r]; i = l-1; j = r;
    for (;;)
      {
        while (less(a[++i], v)) ;
        while (less(v, a[--j]))
          if (j == l) break;
        if (i >= j) break;
        exch(a[i], a[j]);
      }
    exch(a[i], a[r]);
    return i;
  }
```

Issues

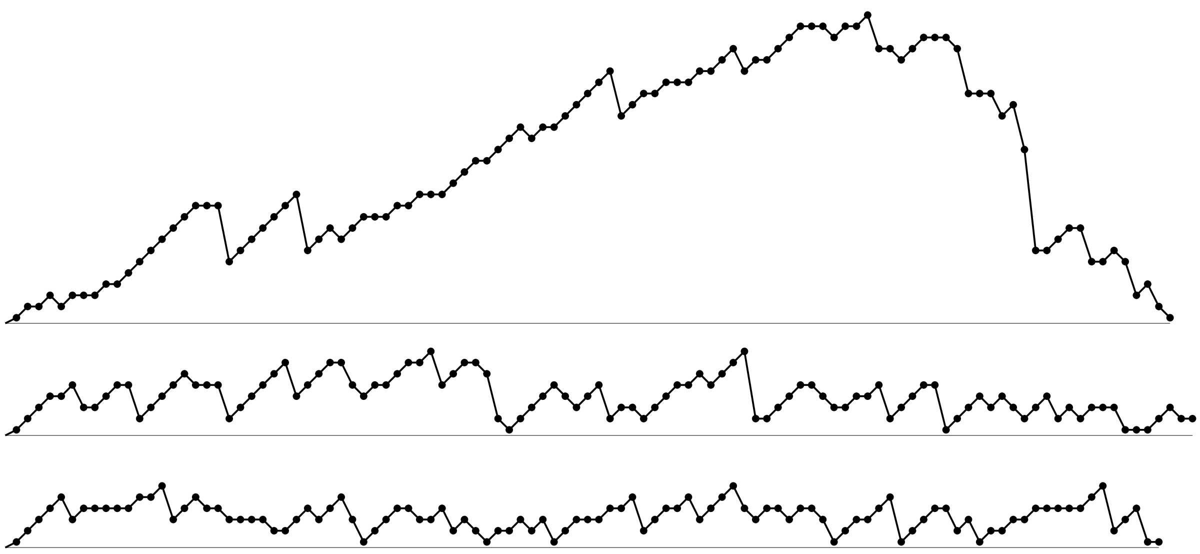
* stop pointers on keys equal to v?
* sentinels or explicit tests for array bounds?
* details of pointer crossing

Quicksort implementation

```
quicksort(Item a[], int l, int r)
  {
    int i;
    if (r > l)
      {
        i = partition(a, l, r);
        quicksort(a, l, i-1);
        quicksort(a, i+1, r);
      }
  }
```

Issues

* overhead for recursion?
* running time depends on input
* worst-case time cost
  (quadratic, a problem)
* worst-case space cost
  (linear, a serious problem)



Nonrecursive Quicksort

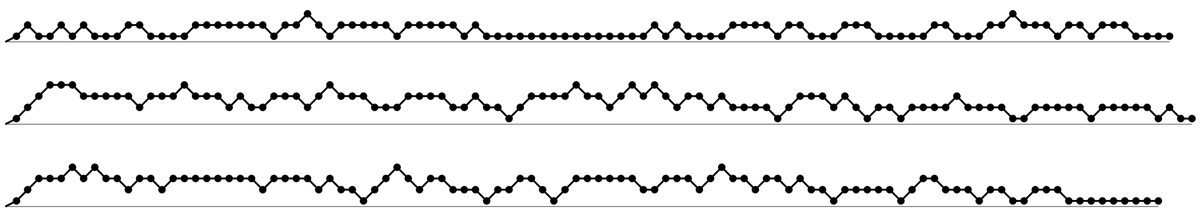
Use explicit stack instead of recursive calls

Sort smaller of two subfiles first

```
#define push2(A, B)  push(A); push(B);
void quicksort(Item a[], int l, int r)
  {
    int i;
    stackinit(); push2(l, r);
    while (!stackempty())
      {
        r = pop(); l = pop();
        if (r <= l) continue;
        i = partition(a, l, r);
        if (i-l > r-i)
          { push2(l, i-1); push2(i+1, r); }
        else
          { push2(i+1, r); push2(l, i-1); }
      }
```

Guaranteed less than (lg N) extra space cost

Still quadratic worst-case time cost



Analysis

Total running time is sum of

cost*frequency

for all the basic operations

Cost depends on machine

Frequency depends on algorithm, input

For Quicksort

- A -- number of partitioning stages
- B -- number of exchanges
- C -- number of comparisons

Cost on a typical machine

$35A + 11B + 4C$

Number of comparisons in the worst case

$N + (N-1) + (N-2) + ... = N(N-1)/2$

Average case analysis

Assume input randomly ordered

* each element equally likely to be partitioning element
* subfiles randomly ordered (if partitioning is "blind")

Average number of comparisions satisfies

```
 C(N) = N+1
     +  (C(1) + C(N-1))/N
     +  (C(2) + C(N-2))/N
     ...
     +  (C(N-1) + C(1))/N
 C(N) = N+1 + 2( C(1)  +  C(2) + ... + C(N-1) )/N
 NC(N) = N(N+1)  + 2( C(1) + C(2) + ... + C(N-1))
 NC(N) - (N-1)C(N-1) = 2N + 2C(N-1)
 NC(N) = (N+1)C(N-1) + 2N
 C(N)/(N+1) = C(N-1)/N + 2/(N+1)
     = 2( 1 + 1/2 + 1/3 + ... 1/(N+1) )
     = 2 ln N + (small error term)
```

THM: Quicksort uses about 2N ln N = (1.38...)N lg
comparisons (and 1/6 as many exchanges)
on average.

Randomized algorithm

* pick random partitioning element
* analysis then valid for any input

Inner loop

Use profiler

* find instructions executed most often
* verify analysis
* streamline program by iterating

```
quicksort(int a[], int l, int r)
   <132659>{
   int v, i, j, t;
   if (<132659>r > l)
      {
         <66329>v = a[r];
         <66329>i = l-1; <66329>j = r;
         for (<66329>;<327102>;<327102>)
            {
               while (<1033228>a[++i] < v) <639797>;
               while (<1077847>a[--j] > v) <684416>;
               if (<393431>i >= j) <66329>break;
          <327102>t = a[i]; a[i] = a[j]; a[j] = t;
            }
         <66329>t = a[i]; a[i] = a[r]; a[r] = t;
         <66329>quicksort(a, l, i-1);
         <66329>quicksort(a, i+1, r);
      }
   <132659>}
```

Another partitioning method

Seeking an easier-to-understand method
(detailed justification omitted)

```
quicksort(int a[], int l, int r)
<133395>{
int v, i, k, t;
if (<133395>r <= l) return;

<66697>v = a[l]; <66697>k = l;
for (<66697>i=l+1; <1976624>i<=r; <1909927>i++)
   if (<1909927>a[i] < v)
   { <934565>t = a[i]; a[i] = a[++k]; a[k] = t; }
   <66697>t = a[k]; a[k] = a[l]; a[l] = t;
   <66697>quicksort(a, l, k-1);
   <66697>quicksort(a, k+1, r);
   }

<133395>}
```

Not much simpler,
three times as many exchanges

Improvements to Quicksort

Median-of-three
- partitioning element closer to center
- estimate median with median of sample
- number of comparisons close to N lg N with large enough sample
- FEWER LARGE FILES
- slightly more exchanges, more overhead

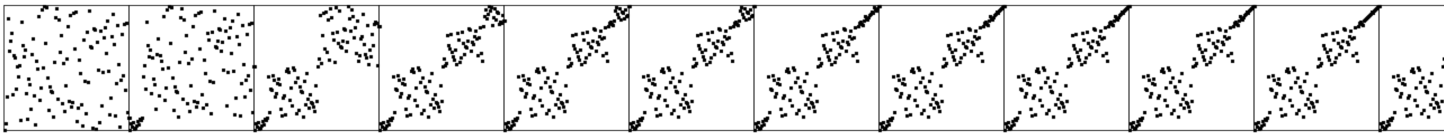
Insertion sort small subfiles
- even Quicksort has too much overhead for files of a few elements
- use insertion sort for tiny files
- (can wait until the end)

Analysis gives best value of parameters
- median of 3 elements
- cut to insertion sort for < 10 elements

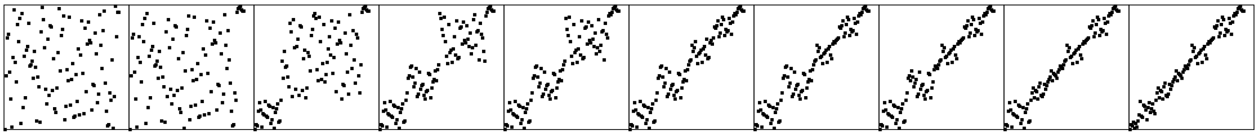
Improvements to Quicksort

Standard

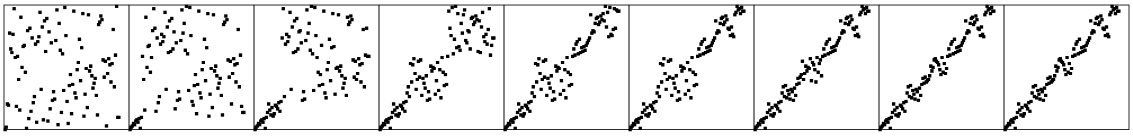
* might have bad partition (1st one in ex.)
* might have good partition (2nd one in ex.)



Cutoff for small subfiles



Median-of-three



Three-way partitioning

Equal keys can affect performance

Suppose all keys are the same

* plain quicksort takes N lg N comparisons (!)
* change partitioning to take N comparisons
* naive method might use N^2 comparisons (!!)

Suppose there are two distinct key values

* reduces to above case for one subfile
* better to complete sort with one partition

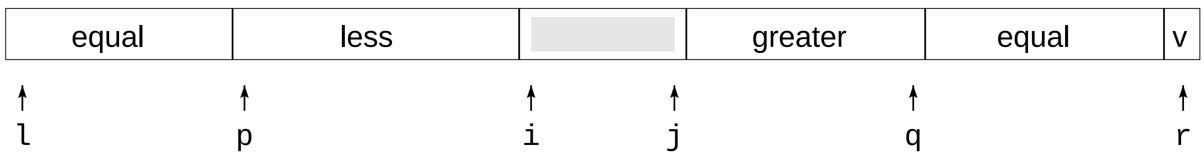
  stop right pointer on left value
  stop left pointer of right value
  exchange

Large file, small number of key values

* reduces to above cases

Partition into three parts

* elements between i and j equal to v
* no larger element left of i
* no smaller element right of j



Three-way partitioning implementation

```
void quicksort(Item a[], int l, int r)
{
   int i, j, k, p, q; Item v;
   if (r <= l) return;

   v = a[r]; i = l-1; j = r; p = l-1; q = r;
   for (;;)
   {
      while (less(a[++i], v)) ;
      while (less(v, a[--j])) if (j == l) break;
      if (i >= j) break;
      exch(a[i], a[j]);
      if (eq(a[i],v)) { p++; exch(a[p],a[i]); }
      if (eq(v,a[j])) { q--; exch(a[q],a[j]); }
   }
   exch(a[i], a[r]); j = i-1; i = i+1;
   for (k = l  ; k < p; k++, j--) exch(a[k], a[j]);
   for (k = r-1; k > q; k--, i++) exch(a[k], a[i]);
   quicksort(a, l, j);
   quicksort(a, i, r);
}
```

Sorting strings

Use three-way partitioning on key characters

actinian	**coenobite**	**actinian**
jeffrey	**conelrad**	**bracteal**
coenobite	**actinian**	**coenobite**
conelrad	**bracteal**	**conelrad**
secureness	**secureness**	**cumin**
cumin	**dilatedly**	**chariness**
chariness	**inkblot**	**centesimal**
bracteal	**jeffrey**	**cankerous**
displease	**displease**	**circumflex**
millwright	**millwright**	**millwright**
repertoire	**repertoire**	**repertoire**
dourness	**dourness**	**dourness**
centesimal	**southeast**	**southeast**
fondler	**fondler**	**fondler**
interval	**interval**	**interval**
reversionary	**reversionary**	**reversionary**
dilatedly	**cumin**	**secureness**
inkblot	**chariness**	**dilatedly**
southeast	**centesimal**	**inkblot**
cankerous	**cankerous**	**jeffrey**
circumflex	**circumflex**	**displease**

Not many disadvantanges

(to be continued: this is a radix sort)