Assignment 3: Vibrating String

COS 496 Due: Thurs., April 11

Basic Part: Ideal String

If $y(x,t)$ is the displacement of a string stretched between two points, and if the displacement is small, vibration is determined by the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The assignment is to simulate a vibrating string using a finite-difference method, starting from various initial conditions. To do this, first discretize time and space by

$$y(m\Delta x, n\Delta t) = y_{mn} = y(x,t)$$

Then use the following central-difference approximation for $\partial^2 y/\partial t^2$ [1, pp. 213*ff*]:

$$\frac{\partial^2 y}{\partial t^2} \approx \frac{y_{m,n-1} - 2y_{mn} + y_{m,n+1}}{(\Delta t)^2}$$

and the analogous approximation for the right-hand side of the wave equation. Then derive an explicit iteration method by finding $y_{m,n+1}$ in terms of values at times n and $n-1$.

Define the dimensionless parameter

$$s = c\frac{\Delta t}{\Delta x}$$

and explore the behavior of your iterative method for the following initial conditions: (a) a sine wave at the zeroth mode; (b) a sine wave at a higher harmonic; and (c) a triangular shape (for a "plucked" string) — and for different values of s. For what values of s is the iteration stable?

Extra Credit

Many of these extensions are from [2]. There is enough here altogether for a classic Master's Thesis – in fact, Ruiz's Master's Thesis [3]. If you have time for extra work on this problem, pick any that interest you. References [2] and [3] are on reserve in the engineering library.

A. Convert your result to sound so you can listen to the string. See the example pendulum code with soundfile output on the course web page.

B. Real strings are subject to a damping force which accounts for dissipation of energy in the form of heat. To a first approximation, the damping force is proportional to the velocity of the string, so the wave equation is modified to become

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - b\frac{\partial y}{\partial t}, \qquad b > 0$$

where b is a constant representing the amount of damping. Incorporate this term into your iteration formula, and explore its effect. How are different harmonics affected? Do higher frequencies die out faster or slower than lower harmonics?

C. Repeat the previous part, but with the damping term of the form

$$+r\frac{\partial^3 y}{\partial t^3}, \qquad r > 0$$

which Hiller and Ruiz claim is the simplest one that can account for loss of energy through radiation. How are the different harmonics now affected?

D. (Stiff string) When stiffness of the string is taken into account, the wave equation becomes

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - S\frac{\partial^4 y}{\partial x^4}, \qquad S > 0$$

where S is a constant depending on properties of the string like Young's modulus of elasticity, cross-sectional area, etc. Try incorporating this into your difference equation. What effect does stiffness have on the fundamental frequency and higher partials of the vibration? (When harmonics are no longer integer multiples of a fundamental frequency, they're called *partials*.)

E. Put together the three additional terms from Parts B-D.

F. Play with the effect of different initial conditions. Can you make the string sound plucked? Bowed?

References

[1] G. D. Smith, *Numerical Solution of Partial Differential Equations*, (third edition), Clarendon Press, Oxford, 1985.

[2] L. Hiller and P. Ruiz, "Synthesizing Musical Sounds by Solving the Wave Equation for Vibrating Objects," *Journal of the Audio Engineering Society* **19**, 6, (June 1971), pp. 462-470 (Part I), and **19**, 7, (August 1971) pp. 542-551 (Part II).

[3] P. Ruiz, "A Technique for Simulating the Vibrations of Strings with a Digital Computer," Master's Thesis, Department of Music, University of Illinois, 1969.

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