COS 496 Simulation ...
Assignment 2 Due 2/22/96
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The Pendulum

A Pendulum consists of a mass (m) suspended on a length (r) of massless cord. For simplicity, the cord can be assumed to be a rigid rod. Assuming no air friction on the mass or cord, and no friction on the pivot where the cord is attached, the forces acting on the mass are: the force due to gravity, a tension force in the cord which always acts in the direction of the cord, and possibly some externally imposed forces (like you grabbing the mass and displacing the pendulum from rest position).

Recalling your **Basic Newtonian Physics**: Force = Mass * Acceleration, $f = ma$, where acceleration is the rate of change of velocity (dv/dt), and velocity is the rate of change of position (dx/dt). Acceleration is thus the second time derivative of position ($a = d^2x/dt^2$). These equations can be extended into a rotational geometry (like our pendulum) by replacing Force with Torque (τ), Mass with Moment of Intertia (I), and Acceleration with Angular Acceleration (α). Torque is computed as the force normal to radius times distance from the pivot point, or in our pendulum problem, $\tau = f_T * r$, where f_T is the tangential component of force acting normal to the cord at a distance r. Moment of inertia is computed about a particular point (the pivot point for our pendulum), and is $I = m * r^2$ for a point mass. Angular acceleration is the second derivative of angular displacement with respect to time ($\alpha = d^2\theta/dt^2$). See following discussion.

Recalling your **Basic Trigonometry**: displacing the mass from rest such that the cord makes an angle θ_0 with the vertical, the force due to gravity is mg, but in a direction vertically downward (not in general along the cord). Thus the force on the mass has a projection in the direction along the cord, and a projection normal to the cord. This latter force, tangential to the circle defined by the cord-radius, is the restoring force which causes the pendulum to return to rest. The restoring force is given by $mg\sin(\theta)$, where g is the acceleration due to gravity, $9.8\frac{m}{s^2}$ (meters per second per second). Note here that once the pendulum is released, θ is a function of time, which is why we assign θ_0 as the *maximum* displacement at time zero. So you'd plug the tangential force component into the $Torque$ expression above. For units to work out, θ is expressed in radians. Also, there is a negative sign reflecting the fact that the restoring force acts opposite the displacement.

Convenient Simplifications: Recall that for small θ, $\sin(\theta) \simeq \theta$, which means the restoring force is roughly linear in the angular displacement.

Using all of the above information, by plugging into the $\tau = I\alpha$ equation, you can get expressions for the Ordinary Differential Equations (ODE), for the exact and approximate cases.

In the real world there are losses which affect the equations and solutions, and possibly external forces. For example, if you were to hold the top of the pendulum in your moving hand, you would have another force term due to the external input.

Your assignment:

1. Set up and simulate the pendulum using discrete time, and continuous position, velocity, and acceleration. Simulate both the exact and approximate solutions. See how the oscillation frequency varies as a function of cord length, mass, and initial displacement. Determine when and how the small-angle approximation breaks down, by assigning an error margin of your choosing, and determine some rules for r, and θ_0 which cause the solution to closely match your maximum error. Investigate how length and initial displacement affect things like the oscillation frequency and waveform shape. Investigate how the time-step affects the simulation. Do one specific set of simulations with length $r = 0.8$ meter, $\theta_0 = 1.0$ radians, and $\Delta t = 0.01$s, 0.1s, 0.5s. Do a simulation with an initial displacement of 0, but an initial velocity of some non-zero amount. Determine if there is an initial velocity which causes the pendulum to swing completely around.

2. Now go out and make your own pendulum (an actual physical pendulum, using a ball and string). For interest, the mass of a U.S. 25 cent piece is 5.67 grams. Run experiments on your physical pendulum, and use the results to add frictional loss to your virtual pendulum. Can you experimentally test the small-angle approximation, and your prior rules for achieving your error margin from part 1?

Extra credit:

- Predict, from physical considerations, the initial velocity required to push the pendulum over the top and check against your simulations.
- Modify your pendulum to include an external input force signal with some deterministic and some random components.
- Derive or look up the period of the nonlinear version of the pendulum (without the $\sin(\theta) \approx \theta$ approximation) and check it against your simulations and experiments.
- Set up and simulate the problem of a hanging rope with mass per unit length ρ. Note that this is similar to the pendulum, but a lot harder. Some famous historical physicists once opted to discretize space along the rope by calling it a "chain."
- Go to a museum and watch the giant pendulum for 112 days. Is it linear or non-linear? Are you?
- Use a pendulum to kill off any extra mice you still have mating in your computer.