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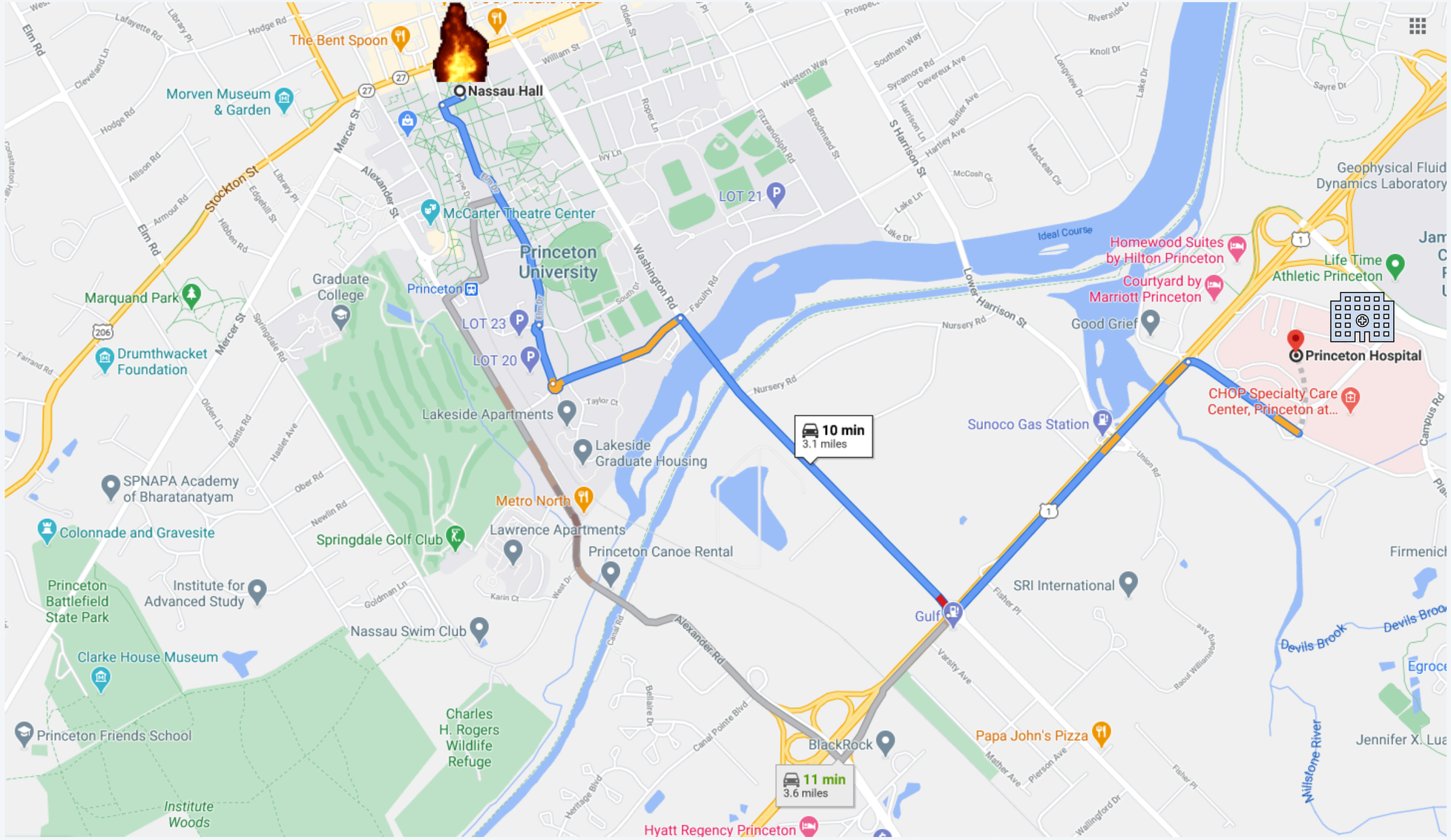
4.4 Shortest Paths

- *properties*
- *APIs*
- *Bellman–Ford algorithm*
- *Dijkstra's algorithm*

A motivating example: route planning in a road network



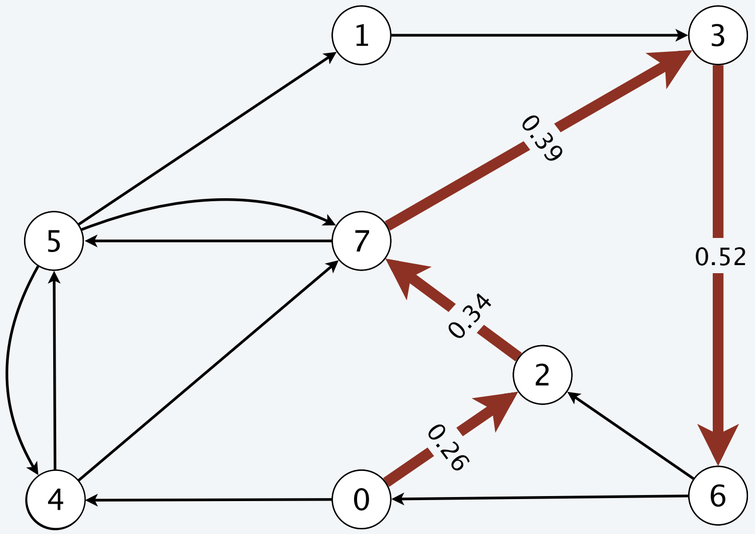
Shortest path in an edge-weighted digraph

Given an edge-weighted digraph, find a shortest path from one vertex s to another vertex t.

length of path = sum of edge weights

edge-weighted digraph

```
4->5  0.35
5->4  0.35
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```



shortest path from 0 to 6

0 → 2 → 7 → 3 → 6

length of path = 1.51

(0.26 + 0.34 + 0.39 + 0.52)

Shortest path variants

Which vertices?

- Source–destination: from one vertex s to another vertex t.
- Single source: from one vertex s to every other vertex.
- Single destination: from every vertex to one vertex t.
- All pairs: from every vertex to every other vertex.

Restrictions on edge weights?

- Non-negative weights. ← *we assume this in today's lecture (except as noted)*
- Euclidean weights.
- Arbitrary weights.

Directed cycles?

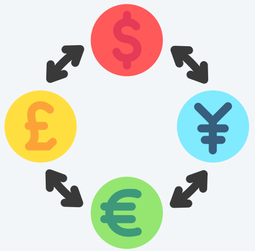
- Prohibit. ← *can derive faster algorithms in DAGs (see next lecture)*
- Allow.

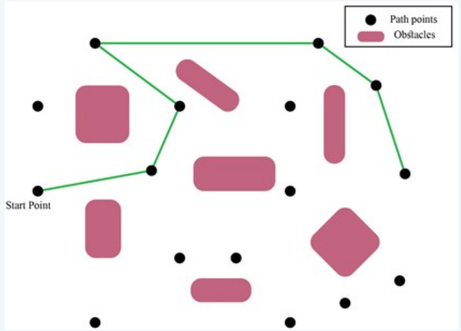
Simplifying assumption.

Each vertex is reachable from s. ← *ensures that shortest path from s to v exists (and that $E \geq V - 1$)*

Shortest path applications

- Seam carving. ⟵ *see Assignment 6*
- Texture mapping.
- Typesetting in TEX.
- Currency exchange.
- Urban traffic planning.
- Robot motion planinng.
- Social network analysis.
- Telecommunication routing.
- Project scheduling (PERT/CPM).
- Optimal pipelining of VLSI chip.
- Packet routing (OSPF, BGP, RIP, IS-IS).
- Route planning (Google maps, car navigation, ...).





Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

Shortest paths: poll 1

Which shortest path variant for route planning by a car navigation system?

Hint: drivers make wrong turns occasionally.

A. Source–destination: from one vertex s to another vertex t.

B. Single source: from one vertex s to every vertex.

C. Single destination: from every vertex to one vertex t.

D. All pairs: from every vertex to every other vertex.

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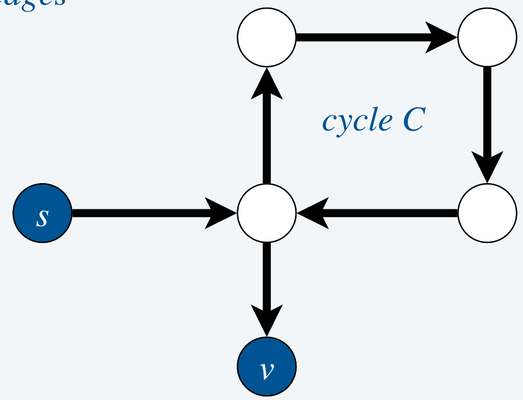
Shortest path properties

Simple paths. There exists a shortest $s \rightsquigarrow v$ path that has no repeated vertices. $\Longrightarrow$ $\leq V - 1$ *edges*

Pf. [by contradiction]

- Let P^* be a shortest $s \rightsquigarrow v$ path with the fewest edges, and assume it has a repeated vertex.
- Remove the cycle C from P^*, yielding a shortest path with fewer edges.

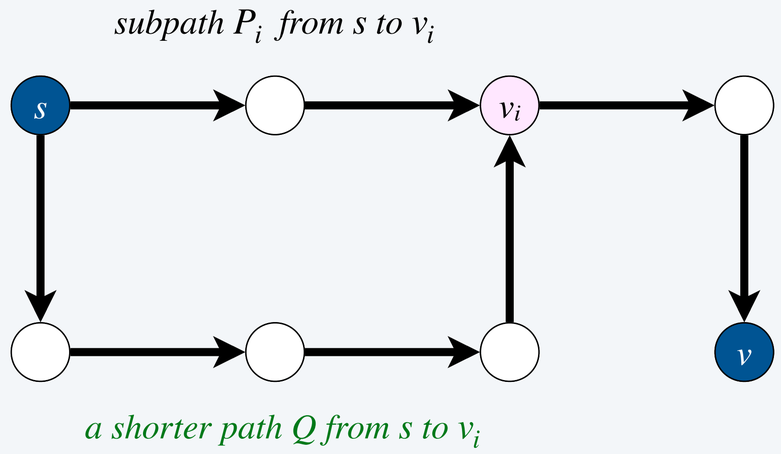


Optimal substructure. Let P^* be a shortest $s \rightsquigarrow v$ path.

Then, any subpath P_i of P^* from s to v_i is a shortest $s \rightsquigarrow v_i$ path.

Pf. [by contradiction]

- Suppose subpath P_i were not a shortest $s \rightsquigarrow v_i$ path.
- Let Q be a shortest $s \rightsquigarrow v_i$ path.
- Replace subpath P_i with Q in P^*, yielding a shorter $s \rightsquigarrow v$ path.



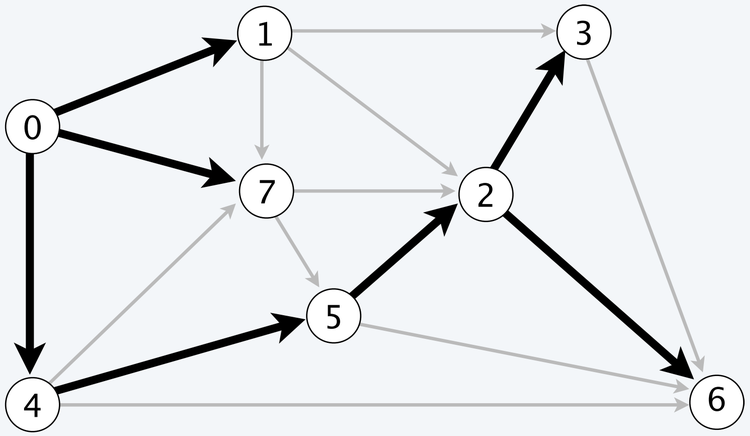
Data structures for single-source shortest paths

Goal. Find a shortest path from s to every other vertex.

Observation 2. A shortest-paths tree solution exists. Why?

Consequence. Can represent shortest paths with two parallel arrays:

- `distTo[v]`: length of a shortest $s \rightsquigarrow v$ path.
- `edgeTo[v]`: last edge on a shortest $s \rightsquigarrow v$ path.



shortest-paths tree from 0

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

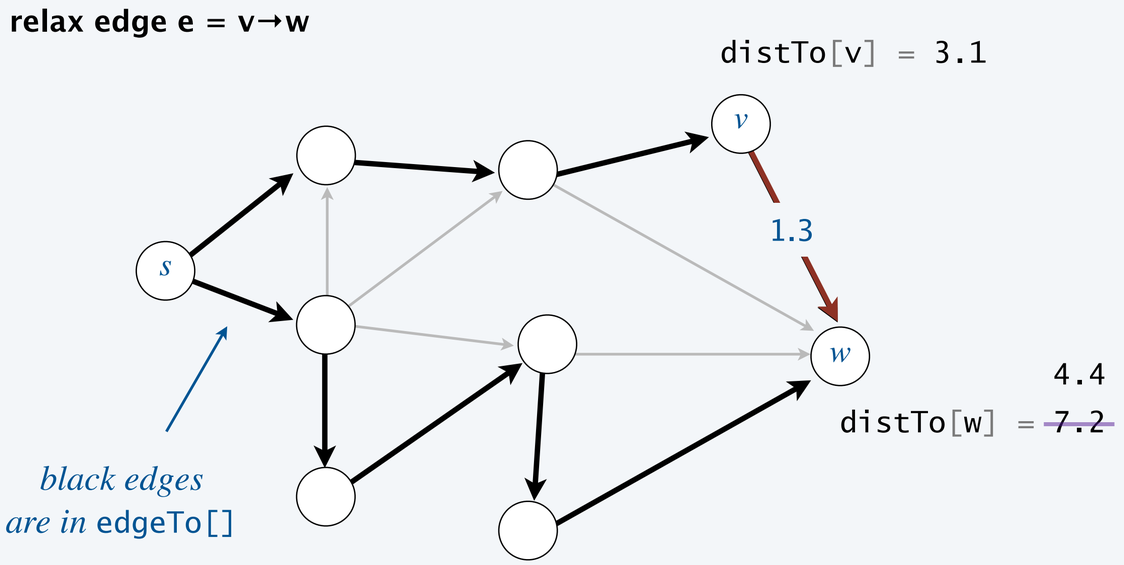
parent-link representation

Edge relaxation

Relax edge $e = v \rightarrow w$.

- `distTo[v]`: length of shortest known $s \rightsquigarrow v$ path.
- `distTo[w]`: length of shortest known $s \rightsquigarrow w$ path.
- `edgeTo[w]`: last edge on shortest known $s \rightsquigarrow w$ path.
- If edge e yields a shorter $s \rightsquigarrow w$ path via v, then update:
  - `distTo[w] = distTo[v] + e.weight()`
  - `edgeTo[w] = e`

context: shortest-path algorithms maintain champion distances and paths (updated through edge relaxations)



Shortest paths: poll 2

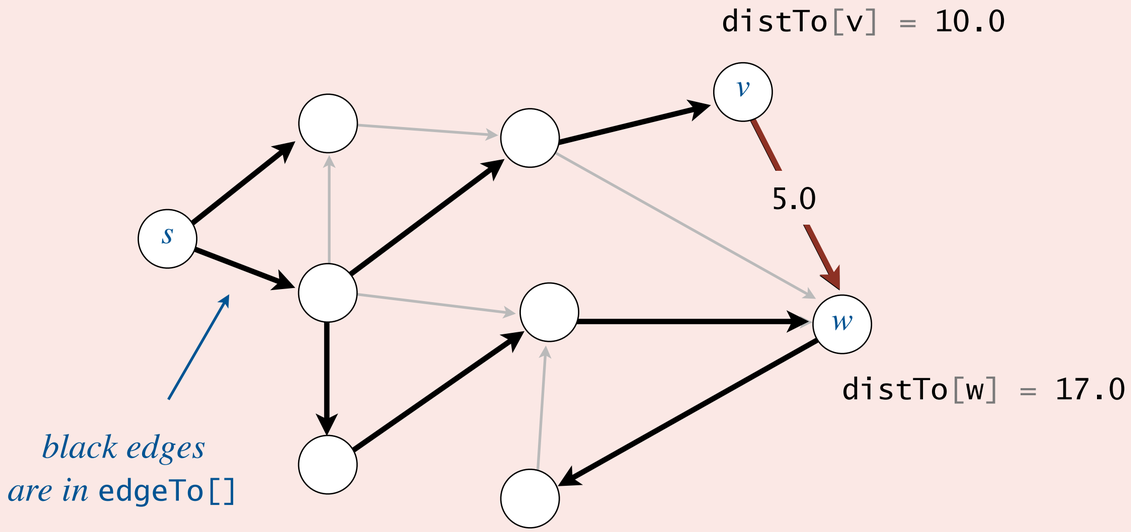
What are the values of `distTo[v]` and `distTo[w]` immediately after relaxing edge $e = v \rightarrow w$?

A. `10.0` and `15.0`

B. `10.0` and `17.0`

C. `12.0` and `15.0`

D. `12.0` and `17.0`



Framework for shortest-paths algorithm

Generic algorithm (to compute a SPT from s)

For each vertex v: distTo[v] = ∞.

For each vertex v: edgeTo[v] = null.

distTo[s] = 0.

Repeat until distTo[v] values converge:

- **Relax any edge.**

Key properties. Throughout the generic algorithm:

- `distTo[v]` is either ∞ or the length of a (simple) $s \rightsquigarrow v$ path.
- `distTo[v]` never increases; it decreases when a shorter $s \rightsquigarrow v$ path is found.
- `edgeTo[]` defines a shortest-paths tree rooted at s.

Framework for shortest-paths algorithm

Generic algorithm (to compute a SPT from s)

For each vertex v: distTo[v] = ∞.

For each vertex v: edgeTo[v] = null.

distTo[s] = 0.

Repeat until distTo[v] values converge:

- **Relax any edge.**

Efficient implementations.

- Which edge to relax next?
- How many edge relaxations needed to guarantee convergence?

Ex 1. Bellman–Ford algorithm.

Ex 2. Dijkstra's algorithm.

Ex 3. Topological sort algorithm. ⟵ *next lecture*

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4.4 Shortest Paths

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Weighted directed edge API

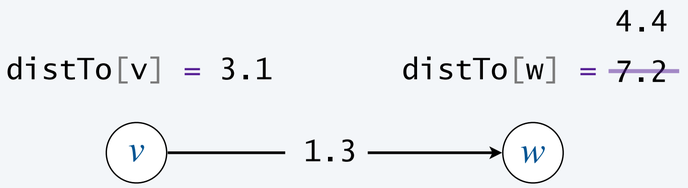
API.

`public class DirectedEdge`

	`DirectedEdge(int v, int w, double weight)`	*create weighted edge v→w*
`int`	`from()`	*tail vertex v*
`int`	`to()`	*head vertex w*
`double`	`weight()`	*weight of edge v→w*
	⋮	⋮

Ex. Relax edge $e = v \rightarrow w$.

```
private void relax(DirectedEdge e) {
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight()) {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
    }
}
```



Weighted directed edge: Java implementation

```
public class DirectedEdge {
   private final int v, w;
   private final double weight;

   public DirectedEdge(int v, int w, double weight) {
     this.v = v;
     this.w = w;
     this.weight = weight;
   }

   public int from() {
      return v;
   }

   public int to() {
      return w;
   }

   public double weight() {
      return weight;
   }
}
```

from() *and* to() *replace* either() *and* other()

Edge-weighted digraph API

API. Same as `EdgeWeightedGraph` except with `DirectedEdge` objects.

`public class EdgeWeightedDigraph`		
	`EdgeWeightedDigraph(int V)`	*edge-weighted digraph with V vertices (and no edges)*
`void`	`addEdge(DirectedEdge e)`	*add edge e*
`Iterable<DirectedEdge>`	`adj(int v)`	*edges incident from v*
`int`	`V()`	*number of vertices*
`int`	`E()`	*number of edges*
	⋮	⋮

Edge-weighted digraph: adjacency-lists implementation in Java

Implementation. Almost identical to `EdgeWeightedGraph`.

```
public class EdgeWeightedDigraph {
   private final int V;
   private final Queue<DirectedEdge>[] adj;

   public EdgeWeightedDigraph(int V) {
     this.V = V;
     adj = (Queue<Edge>[]) new Queue[V];
     for (int v = 0; v < V; v++)
        adj[v] = new Queue<>();
   }

   public void addEdge(DirectedEdge e) {
     int v = e.from();
     adj[v].enqueue(e);
   }

   public Iterable<DirectedEdge> adj(int v) {
      return adj[v];
   }
}
```

add edge e = v→w to only v's adjacency list

Single-source shortest paths API

Goal. Find the shortest path from s to every other vertex.

`public class SP`		
	`SP(EdgeWeightedDigraph digraph, int s)`	*shortest paths from s in digraph*
`double`	`distTo(int v)`	*length of shortest path from s to v*
`boolean`	`hasPathTo(int v)`	*is there a path from s to v ?*
`Iterable <DirectedEdge>`	`pathTo(int v)`	*shortest path from s to v*

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4.4 Shortest Paths

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Bellman–Ford algorithm

Bellman–Ford algorithm

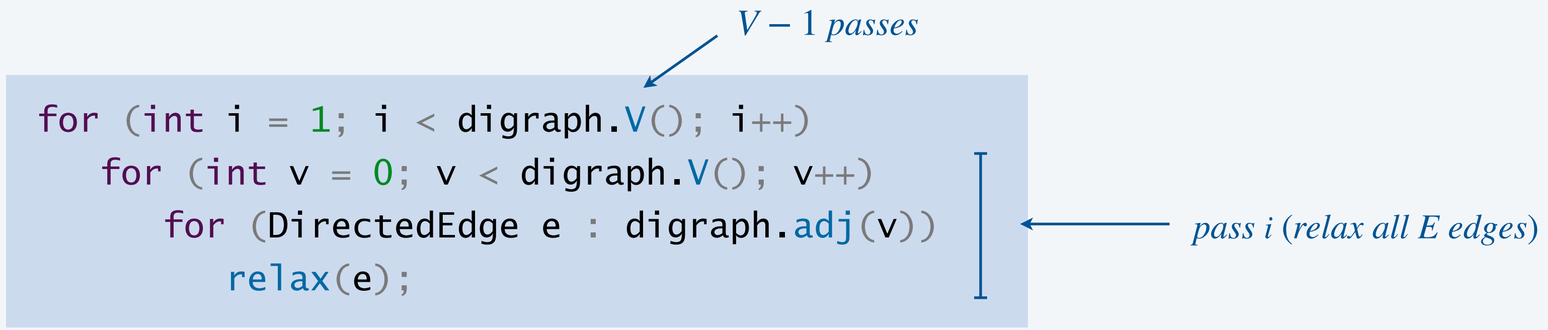
For each vertex v: distTo[v] = ∞.

For each vertex v: edgeTo[v] = null.

distTo[s] = 0.

Repeat V–1 times:

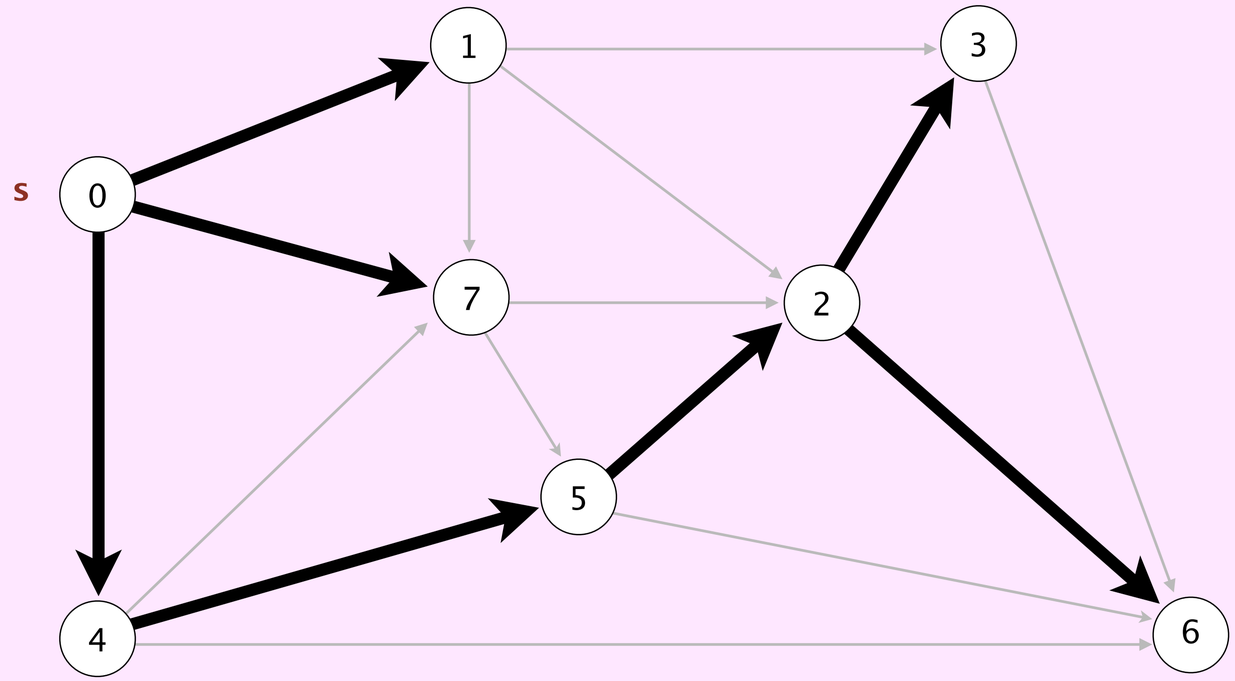
– Relax all E edges.



Performance. Algorithm takes $\Theta(E\,V)$ time and uses $\Theta(V)$ extra space.

Bellman–Ford algorithm demo

Repeat $V - 1$ times: relax all E edges.

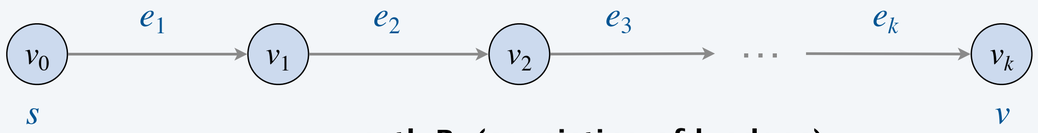


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Bellman–Ford algorithm: proof of correctness

Notation. Let $P^* = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_k$ be a shortest path of k edges from $s = v_0$ to $v = v_k$, and let $P_i = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$ be the subpath consisting of the first i edges in P^*.



Proposition. After pass i of Bellman–Ford, $\texttt{distTo}[v_i] = length(P_i)$. ← *after pass i, have already found a shortest $s \rightsquigarrow v_i$ path*

Pf. [by induction on number of passes i]

- Base case: initially, $\texttt{distTo}[v_0] = 0 = length(P_0)$.
- Inductive hypothesis: after pass i, $\texttt{distTo}[v_i] = length(P_i)$.
- Immediately after relaxing edge e_{i+1} in pass $i+1$, we have

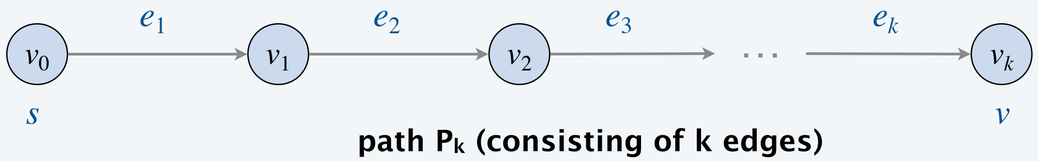
$$\begin{aligned} \texttt{distTo}[v_{i+1}] &\leq \texttt{distTo}[v_i] + weight(e_{i+1}) && \leftarrow \textit{edge relaxation} \\ &= length(P_i) + weight(e_{i+1}) && \leftarrow \textit{inductive hypothesis} \\ &= length(P_{i+1}) \end{aligned}$$

and remains that way for the rest of algorithm ↓

- Since $\texttt{distTo}[v_{i+1}]$ is the length of some $s \rightsquigarrow v_{i+1}$ path, we must have $\texttt{distTo}[v_{i+1}] = length(P_{i+1})$ ▪

Bellman–Ford algorithm: proof of correctness

Notation. Let $P^* = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_k$ be a shortest path of k edges from $s = v_0$ to $v = v_k$, and let $P_i = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_i$ be the subpath consisting of the first i edges in P^*.



Proposition. After pass i of Bellman–Ford, `distTo[`v_i`]` $= length(P_i)$. ⟵ *after pass i, have already found a shortest $s \rightsquigarrow v_i$ path*

Corollary. For each vertex v, Bellman–Ford computes the length of a shortest $s \rightsquigarrow v$ path.

Pf.

- There exists a shortest $s \rightsquigarrow v$ path P^* that is simple. So, P^* contains $k \leq V - 1$ edges.
- From the Proposition, after k passes, `distTo[`v_k`]` $= length(P_k)$. ▪

$v = v_k$ $\qquad$ $P^* = P_k$

Bellman–Ford algorithm: practical improvement

Observation. If `distTo[v]` does not change during pass i,
not necessary to relax any edges incident from v in pass $i + 1$.

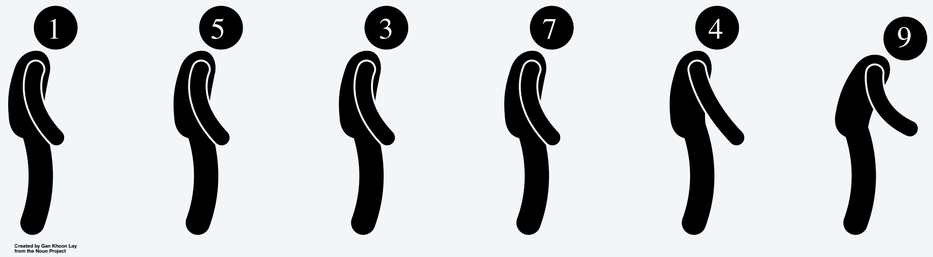
```
private void relax(int v) {
   for (DirectedEdge e : digraph.adj(v))
      relax(e);
}
```

vertex relaxation

Queue-based implementation of Bellman–Ford.

- Perform vertex relaxations. ⟵ *relax vertex v = relax all edges incident from v*
- Maintain queue of vertices whose `distTo[]` values changed since it was last relaxed.

must ensure each vertex is on queue at most once
(or exponential blowup!)



relax in pass $i + 1$

relax in pass i

relax vertex v

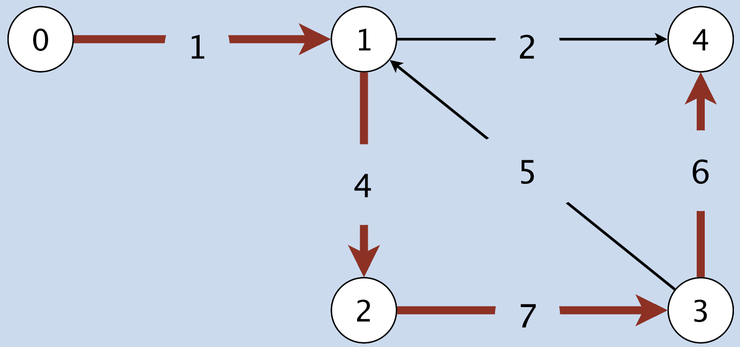
Impact.

- In the worst case, the running time is still $\Theta(E V)$.
- But much faster in practice on typical inputs.

Longest path

Problem. Given a digraph G with positive edge weights and source vertex s, find a longest simple path from s to every other vertex.

Goal. Design an algorithm that takes $O(EV)$ time in the worst case.



longest simple path from 0 to 4

0 → 1 → 2 → 3 → 4

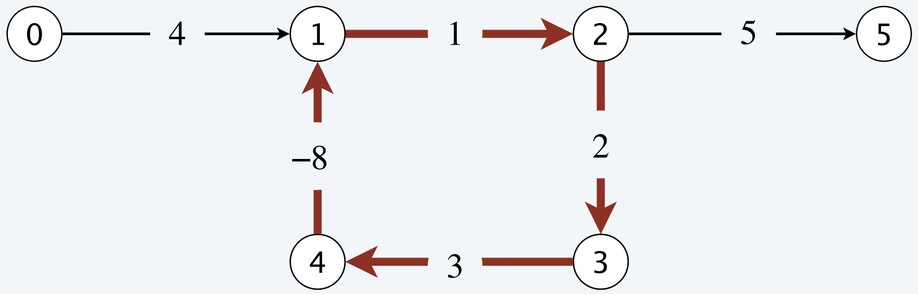
length of path = 18

(1 + 4 + 7 + 6)

Bellman–Ford algorithm: negative weights

Remark. The Bellman–Ford algorithm works even if some edge weights are negative, provided there are no negative cycles.

Def. A negative cycle is a directed cycle whose total length is negative.



negative cycle
(length = 1 + 2 +3 + −8 = −2 < 0)

Negative cycles and shortest paths. Length of path can be made arbitrarily negative by using negative cycle.

$$0 \to 1 \to \underline{2 \to 3 \to 4 \to 1} \to \quad \cdots \quad \to \underline{2 \to 3 \to 4 \to 1} \to 2 \to 5$$

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Edsger Dijkstra: select quote

Dijkstra's algorithm

Dijkstra's algorithm

For each vertex v: distTo[v] = ∞.

For each vertex v: edgeTo[v] = null.

T = ∅.

distTo[s] = 0.

Repeat until all vertices are marked:

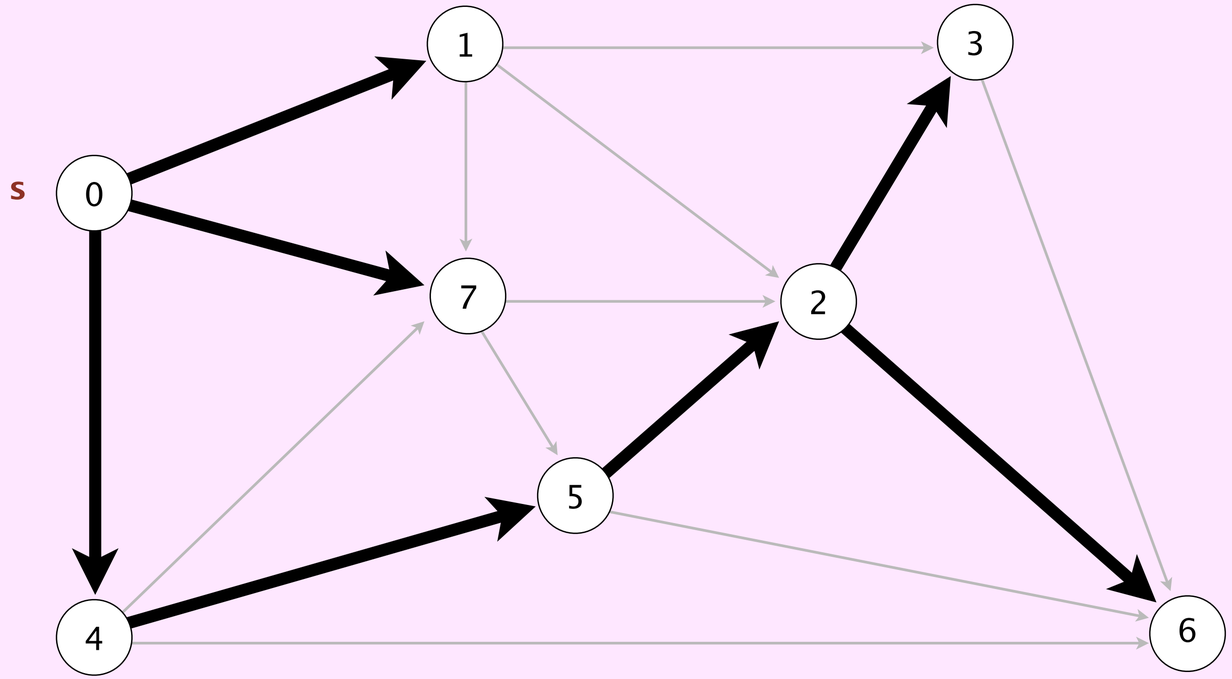
- **Select unmarked vertex v with the smallest distTo[] value.**
- **Mark v.**
- **Relax each edge incident from v.**

Key difference with Bellman–Ford. Each edge gets relaxed exactly once!

Dijkstra's algorithm demo

Repeat until all vertices are marked:

- Select unmarked vertex v with the smallest `distTo[]` value.
- Mark v and relax all edges incident from v.



v	distTo[]	edgeTo[]
0	0.0	-
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2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

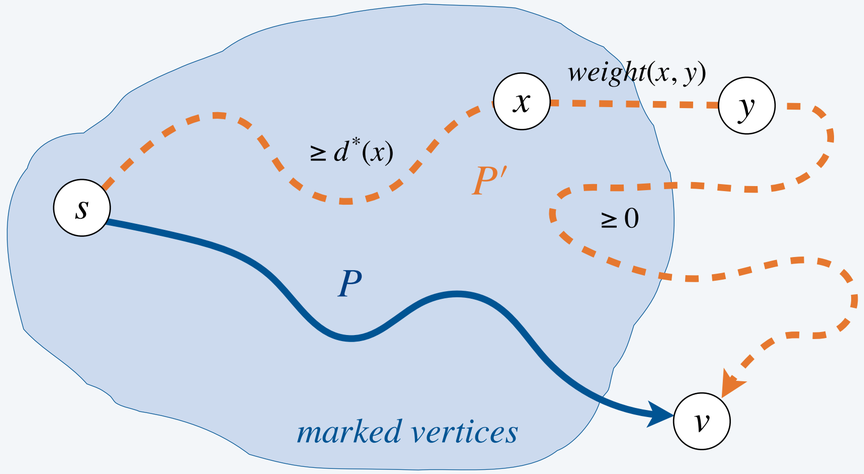
Dijkstra's algorithm: proof of correctness

Invariant. For each marked vertex v : `distTo[v]` $= d^*(v)$.

notation for length of shortest $s \rightsquigarrow v$ path

Pf. [by induction on number of marked vertices]

- Let v be next vertex marked.
- Let P be the $s \rightsquigarrow v$ path of length `distTo[v]`.
- Consider any other $s \rightsquigarrow v$ path P'.
- Let $x \rightarrow y$ be first edge in P' with x marked and y unmarked.
- P' is already as long as P by the time it reaches y :



by definition

$$length(P) = \texttt{distTo[v]}$$

Dijkstra chose v instead of y $\longrightarrow$ $\leq$ `distTo[y]`

vertex x is marked (so it was relaxed) $\longrightarrow$ $\leq$ `distTo[x]` $+ weight(x, y)$

induction $\longrightarrow$ $= d^*(x) + weight(x, y)$

P′ is a path from s to x, followed by edge x→y, followed by non-negative edges $\longrightarrow$ $\leq length(P')$ ▪

Dijkstra's algorithm: proof of correctness

Invariant. For each marked vertex v : `distTo[v]` $= d^*(v)$.

notation for length of shortest $s \rightsquigarrow v$ path

Corollary 1. Dijkstra's algorithm computes shortest path distances.

Corollary 2. Dijkstra's algorithm relaxes vertices in increasing order of distance from s.

generalizes both
level-order traversal in a tree
and breadth-first search in a graph

Dijkstra's algorithm: Java implementation

```
public class DijkstraSP {
   private DirectedEdge[] edgeTo;
   private double[] distTo;

   private IndexMinPQ<Double> pq;

   public DijkstraSP(EdgeWeightedDigraph digraph, int s) {
      edgeTo = new DirectedEdge[digraph.V()];
      distTo = new double[digraph.V()];

      pq = new IndexMinPQ<Double>(digraph.V());

      for (int v = 0; v < digraph.V(); v++)
         distTo[v] = Double.POSITIVE_INFINITY;
      distTo[s] = 0.0;

      pq.insert(s, 0.0);
      while (!pq.isEmpty()) {
         int v = pq.delMin();
         for (DirectedEdge e : digraph.adj(v))
            relax(e);
      }
   }
}
```

PQ that supports decreasing the key (stay tuned)

PQ contains the unmarked vertices with finite `distTo[]` *values*

relax vertices in increasing order of distance from s

Dijkstra's algorithm: Java implementation

When relaxing an edge $e = v \rightarrow w$, also update PQ:

- Found first $s \rightsquigarrow w$ path : add vertex w to PQ.
- Found better $s \rightsquigarrow w$ path: decrease priority associated with vertex w in PQ.

```
private void relax(DirectedEdge e) {
   int v = e.from(), w = e.to();
   if (distTo[w] > distTo[v] + e.weight()) {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
       if (!pq.contains(w)) pq.insert(w, distTo[w]);
       else                 pq.decreaseKey(w, distTo[w]);
   }
}
```

update PQ

index *priority*

Q. How to efficiently implement DECREASE-KEY operation in a priority queue?

Indexed priority queue (Section 2.4)

Associate an index between 0 and $n - 1$ with each key in a priority queue.

- Insert a key associated with a given index.
- Delete a minimum key and return associated index.
- Decrease the key associated with a given index.

for Dijkstra's algorithm:
$n = V$,
index = vertex,
key = distance from s

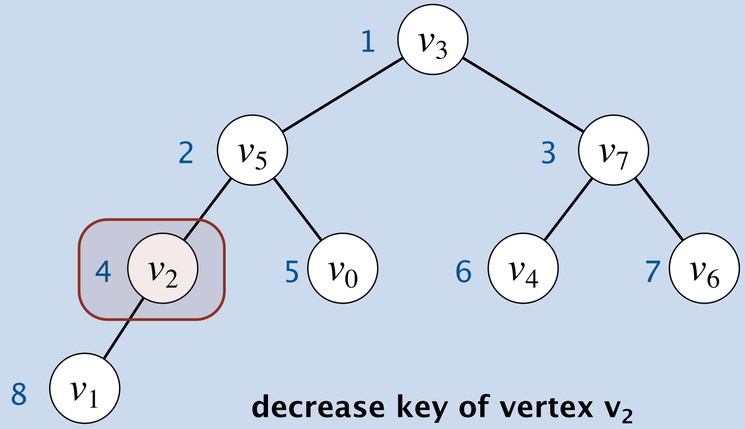
`public class IndexMinPQ<Key extends Comparable<Key>>`

	`IndexMinPQ(int n)`	*create PQ with indices $0, 1, \ldots, n - 1$*
`void`	`insert(int i, Key key)`	*associate key with index i*
`int`	`delMin()`	*remove min key and return associated index*
`void`	`decreaseKey(int i, Key key)`	*decrease the key associated with index i*
`boolean`	`contains(int i)`	*does the priority queue contain index i ?*
`boolean`	`isEmpty()`	*is the priority queue empty ?*
	⋮	⋮

Decreasing the key in a binary heap

Goal. Implement DECREASE-KEY operation in a min-oriented binary heap.

	0	1	2	3	4	5	6	7	8
pq[]	–	v_3	v_5	v_7	v_2	v_0	v_4	v_6	v_1



decrease key of vertex v_2

Decreasing the key in a binary heap

Goal. Implement DECREASE-KEY operation in a min-oriented binary heap.

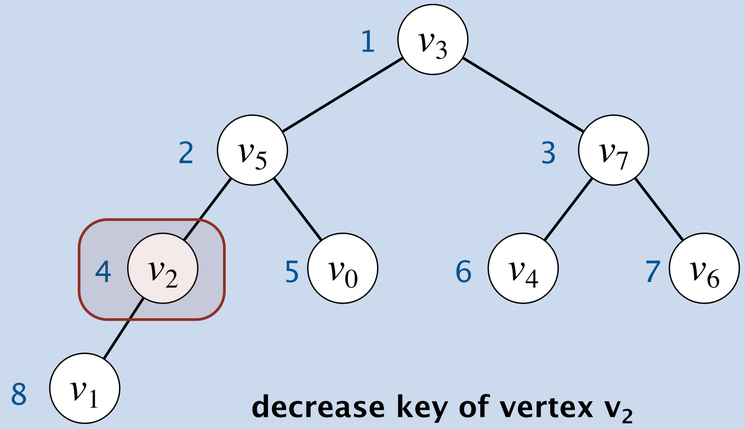
Solution.

- Find vertex in binary heap. How?
- Change priority of vertex and call `swim()` to restore heap invariant.

Extra data structure. Maintain an inverse array `qp[]` that maps from the vertex to the binary heap node index.

	0	1	2	3	4	5	6	7	8
pq[]	–	v_3	v_5	v_7	v_2	v_0	v_4	v_6	v_1
qp[]	5	8	4	1	6	2	4	3	–
keys[]	1.0	2.0	3.0	0.0	6.0	8.0	4.0	2.0	–

vertex 2 has priority 3.0 and is at heap index 4



decrease key of vertex v_2

Dijkstra's algorithm: which priority queue?

Number of PQ operations: V INSERT, V DELETE-MIN, $\leq E$ DECREASE-KEY.

PQ implementation	INSERT	DELETE-MIN	DECREASE-KEY	total
unordered array	1	V	1	V^2
binary heap	$\log V$	$\log V$	$\log V$	$E \log V$
d-way heap	$\log_d V$	$d \log_d V$	$\log_d V$	$E \log_{E/V} V$
Fibonacci heap	1 †	$\log V$ †	1 †	$E + V \log V$

† *amortized*

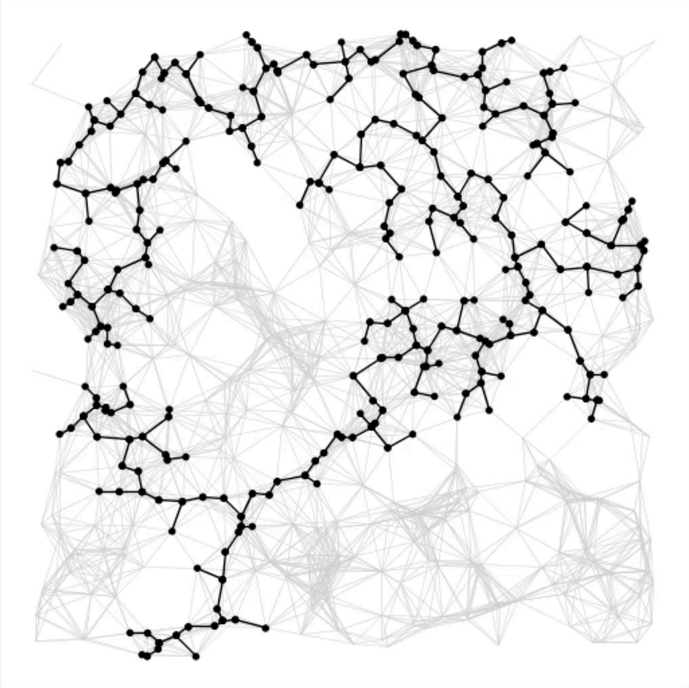
Bottom line.

- Array implementation optimal for complete digraphs.
- Binary heap much faster for sparse digraphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but probably not worth implementing.

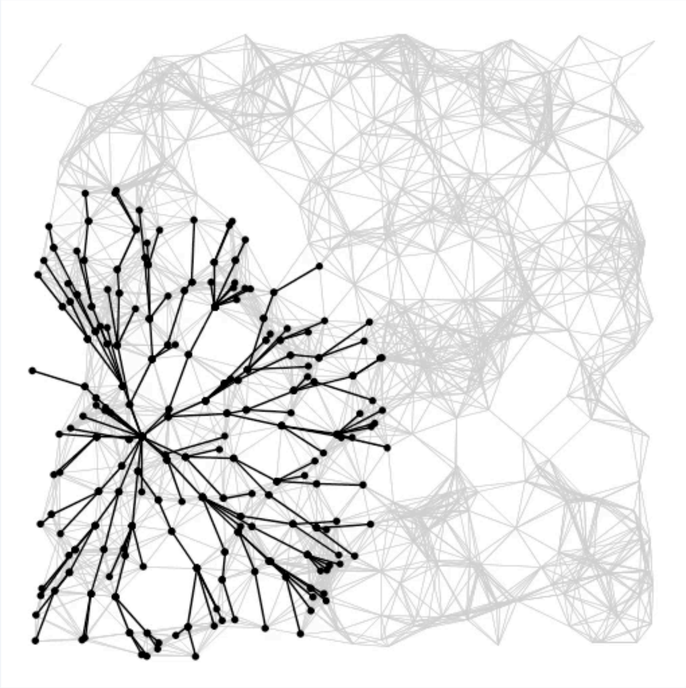
Priority-first search

Observation. Prim and Dijkstra are essentially the same algorithm.

- Prim: Choose next vertex that is closest to any vertex in the tree (via an undirected edge).
- Dijkstra: Choose next vertex that is closest to the source vertex (via a directed path).



Prim's algorithm



Dijkstra's algorithm

Algorithms for shortest paths

Variations on a theme: vertex relaxations.

- Bellman–Ford: relax all vertices; repeat $V - 1$ times.
- Dijkstra: relax vertices in order of distance from s.
- Topological sort: relax vertices in topological order. ⟵ *see Section 4.4 and next lecture*

algorithm	worst-case running time	negative weights †	directed cycles
Bellman–Ford	$\Theta(E\ V)$	✔	✔
Dijkstra	$\Theta(E \log V)$		✔
topological sort	E	✔	

† *no negative cycles*

Which shortest paths algorithm to use?

Select algorithm based on properties of edge-weighted digraph.

- Non-negative weights: Dijkstra.
- Negative weights (but no "negative cycles"): Bellman-Ford.
- DAG: topological sort.

algorithm	worst-case running time	negative weights †	directed cycles
Bellman-Ford	$\Theta(E\ V)$	✔	✔
Dijkstra	$\Theta(E \log V)$		✔
topological sort	E	✔	

† *no negative cycles*

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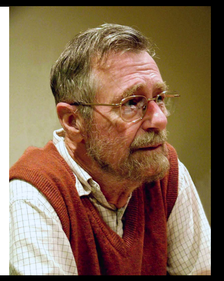
A final thought

> *Do only what only you can do.*
>
> — Edsger W. Dijkstra



A final thought