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3.4 Hash Tables

- *hash functions*
- *separate chaining*
- *linear probing*
- *context*

Symbol table implementations: summary

implementation	worst case			typical case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
red-black BST	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`
hashing	n	n	n	1 †	1 †	1 †		`equals()` `hashCode()`

† *subject to certain technical assumptions*

Q. Can we do better?

A. Yes, but only with different access to the symbol table keys.

Hashing: basic plan

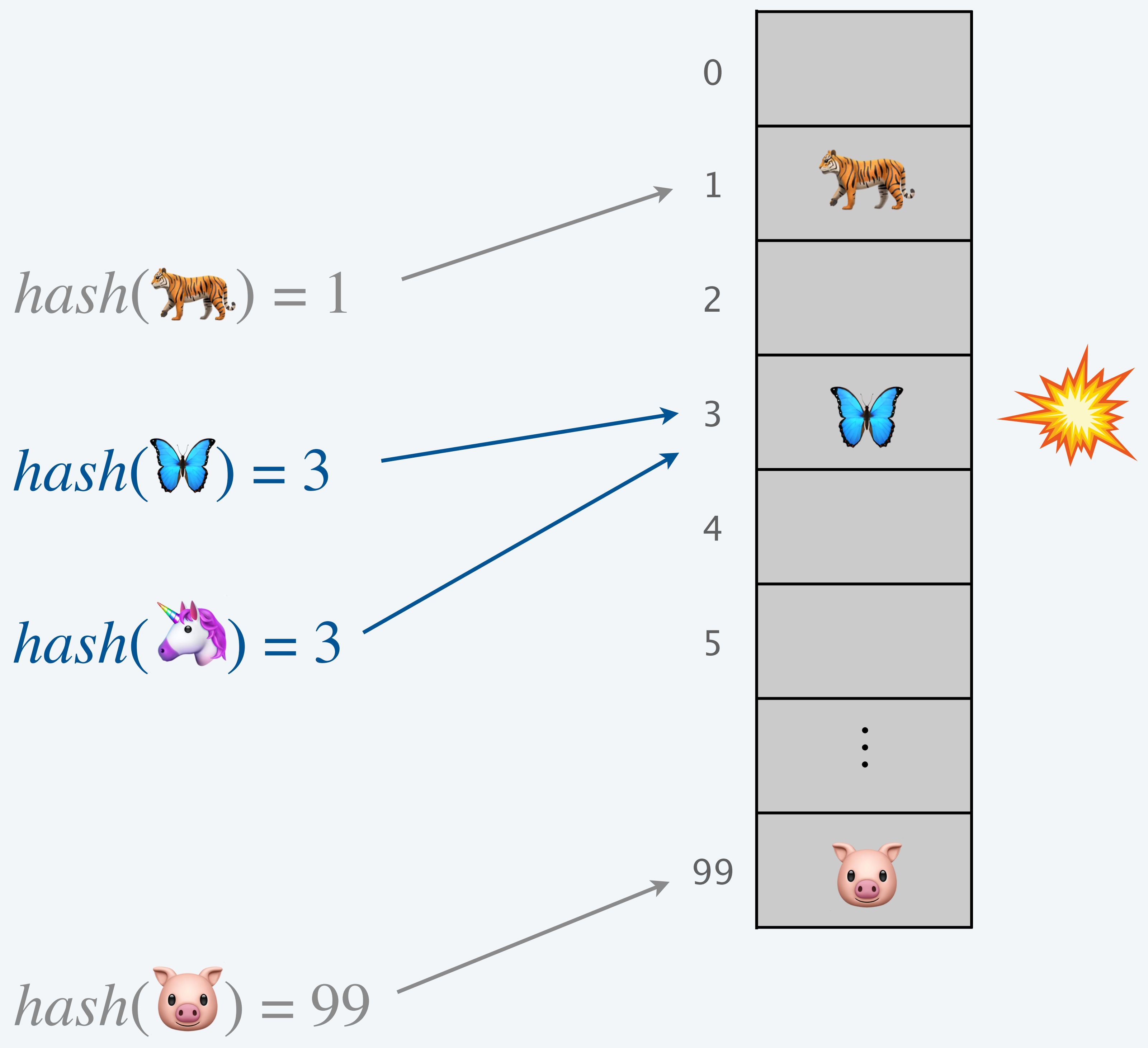
Save key-value pairs in an array, using a hash function to determine index of each key.

Hash function: Mathematical function that maps (hashes) a key to an array (table) index.

Collision: Two distinct keys that hash to the same index.

Issue. Collisions are typically unavoidable.

- How to limit collisions?
  [good hash functions]
- How to accommodate collisions?
  [novel algorithms and data structures]



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3.4 Hash Tables

- *hash functions*
- *separate chaining*
- *linear probing*
- *context*

Designing a hash function

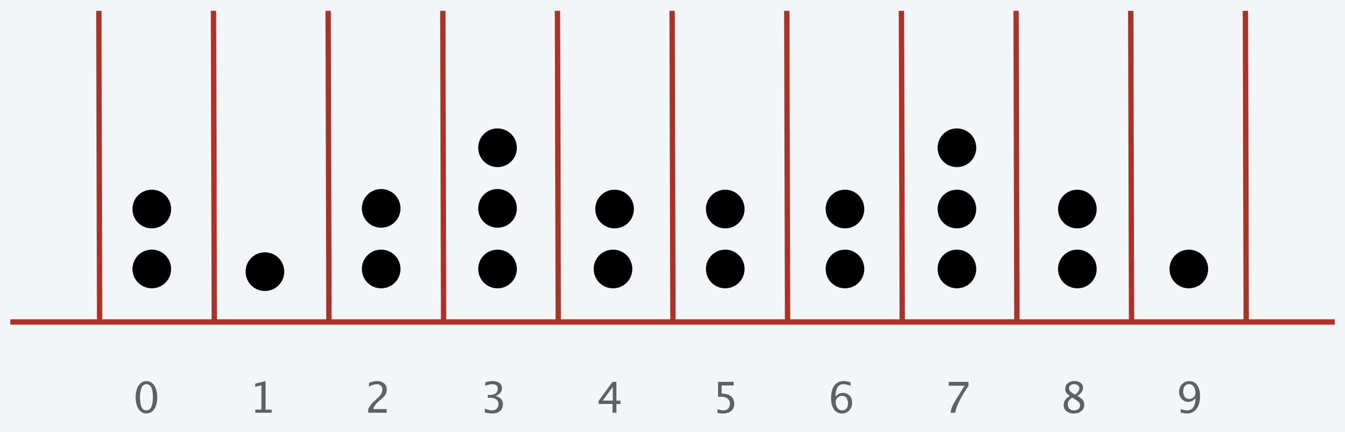
Required properties. [for correctness]

m = table size

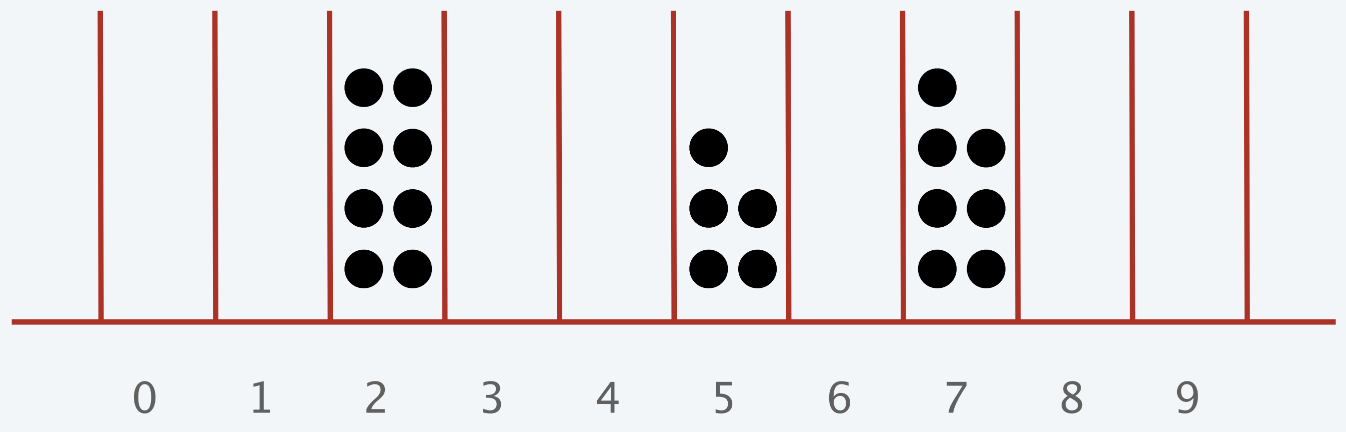
- Valid indices: each key hashes to a table index between 0 and $m - 1$.
- Deterministic: hashing the same key twice yields the same index.

Desirable properties. [for performance]

- Very fast to compute.
- Distributes the keys uniformly: for any subset of n keys to be hashed, each table index gets approximately n / m keys.



leads to good hash-table performance
(m = 10, n = 20)



leads to poor hash-table performance
(m = 10, n = 20)

Designing a hash function

Required properties. [for correctness]

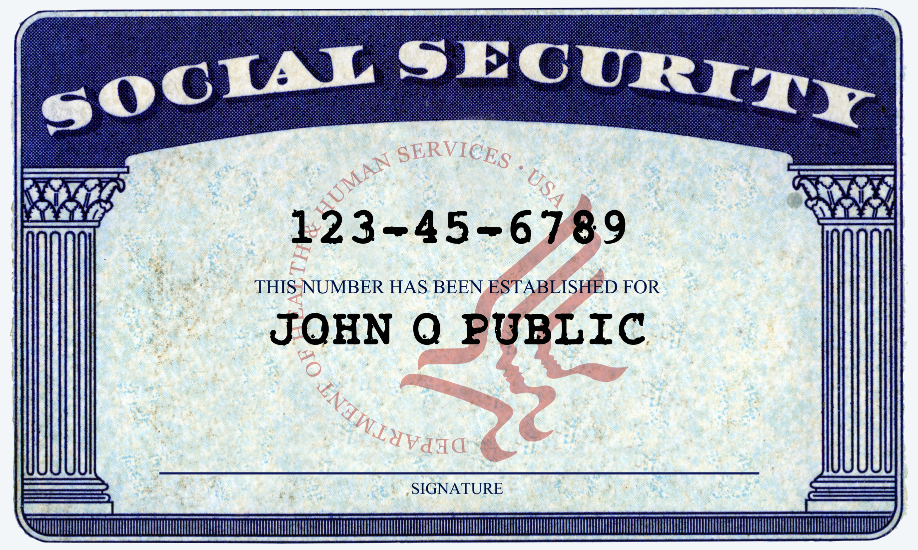
- Valid indices: each key hashes to a table index between 0 and $m - 1$.
- Deterministic: hashing the same key twice yields the same index.

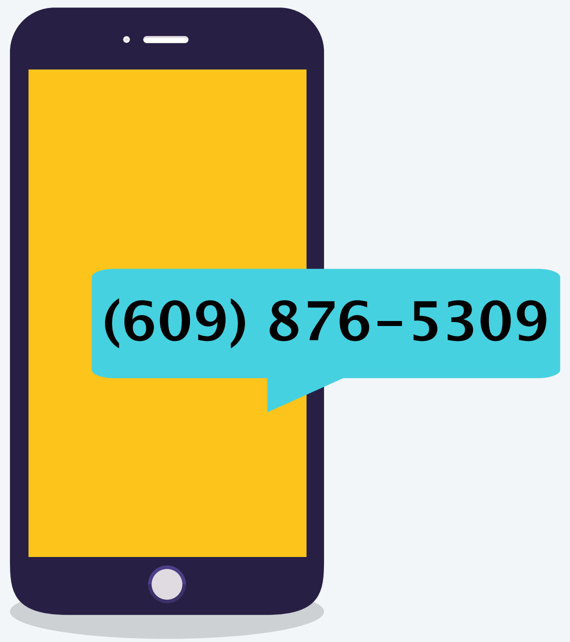
Desirable properties. [for performance]

- Very fast to compute.
- Distributes the keys uniformly: for any subset of n keys to be hashed, each table index gets approximately n / m keys.

Ex 1. $[m = 10{,}000]$ Last 4 digits of U.S. Social Security number.

Ex 2. $[m = 10{,}000]$ Last 4 digits of phone number.





Which is the last digit of your day of birth?

- **A.** 0 or 1
- **B.** 2 or 3
- **C.** 4 or 5
- **D.** 6 or 7
- **E.** 8 or 9

Which is the last digit of your year of birth?

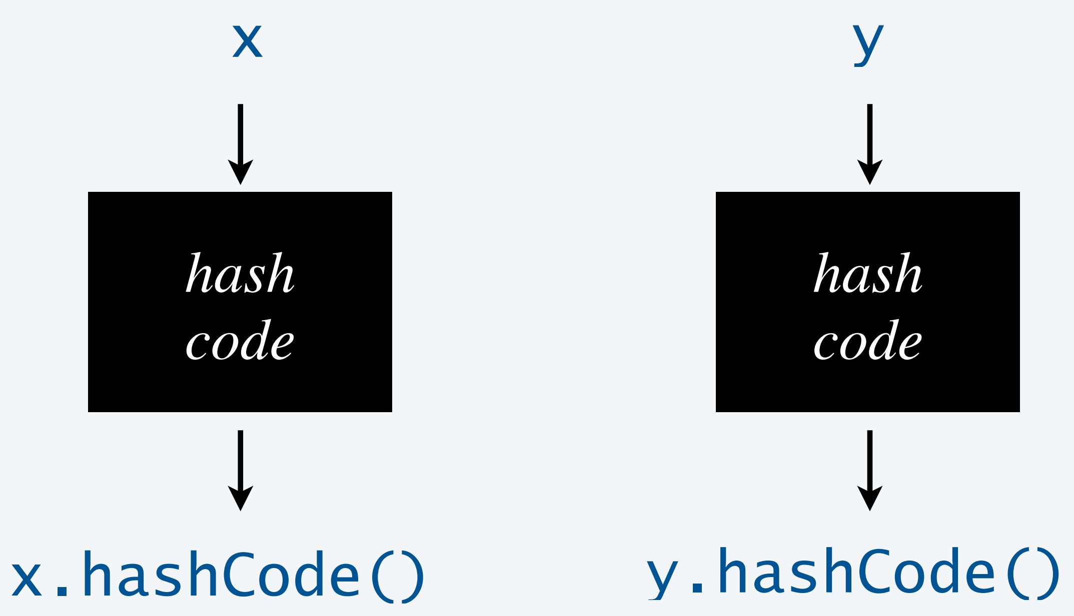
- **A.** 0 or 1
- **B.** 2 or 3
- **C.** 4 or 5
- **D.** 6 or 7
- **E.** 8 or 9

Java's hashCode() method

All Java classes inherit a method `hashCode()`, which returns a 32-bit `int`.

Required. [for correctness] If `x.equals(y)`, then `x.hashCode() == y.hashCode()`.

Highly desirable. [for efficiency] If `!x.equals(y)`, then `x.hashCode() != y.hashCode()`.



Customized implementations. `Integer`, `Double`, `String`, `java.net.URL`, …

Legal (but highly undesirable) implementation. Always return `17`.

User-defined types. Requires some care to design.

Implementing hashCode(): integers and doubles

Java library implementations

```
public final class Integer {
   private final int value;
   ...

   public int hashCode() {
      return value;
   }

}
```

```
public final class Double {
   private final double value;
   ...

   public int hashCode() {
      long bits = doubleToLongBits(value);
      return (int) (bits ^ (bits >>> 32));
   }
}
```

convert to IEEE 64-bit representation

if used only least significant 32 bits, all integers between -2^{21} *and* 2^{21} *would have same hash code* (0)

xor most significant 32-bits with least significant 32-bits

Implementing hashCode(): user-defined types

$31x + y$ rule.

- Initialize hash to 1.
- Repeatedly multiply hash by 31 and add hash of each significant field.

origin of rule remains a mystery, but works well in practice

```
public final class Transaction {
   private final String who;
   private final Date when;
   private final double amount;

   ...

   public int hashCode() {
      int hash = 1;
      hash = 31*hash + who.hashCode();
      hash = 31*hash + when.hashCode();
      hash = 31*hash + ((Double) amount).hashCode();
      return hash;
   }

}
```

for reference types: use hashCode()

for primitive types: use hashCode() *of wrapper type*

Implementing hashCode(): user-defined types

$31x + y$ rule.

- Initialize hash to 1.
- Repeatedly multiply hash by 31 and add hash of each significant field.

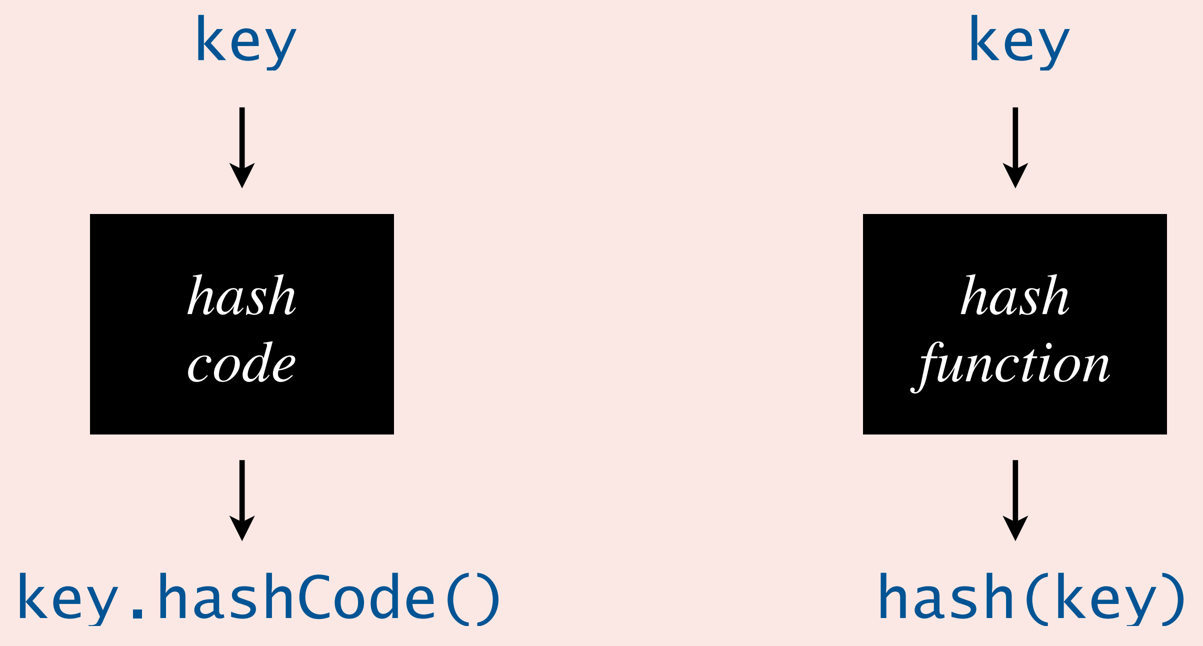
```
public final class Transaction {
   private final String who;
   private final Date when;
   private final double amount;

   ...

   public int hashCode() {
      return Objects.hash(who, when, amount);
   }

}
```

a varargs method that applies $31x + y$ rule to its arguments

Practice. This approach works reasonably well; used in Java libraries.

Hash tables: poll 3

Which Java function maps hashable keys to integers between 0 and $m - 1$?



A.

```
private int hash(Key key) {
   return key.hashCode() % m;
}
```

B.

```
private int hash(Key key) {
  return Math.abs(key.hashCode()) % m;
}
```

C. Both A and B.

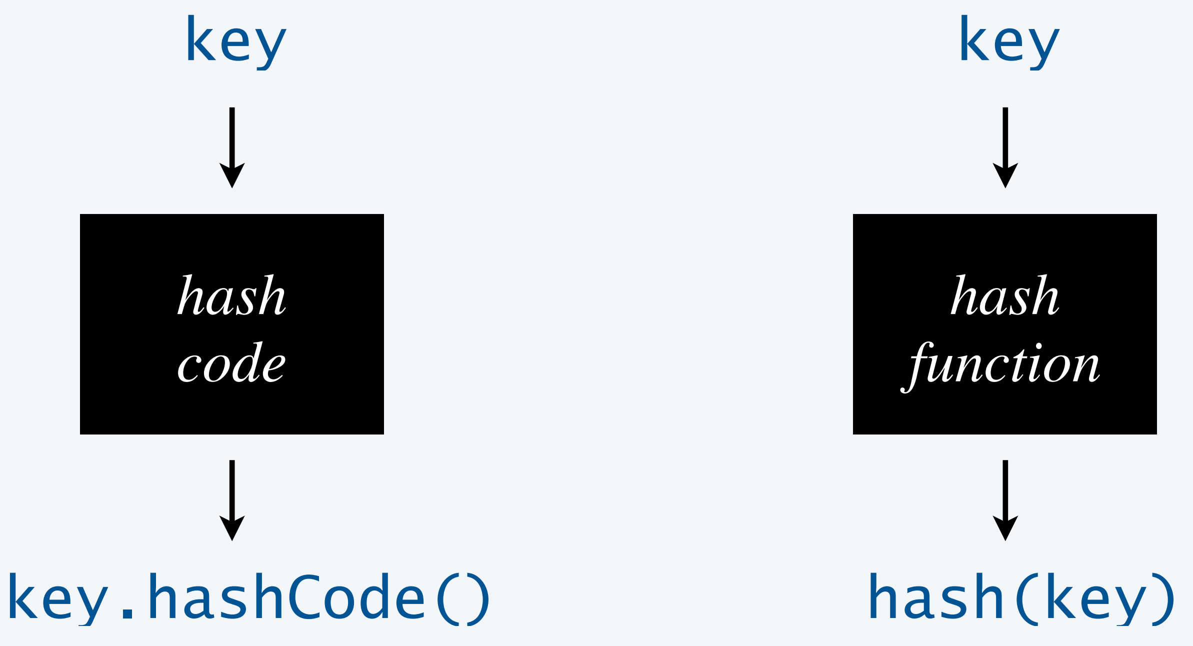
D. Neither A nor B.

Modular hashing

Hash code. An `int` between -2^{31} and $2^{31} - 1$.

Hash function. An `int` between 0 and $m - 1$ (for use as a table index).

m typically a prime or a power of 2




```java
private int hash(Key key) {
    return key.hashCode() % m;
}
```

bug ← *the remainder operator can evaluate to a negative integer*

```java
private int hash(Key key) {
  return Math.abs(key.hashCode()) % m;
}
```

1-in-a-billion bug ← `hashCode()` *of* `"polygenelubricants"` *and* `new Double(-0.0)` *is* -2^{31}

```java
private int hash(Key key) {
    return Math.abs(key.hashCode() % m);
}
```

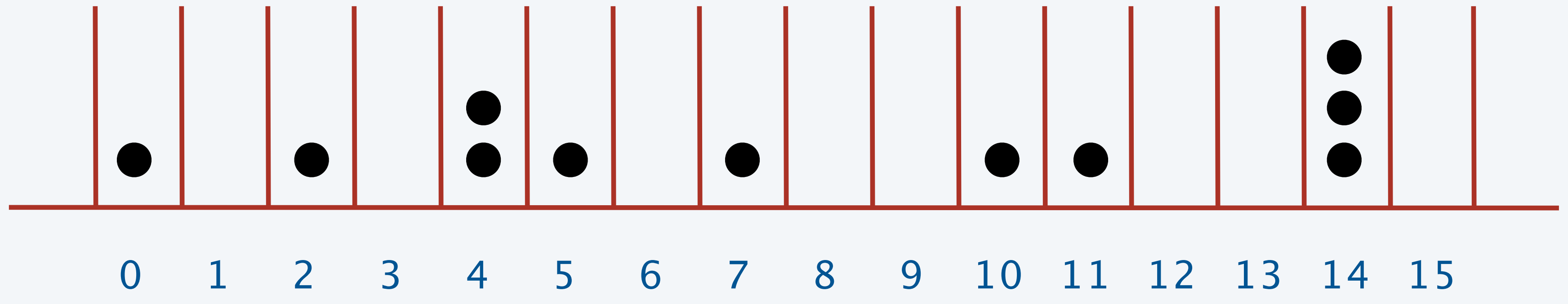
correct

Uniform hashing assumption

Uniform hashing assumption. Each key is equally likely to hash to any of m possible indices.

and independently of other keys

Balls-into-bins model. Toss n balls uniformly at random into m bins.



m = 16 bins, n = 11 balls

Bad news. [birthday problem]

- In a random group of $n = 23$ people, more likely than not that two (or more) share the same birthday ($m = 365$).
- Expect two balls in the same bin after $\sim \sqrt{\pi m / 2}$ tosses.

$\Longrightarrow$ collisions are unavoidable

23.9 when m = 365

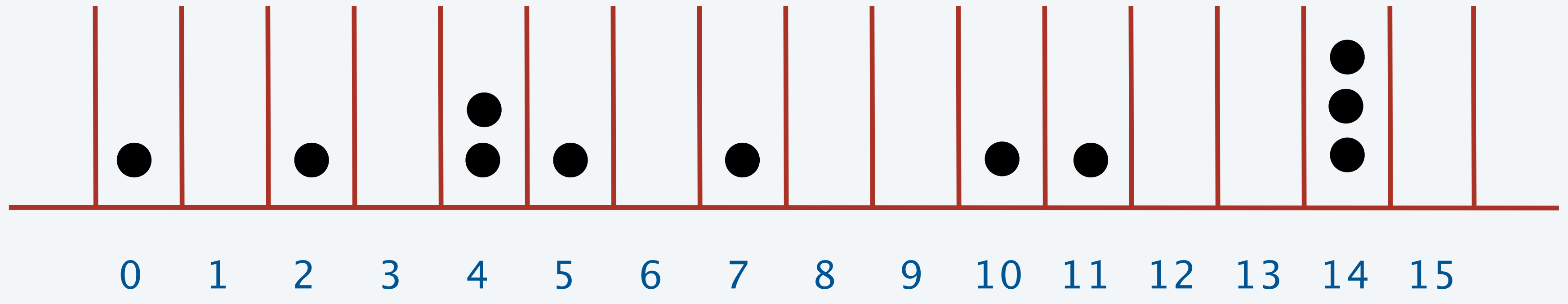
unless m is huge

Uniform hashing assumption

Uniform hashing assumption. Each key is equally likely to hash to any of m possible indices.

and independently of other keys

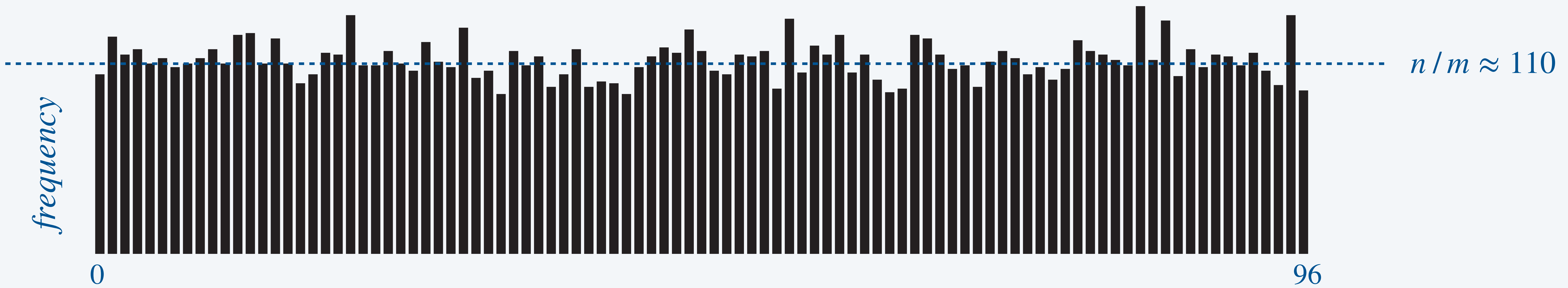
Balls-into-bins model. Toss n balls uniformly at random into m bins.



m = 16 bins, n = 11 balls

Good news. [load balancing]

- Mean number of balls per bin $= n / m$, variance $\sim n / m$. ← *Binomial*$(n, 1 / m)$
- Expect most bins to have approximately n / m balls.



hash value frequencies for words in Tale of Two Cities (m = 97, n = 10,679)

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3.4 Hash Tables

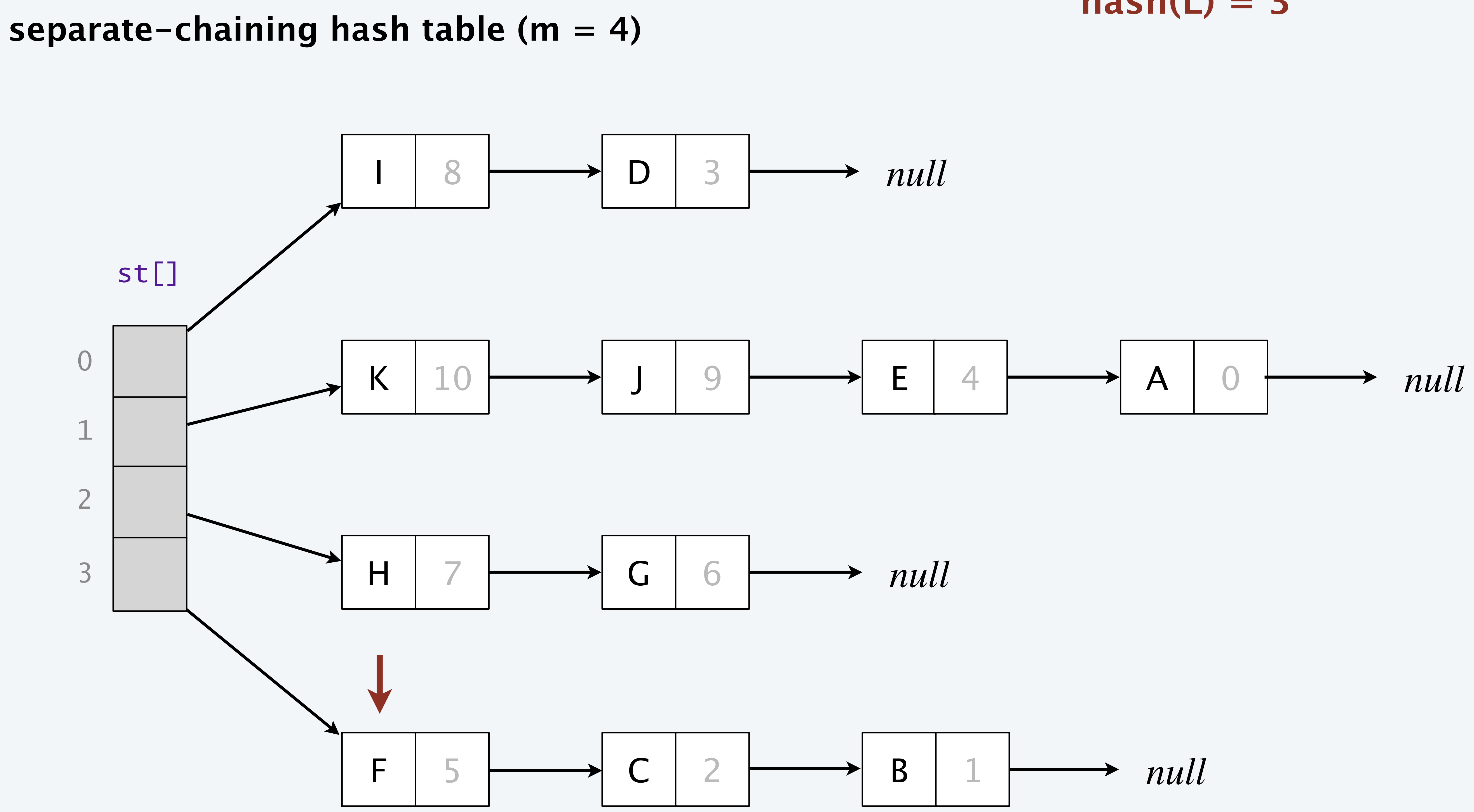
- hash functions
- separate chaining
- linear probing
- context

Separate-chaining hash table

Use an array of m singly linked lists.

- Hash: map key to table index i between 0 and $m - 1$.
- Insert: add key-value pair at front of chain i (if not already in chain).

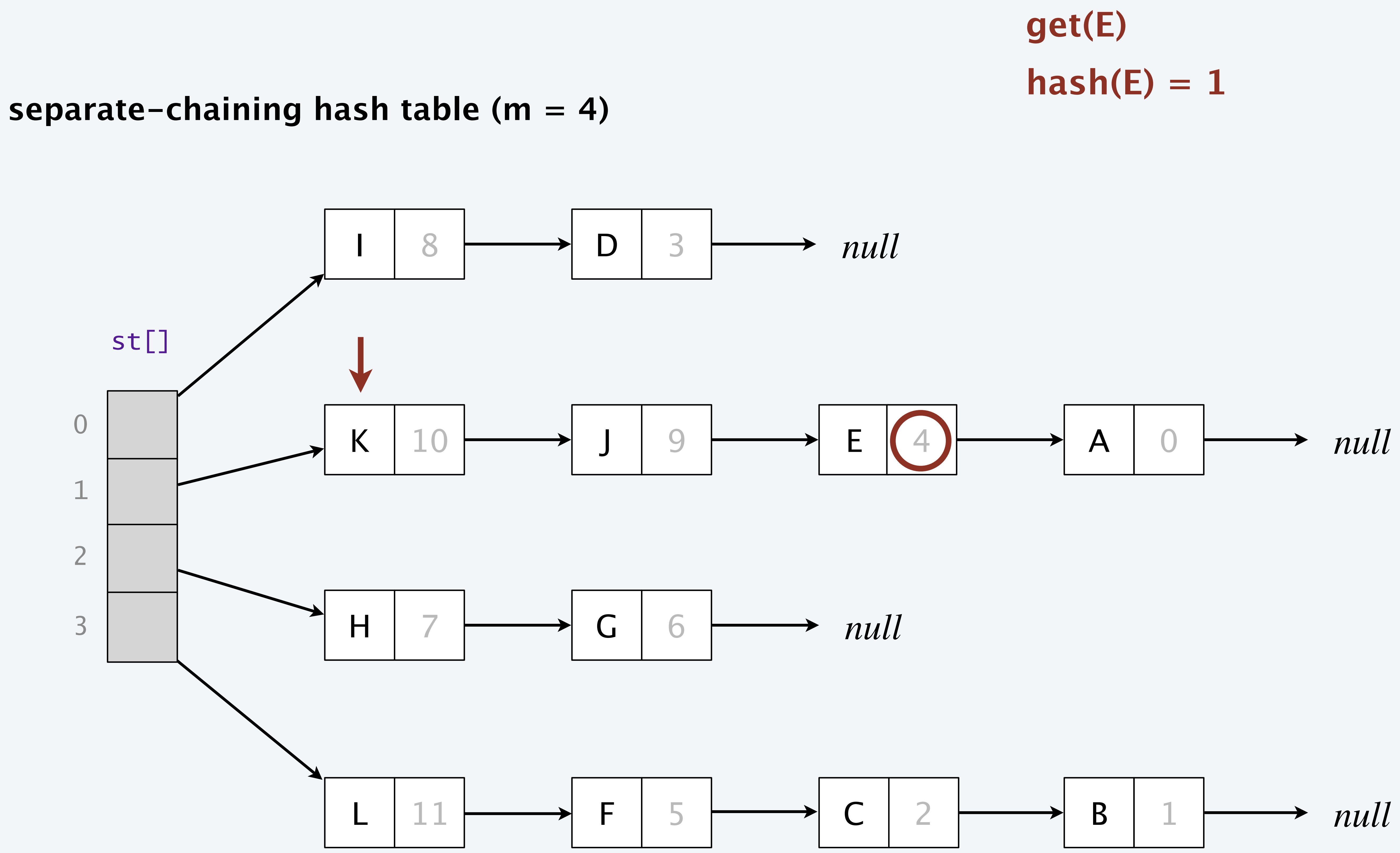
put(L, 11)
hash(L) = 3



Separate-chaining hash table

Use an array of m singly linked lists.

- Hash: map key to table index i between 0 and $m-1$.
- Insert: add key-value pair at front of chain i (if not already in chain).
- Search: perform sequential search in chain i.



Separate-chaining hash table: Java implementation

```
public class SeparateChainingHashST<Key, Value> {
   private int m = 128;                     // number of chains
   private Node[] st = new Node[m];         // array of chains

   private static class Node {
      private Object key;
      private Object val;
      private Node next;
      ...
   }

   private int hash(Key key)
   {  /* as before */  }

   public Value get(Key key) {
      int i = hash(key);
      for (Node x = st[i]; x != null; x = x.next)
         if (key.equals(x.key)) return (Value) x.val;
      return null;
   }

}
```

array resizing code omitted

no generic array creation (declare key and value of type `Object`*)*

Separate-chaining hash table: Java implementation

```java
public class SeparateChainingHashST<Key, Value> {
   private int m = 128;                // number of chains
   private Node[] st = new Node[m];    // array of chains

   private static class Node {
      private Object key;
      private Object val;
      private Node next;
      ...
   }

   private int hash(Key key)
   {  /* as before */  }

   public void put(Key key, Value val) {
      int i = hash(key);
      for (Node x = st[i]; x != null; x = x.next)
         if (key.equals(x.key)) { x.val = val; return; }
      st[i] = new Node(key, val, st[i]);
   }

}
```

Hash tables: poll 4

What is worst-case number of probes to search for a key in a separate-chaining hash table with n keys and $m = n$ chains?

calls to either `equals()` *or* `hashCode()`

A. $\Theta(1)$

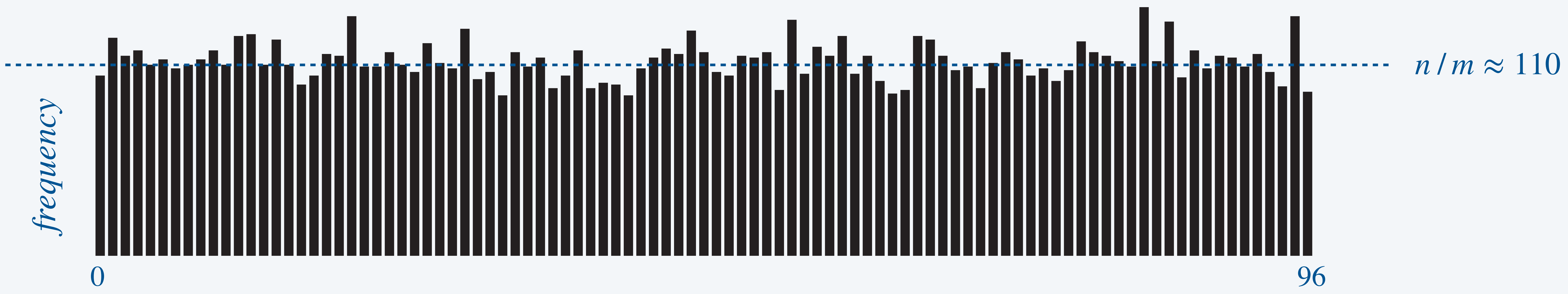
B. $\Theta(\log n)$

C. $\Theta(n)$

D. $\Theta(n^2)$

Analysis of separate chaining

Recall load balancing: Under the uniform hashing assumption, the length of a chain is tightly concentrated around its mean $= n / m$.



hash value frequencies for words in Tale of Two Cities (m = 97, n = 10,679)

Consequence. Expected number of probes for search/insert is $\Theta(n / m)$.

m times faster than sequential search

- m too small $\Longrightarrow$ chains too long.
- m too large $\Longrightarrow$ too many empty chains.
- Typical choice: $m \sim \frac{1}{4} n \Longrightarrow \Theta(1)$ time for search/insert.

average length of a chain = 4

Resizing in a separate-chaining hash table

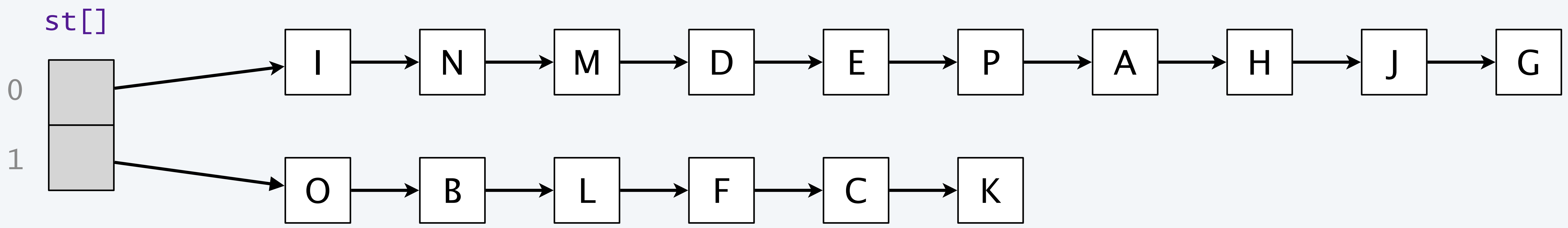
Goal. Resize array so that the average length of a chain is $\Theta(1)$.

- Double length m of array when $n / m \geq 8$.
- Halve length m of array when $n / m \leq 2$.
- Note: must rehash all keys when resizing.

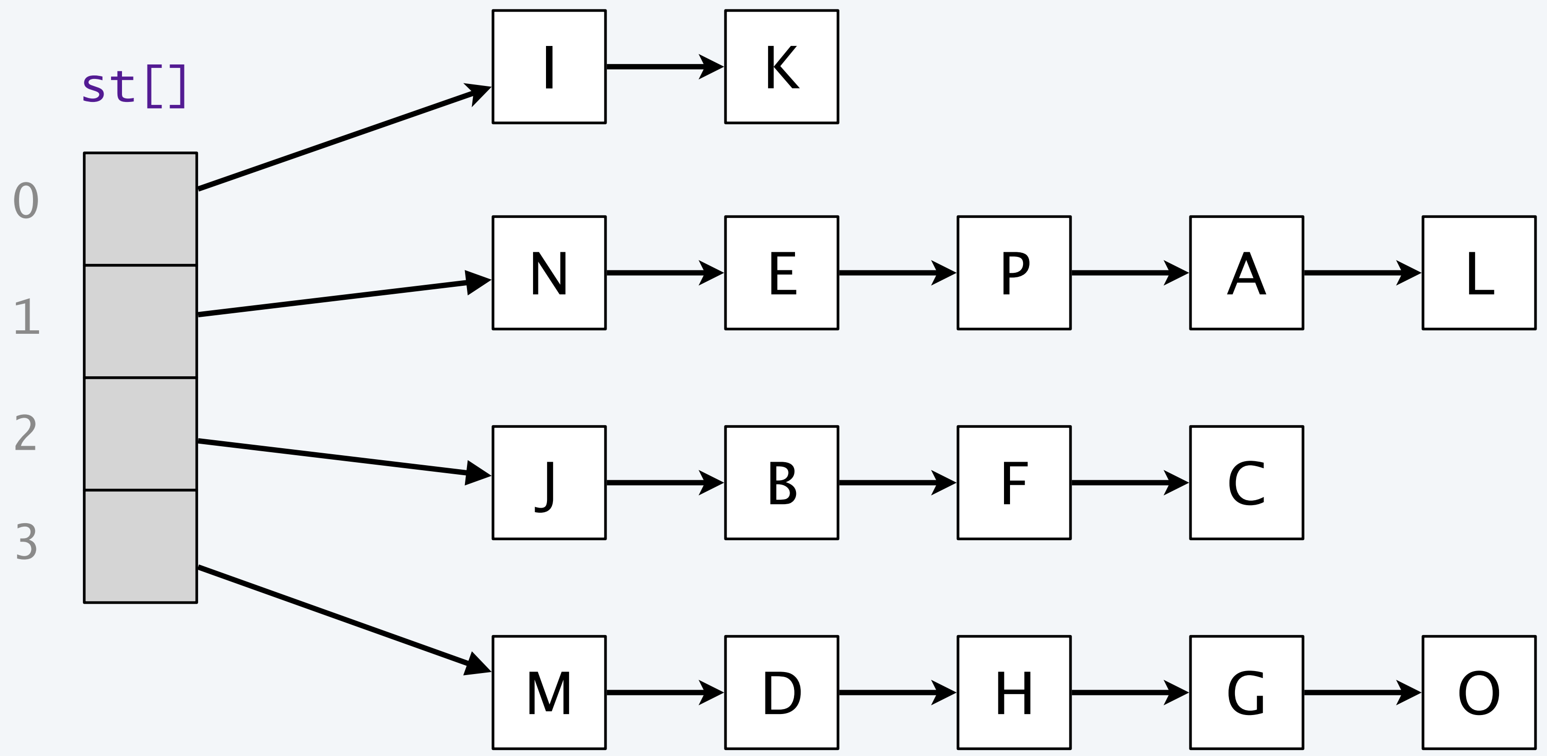
average length of a chain is between 2 and 8

`x.hashCode()` *does not change; but* `hash(x)` *typically does*

before resizing (n/m = 8)



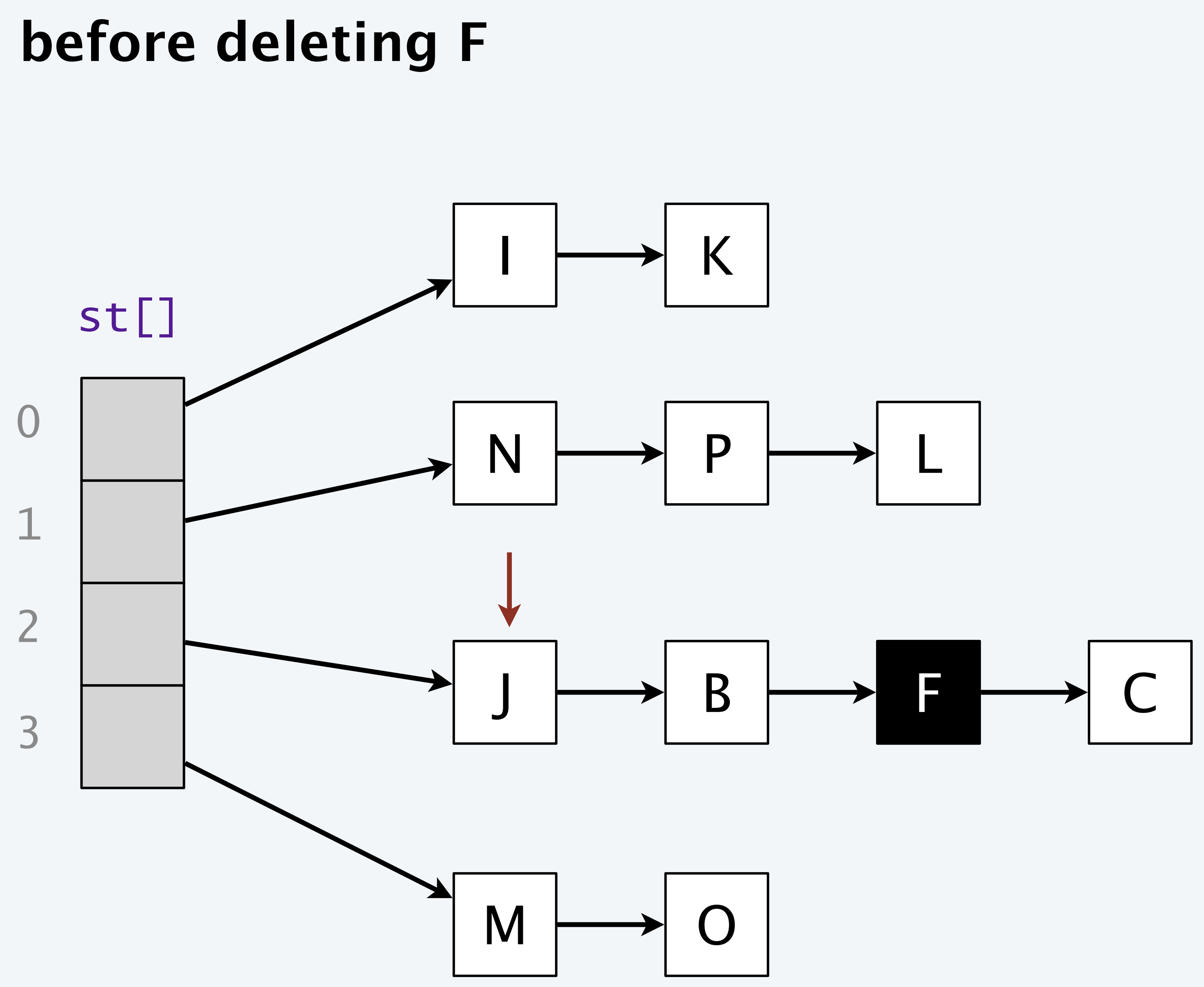
after resizing (n/m = 4)



Deletion in a separate-chaining hash table

Q. How to delete a key (and its associated value)?

A. Easy: need to consider only linked list containing key.



Symbol table implementations: summary

implementation	worst case			typical case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
red-black BST	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`
separate chaining	n	n	n	1 †	1 †	1 †		`equals()` `hashCode()`

can achieve $\Theta(1)$ probabilistic, amortized guarantee by choosing a hash function at random (see "universal hashing")

† *under uniform hashing assumption*

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3.4 Hash Tables

- *hash functions*
- *separate chaining*
- *linear probing*
- *context*

Linear-probing hash table: insertion

- Maintain key-value pairs in two parallel arrays, with one key per cell.
- Resolve collisions by linear probing:
  search successive cells until either finding the key or an unused cell.

Inserting into a linear-probing hash table.

linear-probing hash table

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C		H	L		E				R	X
								K								
								14								
vals[]	11	10			9	5		6	12		13				4	8

put(K, 14)
hash(K) = 7

Linear-probing hash table: search

- Maintain key-value pairs in two parallel arrays, with one key per cell.
- Resolve collisions by linear probing:
  search successive cells until either finding the key or an unused cell.

Searching in a linear-probing hash table.

Linear-probing hash table demo

Hash. Map key to integer i between 0 and $m - 1$.

Insertion. Put at table index i if free; if not, try $i + 1$, $i + 2$, …

Search. Search table index i; if occupied but no match, try $i + 1$, $i + 2$, …

Note. Array length m must be greater than number of key-value pairs n.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
`keys[]`	P	M			A	C	S	H	L		E				R	X

Linear-probing symbol table: Java implementation

```
public class LinearProbingHashST<Key, Value> {
   private int m = 32768;
   private Value[] vals = (Value[]) new Object[m];
   private Key[]   keys = (Key[])   new Object[m];

   private int hash(Key key)
   {  /* as before */  }

   private void put(Key key, Value val) { /* next slide */  }

   public Value get(Key key) {
      for (int i = hash(key); keys[i] != null; i = (i+1) % m) {
         if (key.equals(keys[i]))
            return vals[i];
      }
      return null;
   }

}
```

array resizing code omitted

Linear-probing symbol table: Java implementation

```
public class LinearProbingHashST<Key, Value> {
   private int m = 32768;
   private Value[] vals = (Value[]) new Object[m];
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   private int hash(Key key)
   {  /* as before */  }

   public Value get(Key key) { /* previous slide */  }

   public void put(Key key, Value val) {
      int i;
      for (i = hash(key); keys[i] != null; i = (i+1) % m) {
         if (keys[i].equals(key))
             break;
      }
      keys[i] = key;
      vals[i] = val;
   }

}
```

Hash tables: poll 6

Under the uniform hashing assumption, where is the next key most likely to be added in this linear-probing hash table (no resizing)?

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
H	I	M	N		A	D		K			A	B	E	F	G	J		C	L

A. Index 4.

B. Index 17.

C. Either index 4 or 17.

D. All open indices (4, 7, 9, 10, 17) are equally likely.

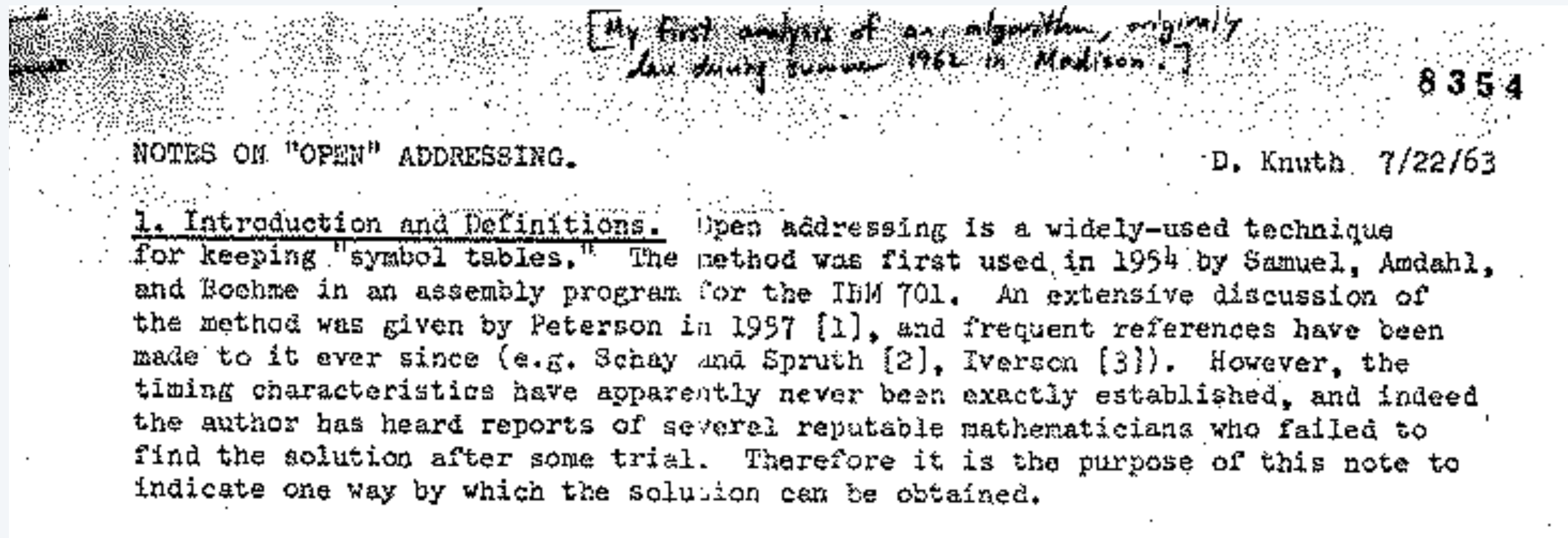
Analysis of linear probing

Proposition. Under uniform hashing assumption, the average # of probes in a linear-probing hash table of size m that contains $n = \alpha \cdot m$ keys is at most

$$\frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right) \qquad \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

search hit — search miss / insert

Pf. [beyond course scope]

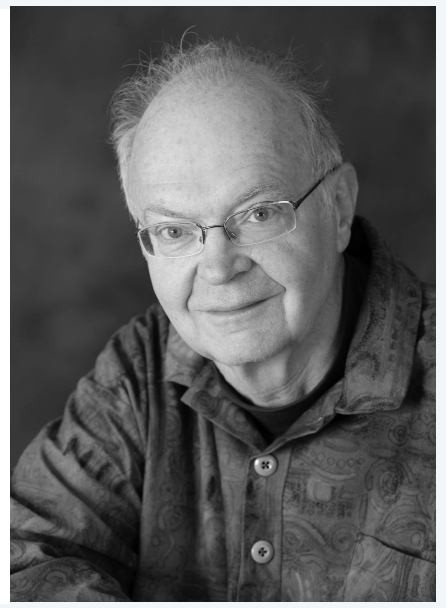


[My first analysis of an algorithm, originally done during summer 1962 in Madison.]

8354

NOTES ON "OPEN" ADDRESSING. D. Knuth 7/22/63

1. Introduction and Definitions. Open addressing is a widely-used technique for keeping "symbol tables." The method was first used in 1954 by Samuel, Amdahl, and Boehme in an assembly program for the IBM 701. An extensive discussion of the method was given by Peterson in 1957 [1], and frequent references have been made to it ever since (e.g. Schay and Spruth [2], Iverson [3]). However, the timing characteristics have apparently never been exactly established, and indeed the author has heard reports of several reputable mathematicians who failed to find the solution after some trial. Therefore it is the purpose of this note to indicate one way by which the solution can be obtained.



Parameters.

- m too large $\Longrightarrow$ wastes space (empty array entries).
- m too small $\Longrightarrow$ search time blows up.
- Typical choice: $\alpha = n / m \sim \frac{1}{2}$. ← *# probes for search hit is about 3 / 2*; *# probes for search miss is about 5 / 2*

Deletion in a linear-probing hash table

Q. How to delete a key-value pair from a linear-probing hash table?

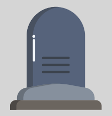
A. Requires some care: can't simply null out array entries.

before deleting S

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C	S	H	L		E				R	X
vals[]	10	9			8	4	0	5	11		12				3	7

search no longer works
(e.g., if hash(H) = 4)

after deleting S ?

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C		H	L		E				R	X
vals[]	10	9			8	4		5	11		12				3	7

"tombstone"
(skip for search; reuse for insert)

ST implementations: summary

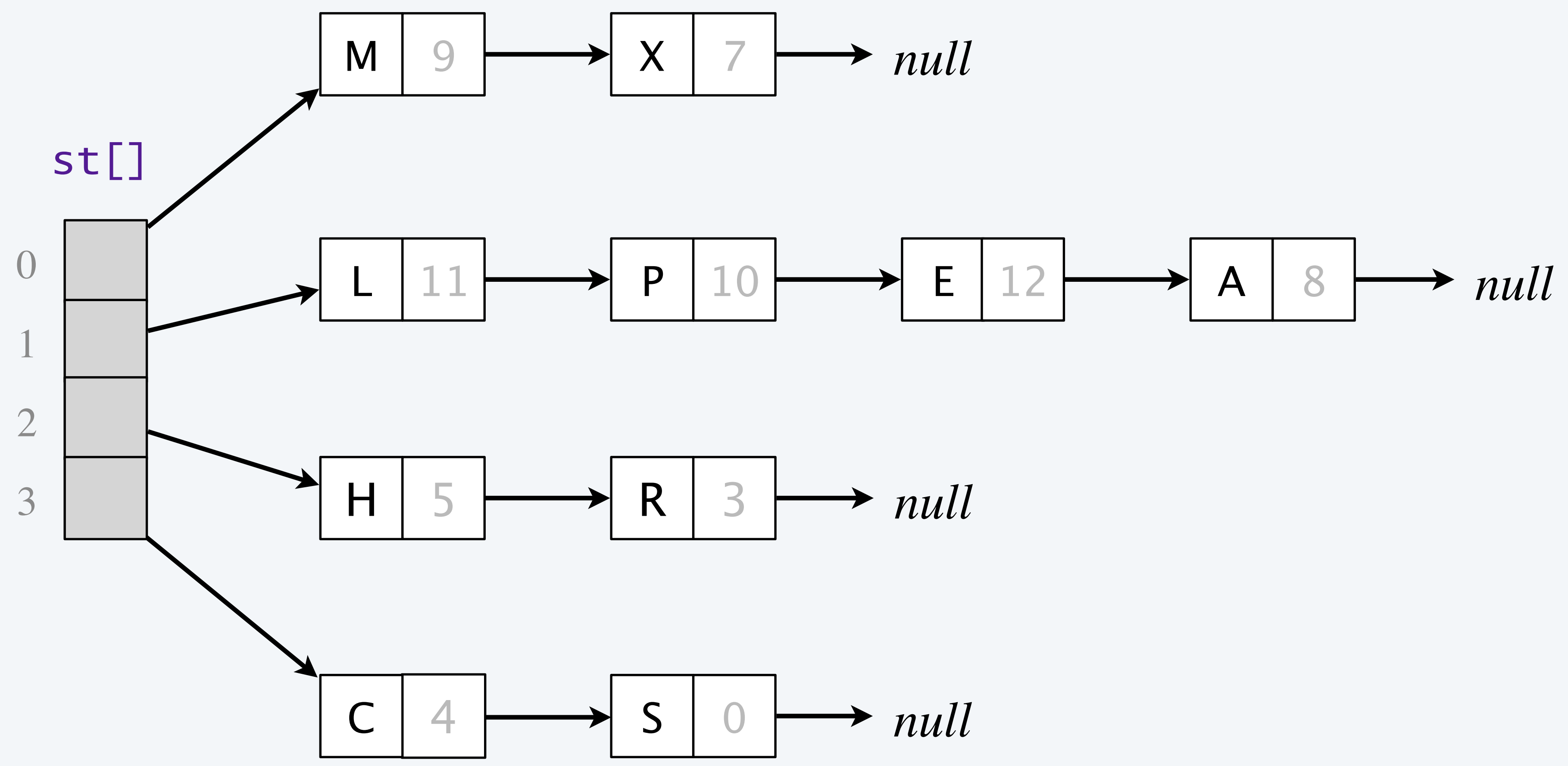
implementation	worst case			typical case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
red-black BST	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`
separate chaining	n	n	n	$1^{\dagger}$	$1^{\dagger}$	$1^{\dagger}$		`equals()` `hashCode()`
linear probing	n	n	n	$1^{\dagger}$	$1^{\dagger}$	$1^{\dagger}$		`equals()` `hashCode()`

† *under uniform hashing assumption*

Separate chaining vs. linear probing

Separate chaining.

- Performance degrades gracefully.
- Clustering less sensitive to poorly-designed hash function.



Linear probing.

- Unrivaled data locality.
- More probes because of clustering.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C	S	H	L		E				R	X
vals[]	10	9			8	4	0	5	11		12				3	7

3-Sum (revisited)

3-SUM. Given n distinct integers, find three such that $a + b + c = 0$.

Goal. $\Theta(n^2)$ expected time; $\Theta(n)$ extra space.

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3.4 Hash Tables

- ‣ *hash functions*
- ‣ *separate chaining*
- ‣ *linear probing*
- ‣ *context*

Hashing: variations on the theme

Many many improved versions have been studied.

Use different probe sequence, i.e., not $h(k)$, $h(k) + 1$, $h(k) + 2$, … ← *eliminates primary clustering, which enables higher load factor / less memory (but sacrifices data locality)*

[quadratic probing, double hashing, pseudo-random probing, …]

Google Swiss Table (quadratic probing) · *Facebook F14* (double hashing) · *Python 3* (pseudo-random probing)

During insertion, relocate some of the keys already in the table. ← *reduces worst-case time for search*

[Cuckoo hashing, Robin Hood hashing, Hopscotch hashing, …]

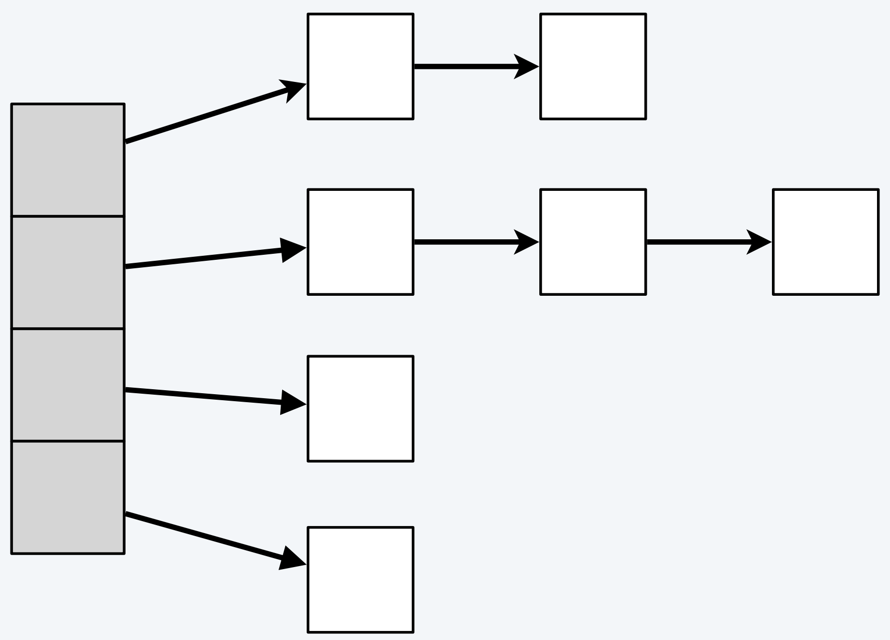
Insert tombstones prophylactically, to avoid primary clustering. ← *eliminates primary clustering; maintains data locality*

[graveyard hashing]

Hash tables vs. balanced search trees

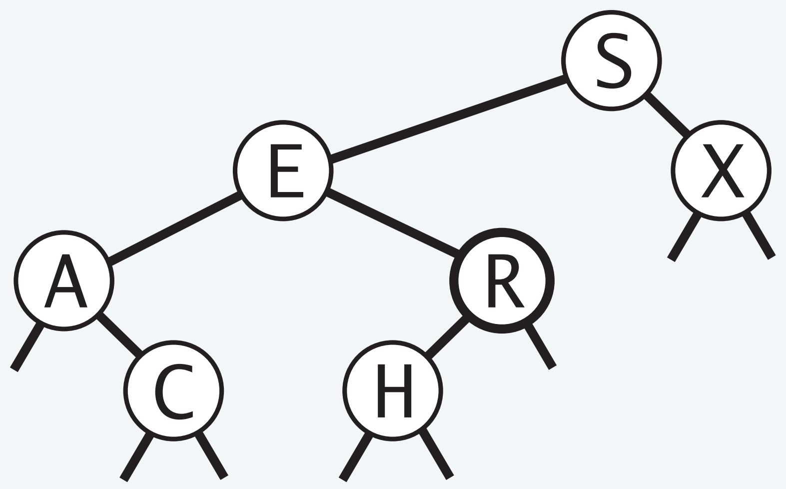
Hash tables.

- Simpler to code.
- Typically faster in practice.



Balanced search trees.

- Stronger performance guarantees.
- Support for ordered ST operations.



Java collections library includes both.

- BSTs: `java.util.TreeMap`. ← *red–black BST*
- Hash tables: `java.util.HashMap`, `java.util.IdentityHashMap`.

separate chaining (if chain gets too long, use red–black BST for chain)

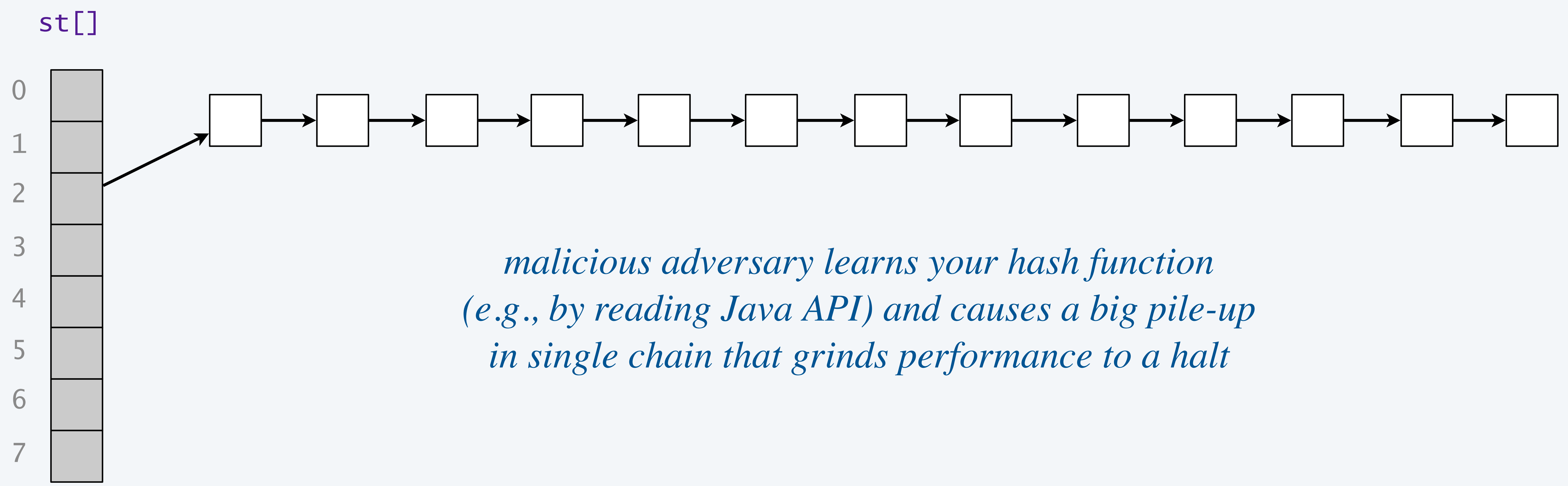
linear probing

Algorithmic complexity attacks

Q. Is the uniform hashing assumption important in practice?

A1. Yes: aircraft control, nuclear reactor, pacemaker, HFT, missile-defense system, …

A2. Yes: denial-of-service (DoS) attacks.



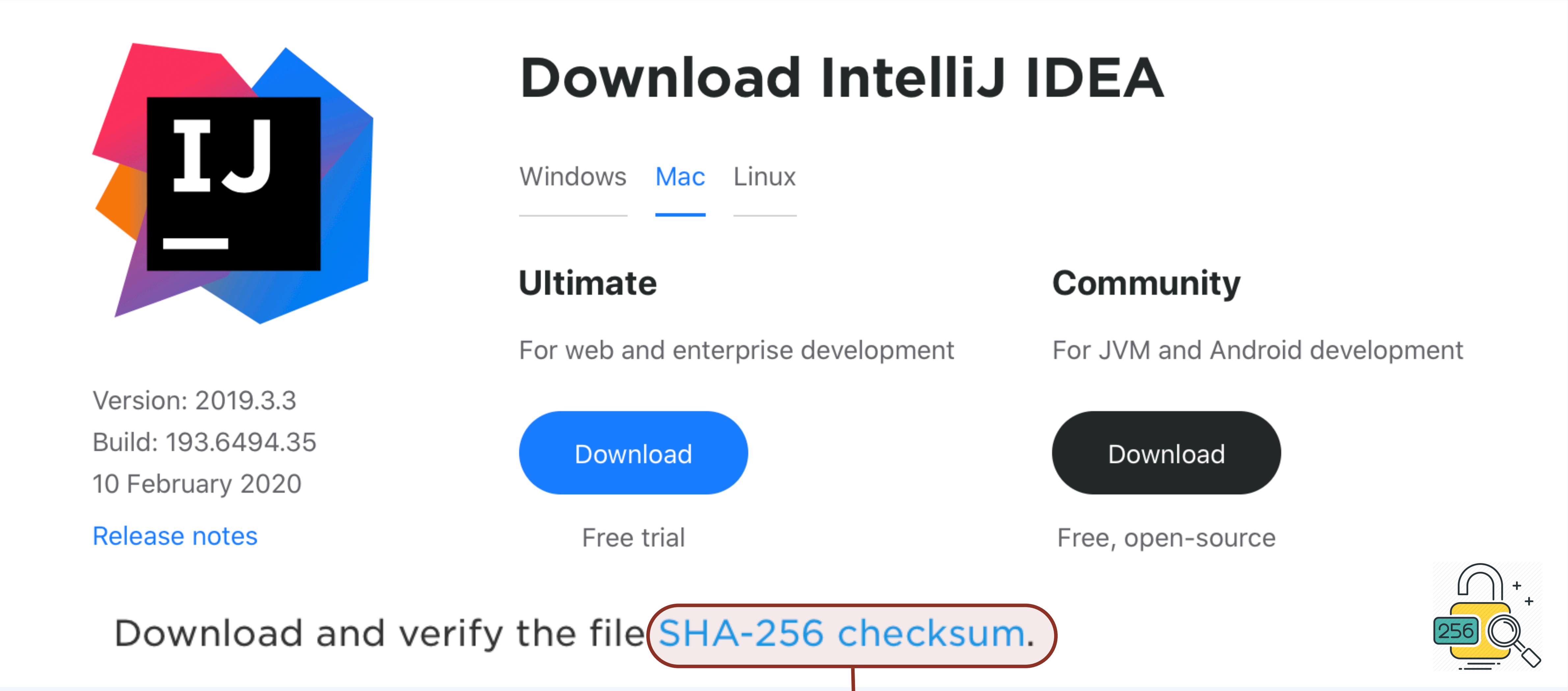
Real-world exploits. [Crosby-Wallach 2003]

- Linux 2.4.20 kernel: save files with carefully chosen names.
- Bro server: send carefully chosen packets to DoS the server, using less bandwidth than a dial-up modem.

Hashing: beyond symbol tables

File verification. When downloading a file from the web:

- Vendor publishes hash of file.
- Client checks whether hash of downloaded file matches.
- If mismatch, file corrupted. ⟵ *(e.g., error in transmission or infected by virus)*



c62ed2df891ccbb40d890e8a0074781801f086a3091a4a2a592a96afaba31270

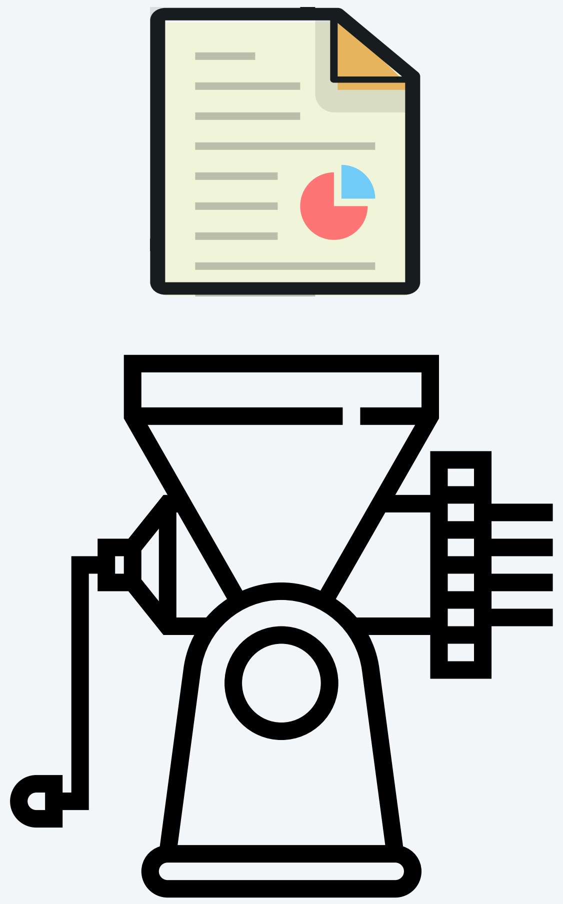
```
~/cos226/hash> sha256sum ideaIC-2019.3.dmg
c62ed2df891ccbb40d890e8a0074781801f086a3091a4a2a592a96afaba31270
```

Hashing: cryptographic applications

One-way hash function. "Hard" to find a key that will hash to a target value (or two keys that hash to same value).

Ex. MD5, SHA-1, SHA-256, SHA-512, SHA3-512, Whirlpool, BLAKE3, …

known to be insecure



c62ed2df891ccbb40d890e8a0074781801f086a3091a4a2a592a96afaba31270

fixed-length hash (256 bits)

Applications. File verification, digital signatures, cryptocurrencies, password authentication, blockchain, non-fungible tokens, Git commit identifiers, …

Credits

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Donald Knuth	Hector Garcia-Molina	

A final thought

" *Programmers waste enormous amounts of time thinking about, or worrying about, the speed of noncritical parts of their programs, and these attempts at efficiency actually have a strong negative impact when debugging and maintenance are considered.*

We should forget about small efficiencies, say about 97% of the time: premature optimization is the root of all evil.

Yet we should not pass up our opportunities in that critical 3%. "

— Donald Knuth