COS 445 - PSet 2

Due online Monday, February 16th at 11:59 pm

Problem 1: Condorcet Losers (30 points)

A *Condorcet loser* is a candidate who loses to every other candidate in a head-to-head election; formally, $a \in A$ is a Condorcet loser if, for every $b \in A \setminus \{a\}$, a strict majority of voters prefer b to a.

Part a (15 points)

Construct a preference profile with 3 candidates where the Condorcet loser is not the candidate with the lowest Borda point count.

Part b (15 points)

Construct a preference profile with 3 candidates where the Plurality winner is the Condorcet loser.

Extra Credit. Prove that in any preference profile on 3 candidates, a Condorcet loser is not the Borda winner.

Problem 2: Truthfulness in Participatory Budgeting (20 points)

Consider the Participatory Budgeting problem with a single modification: the utility of voter j for spending $\vec{x}$ is defined as $\sum_{i \in S_j} x_i^2 c_i$ (in particular, voter much prefers a finished project to a half-finished project at twice the cost).

Recall the setup of the Participatory Budgeting scheme, now with this modification: there is a budget $B > 0$ to be allocated among a set $[m]$ of projects, where each $i \in [m]$ has cost $c_i > 0$; an allocation is a vector $\vec{x} = (x_1, \dots, x_m) \in [0, 1]^m$ such that $\sum_i x_i c_i \leq B$. Each voter $j \in [n]$ has a set $S_j \subseteq [m]$ of projects they would like to fund, and their utility for allocation $\vec{x}$ is $\sum_{i \in S_j} x_i^2 c_i$.

Recall, also, the Participatory Budgeting Rule. Note that this is exactly the same mechanism seen in Lecture 6, only repeated in a precise manner.

- Each $i \in [n]$ reports a set $T_j \subseteq [m]$ of projects they are interested in.
- Sort projects in non-increasing order by $|\{j \in [n] : j \in T_i\}|$, the number of voters who select them (tie-break lexicographically). Call this order $(i_1, i_2, \dots, i_n)$.
- Fund projects starting from the front of the list until the budget runs out: for each $k \in [m]$, let $B_k = \sum_{\ell < k} x_{i_\ell} c_{i_\ell}$ be the budget allocated so far and allocate $x_{i_k} = \min\{(B - B_k)/c_{i_k}, 1\}$.

Prove that Participatory Budgeting under this utility function is not strategy-proof.

Problem 3: Switching Stable Matchings (50 points)

Recall the following definitions regarding stable matchings. Note that these are all the same standard definitions that you're familiar with from earlier in the course, but they are repeated here in an extra-precise manner so that the question being asked is clear.

Definition 1 (Stable Matching Preferences)**.** *A stable matching instance has n students and n universities, and defines a strict preference ordering $\succ_s$ for each student s and $\succ_u$ for each university u. We use $\vec{\succ}$ to refer to the list of all $2n$ preferences, and call $\vec{\succ}$ "the preferences."*

Definition 2 (Blocking pair)**.** *A pair (s, u) is a blocking pair for matching M under preferences $\vec{\succ}$ if $s \succ_u M(u)$ and $u \succ_s M(s)$. That is, (s, u) is a blocking pair for matching M under preferences $\vec{\succ}$ if s strictly prefers u to their partner in M, and u strictly prefers s to their partner in M as well.*

Definition 3 (Stable Matching)**.** *A matching M is stable for preferences $\vec{\succ}$ if there are no blocking pairs for M under $\vec{\succ}$. That is, for all pairs (s, u), (s, u) is not a blocking pair for M under $\vec{\succ}$.*

For any stable matching preferences $\vec{\succ}$, and any two matchings M, M', let $A(M, M')$ denote the set of students who *strictly* prefer their match in M to their match in M' (that is, $A(M, M')$ is the set of students for whom $M(s) \succ_s M'(s)$), and let $B(M, M')$ denote the set of universities who *strictly* prefer their match in M' to their match in M (that is, $B(M, M')$ is the set of universities for whom $M'(u) \succ_u M(u)$).

Part a (30 points)

Prove that for all preferences $\vec{\succ}$, and all M, M' that are both stable for $\vec{\succ}$, that in the matching M, every student in $A(M, M')$ is matched to a university in $B(M, M')$.

Part b (20 points)

Prove that for all preferences $\vec{\succ}$, and all M, M' that are both stable for $\vec{\succ}$, that in *both M and M'*, every student in $A(M, M')$ is matched to a university in $B(M, M')$ and every university in $B(M, M')$ is matched to a student in $A(M, M')$.