# COS 445 - Warmup PSet

Due online Monday, February 2nd at 11:59 pm

**Instructions:**

- Submit your solution as a single PDF to Gradescope. If you collaborated with other students, or consulted an outside resource, submit a (very brief) collaboration statement as well.
- This homework is considerably shorter and simpler than all others. Its purpose is to help you gauge the background expected at the beginning of the course. You may wish to reference the cheatsheet.
- Please reference the course collaboration policy here.

## Problem 1: Basic Probability I (10 points)

(a) Compute the expectation of the discrete random variable $X$, where $X = i$ with probability $2^{-i}$, for all integers $i \geq 1$.

(b) Compute the expectation of the non-negative, continuous random variable $Y$ with CDF $F_Y(x) = 1 - 1/(x+1)^3$, $x \geq 0$.

(c) Compute the expectation of the random variable $Z = X + Y$.

## Problem 2: Basic Probability II (10 points)

Suppose that there are $n$ balls and $n$ bins. Each ball is thrown, independently, into a uniformly random bin.

(a) What is the probability that Bin 1 is empty?

(b) What is the expected number of empty bins?

(c) What is the expected number of bins which contain exactly two balls?

## Problem 3: Basic Continuous Optimization (10 points)

(a) Let $f(x_1, x_2) = x_1^2 - x_1x_2 + x_2^2$. Minimize $f(x_1, x_2)$ over all $(x_1, x_2) \in \mathbb{R}^2$ (and prove that it is the minimum).

(b) Let $f(x_1, x_2) = x_1 - x_2 + x_1x_2$. Maximize $f(x_1, x_2)$ over the range $[-1, 1] \times [-1, 1]$ (and prove that it is the maximum).

## Problem 4: Basic Proofs I (10 points)

You're trying to collect all $n$ distinct cards from your favorite trading card game. The only way to obtain new cards is to purchase a sealed pack of one uniformly random card. After you buy a pack, you open it, see the card inside, and add it to your collection. If you have at least one copy of each of the $n$ cards, you stop. Otherwise, you purchase a new pack. Prove that the expected number of packs you purchase is $\Theta(n \log n)$.[1][2][3]

**Note:** Check out Section 4.2 of the cheatsheet for stylistic tips on structuring a multi-step proof.

## Problem 5: Basic Proofs II (10 points)

Every pair of people in the world are either buddies or not buddies (and if Alice is buddies with Bob, then Bob is buddies with Alice). Say that a set $S$ of people is *buddy-full* if everyone in $S$ is buddies with everyone else in $S$ (that is, for all $u \in S, v \in S, v \neq u$, $u$ and $v$ are buddies). Say that a set $S$ is *buddy-free* if no one in $S$ is buddies with anyone else in $S$ (that is, for all $u \in S, v \in S, v \neq u$, $u$ and $v$ are not buddies). Prove that if six people are together in a room, then there is either a buddy-full set of size 3, or a buddy-free set of size 3. Prove that if only five people are in the room, then it's possible that that there is no buddy-full set of size 3, nor a buddy-free set of size 3.

**Note:** Check out Section 4.2 of the cheatsheet for stylistic tips on structuring a multi-step proof.

[1] Recall that $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. Recall further that $f(n) = O(g(n))$ if there exist absolute constants $C, n_0$ such that $f(n) \leq C \cdot g(n)$ for all $n \geq n_0$. $f(n) = \Omega(g(n))$ if there exist absolute constants $C, n_0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

[2] You may use without proof the fact that $\ln(n+1) + 1 > \sum_{i=1}^{n} 1/i > \ln(n+1)$.

[3] You may also use the following fact without proof: if a coin is heads with probability $p$ (independently on each flip), then the expected number of flips until seeing a heads is $1/p$.

# Extra Credit: Fun with Coupling

Recall that extra credit will contribute to your participation. Some extra credits are **quite** challenging. We do not suggest attempting the extra credit problems for the sake of your grade, but only to engage deeper with the course material. If you are interested in pursuing an IW/thesis in CS theory, the extra credits will give you a taste of what that might be like.[4]

For both parts of this question, it will take you some time to figure out the right proof approach. Once you figure out a good proof approach, however, there is a short proof which is very easy to follow. For full credit, your proof should be comparably easy to follow.[5]
**Hint:** For any part where you are proving a claim, you should try to use a coupling argument. If you don't remember what a coupling argument is, you can find a definition in the cheatsheet.

## Part a

Let $D$ be any distribution which samples only (strictly) positive numbers. For all integers $i \geq 1$, let $X_i$ denote an independent sample from $D$. For any $t \geq 0$, let $N(t)$ be such that:

- $\sum_{i=1}^{N(t)} X_i \geq t$.
- $\sum_{i=1}^{N(t)-1} X_i < t$.

That is, $N(t)$ is the smallest index such that the first $N(t)$ random variables sum to $\geq t$. Observe that $N(t)$ is itself a random variable. Finally, define $f(t) := \mathbb{E}[N(t)]$. Prove that, for all $s, t \geq 0$, $f(s) + f(t) \geq f(s+t)$.

## Part b

Let $S$ be any finite set whose elements are non-negative real numbers, and let $n := |S|$, and let $A$ be the sum of the elements in $S$. Let $X_1, \ldots, X_n$ be random variables equal to the elements of $S$ in uniformly random order (i.e., sample the elements of $S$ without replacement, one at a time). For any real number $t \in [0, A]$, again define $N(t)$ to be such that

- $\sum_{i=1}^{N(t)} X_i \geq t$.
- $\sum_{i=1}^{N(t)-1} X_i < t$.

That is, $N(t)$ is the smallest index such that the first $N(t)$ random variables sum to $\geq t$. Observe that $N(t)$ is itself a random variable, and it is well-defined for $t \in [0, A]$. Again, define $f(t) := \mathbb{E}[N(t)]$. Prove the following claim, or find (and analyze) a counterexample: for all $s, t \geq 0$ such that $s + t \leq A$, $f(s) + f(t) \geq f(s+t)$.[6]

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[4]Keep in mind, of course, that you will do an IW/thesis across an entire semester/year, and you are doing the extra credit in a week. Whether or not you make progress on the extra credit in a week is not the important part — it's whether or not you enjoy the process of tackling an extremely open-ended problem with little idea of where to get started.

[5]Thank you to Nick Arnosti for originally suggesting this question.

[6]Observe that if the random variables were sampled *with* replacement, this claim would be a special case of Part a.