Precept Outline

- Review of Lectures 21 and 22:
  - Randomness
  - Multiplicative Weights
  - Decision Stumps and Boosting

A. Review: Randomness and Multiplicative Weights

Your preceptor will briefly review key points of this week's lectures.

B. Weak Learners and Boosting

In this problem, we will work through a small example of the *weak learner* you will be required to implement in the final programming assignment.

A **decision stump** is a very simple kind of binary classifier for points in k-dimensional space. Its decision depends on three values:

- the **dimension predictor** d_p, an integer between 0 and $k-1$;
- the **value predictor** v_p, an integer; and
- the **sign predictor** $s_p \in \{0, 1\}$.

With these three values, the decision stump outputs a prediction for the **label** (i.e., either 0 or 1) of a sample point $\mathbf{x} = (x_0, x_1, \ldots, x_{k-1})$ as follows:

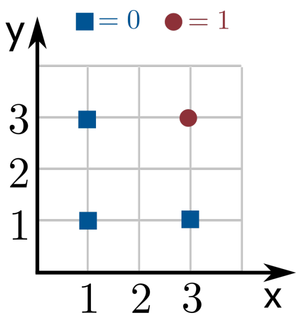
- if $s_p = 0$, output 0 if $x_{d_p} \le v_p$ (and output 1 if $x_{d_p} > v_p$);
- if $s_p = 1$, output 1 if $x_{d_p} \le v_p$ (and output 0 if $x_{d_p} > v_p$).

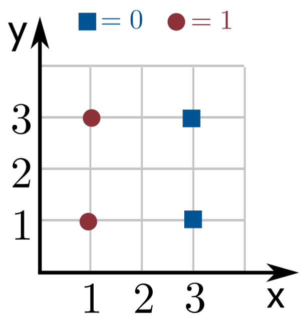
(In the following examples the dimension is $k = 2$, so we can plot the points.)

For example, if $d_p = 1$, $v_p = 0$ and $s_p = 1$, the predicted labels of $\mathbf{x} = (0, 0)$, $\mathbf{y} = (100, -2)$ and $\mathbf{z} = (-100, 1)$ are 1, 1 and 0, respectively.

In the dataset examples below, count the number of correctly classified points (i.e., points whose predicted label matches the actual label) for the two decision stumps with the following values:

1. $d_p = 1$, $v_p = 2$ and $s_p = 0$;
2. $d_p = 0$, $v_p = 1$ and $s_p = 1$.





(Blue squares denote points labeled 0 and red circles denote points labeled 1. Dimension 0 corresponds to coordinates in the x axis, while dimension 1 corresponds to the y axis.)

Additionally, determine which one is the best weak learner, i.e. the one that classifies the most points correctly.

Unfortunately, no decision stump can classify all points correctly in (the first dataset of) the previous problem. So we will try to get around this by combining multiple decision stumps.

Boosting is a technique that enables us to increase the accuracy of a weak learner (like a decision stump). To apply it, we first assign a weight to each one of the 4 points, which initially is $1/4$ (in general, $1/n$ weight for n points). Now we work in iterations, each of which creates a new decision stump based on the current weights and updates them at the end. After T iterations, we have T decision stumps. To classify a new point, we take the majority decision of each one of the T decision stumps (i.e., if more than half of decision stumps predict 0, then so does the boosted classifier; and likewise for 1).

Each boosting iteration does the following:

- creates a new decision stump for the dataset with the current weights;
- doubles the weights of misclassified points; and
- renormalizes the weights (i.e., divide each by the sum of all so that they sum to 1 again).

Each decision stump we create chooses d_p, v_p and s_p to maximize the *weight* (rather than number) of correctly classified points.

Run the boosting algorithm in the first dataset above for 3 iterations. Verify that the resulting decision stumps now correctly label all points (when taking the majority decision).

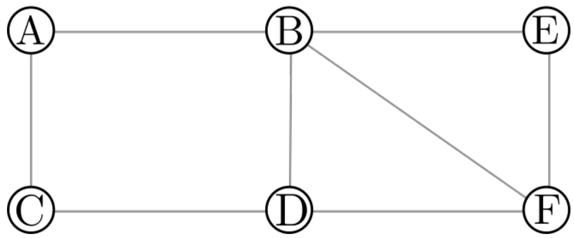
C. Global Mincut

Recall the global mincut problem: you are given a connected, unweighted, undirected graph G. A cut is a set of edges which, if removed, disconnects G. The goal is to find the cut that uses the fewest edges.

In lecture you learned one way of solving this problem: Karger's algorithm. It can be summarized in three steps:

- Assign a random weight (uniform between 0 and 1) to each edge.
- Run Kruskal's MST algorithm until 2 connected components are left (i.e., add all but the last MST edge).
- Output the cut defined by the 2 connected components.

Consider the following graph and set of random edge weights. Run Karger's algorithm with these edge weights and find the global cut it produces. Is it a mincut? If not, how many crossing edges does the mincut have?



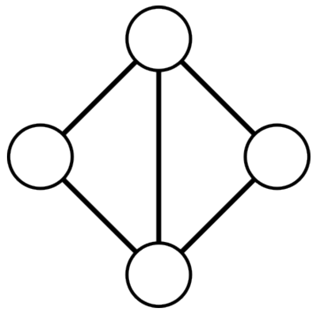
edge	random weight
A—B	0.2
A—C	0.1
B—D	0.7
B—E	0.6
B—F	0.5
C—D	0.8
D—F	0.4
E—F	0.3

D. Optional bonus problems

Part 1: Random spanning trees

In this problem, we'll see how to generate a uniformly random spanning tree in a graph. We'll consider a few natural ideas and see how they behave on a small graph. (Maze generation is one cool application of random spanning trees, but they even show up in quantum field theory!)

First, let's consider a small example where we'll test our candidate algorithms:



Draw *all* spanning trees of the graph above. What are the probabilities of these trees under the uniform distribution?

Consider the following algorithm for sampling spanning trees from a graph: start with a tree containing a single (arbitrary) vertex. At each iteration,

1. choose an edge uniformly at random from the edges with exactly one endpoint in the tree;
2. add the edge to the tree; and
3. output the tree once there are no more edges to add.

Show that this algorithm does *not* generate uniformly random spanning trees. *Hint: show that there is a spanning tree that can only be obtained if its edges are sampled in one particular order.*

Consider the following alternative:

1. apply a uniformly random permutation to the edges;
2. run Kruskal's algorithm (without sorting – add edges in the order given by the permutation); and
3. output the resulting spanning tree.

Show that this algorithm does *not* generate uniformly random spanning trees. *Hint: pick one spanning tree and count how many permutations make the algorithm not select the missing edges.*

Note. Aldous-Broder and Wilson's algorithms are two examples of algorithms that sample uniform spanning trees; but they use random walks, where the algorithm may get stuck revisiting nodes in a partial tree over and over again. (The probability they do do for t steps decreases exponentially with t: they're Las Vegas algorithms, guaranteed to output a uniformly random spanning tree but whose runtime is random.) This makes the analysis more difficult, but the algorithms have been proven to work.

Bottom line: sampling spanning trees is (also) hard!