COS 445 - PSet 2

Due online Monday, February 26th at 11:59 pm

Instructions:

- Some problems will be marked as *no collaboration* problems. This is to make sure you have experience solving a problem start-to-finish by yourself in preparation for the midterms/final. You cannot collaborate with other students or the Internet for these problems (you may still use the referenced sources and lecture notes). You may ask the course staff clarifying questions, but we will generally not give hints.
- Submit your solution to each problem as a **separate PDF** to codePost. Please make sure you're uploading the correct PDFs to the correct locations![1] If you collaborated with other students, or consulted an outside resource, submit a (very brief) collaboration statement as well. Please anonymize your submission, although there are no repercussions if you forget.
- The cheatsheet gives problem solving tips, and tips for a "good proof" or "partial progress."
- Please reference the course collaboration policy here.

For convenience, we restate some definitions used in this problem set.

[1] We will assign a minor deduction if we need to maneuver around the wrong PDFs. Please also note that depending on if/how you use Overleaf, you may need to recompile your solutions in between downloads to get the right files.

Problem 1: Vote Cancelling (20 points, no collaboration)

In this problem we define another desirable property of voting rules: that certain configurations of votes *cancel out*. There are n voters and m candidates. As a reminder, here are the rules we will be interested in for this problem.

Definition 1 (Plurality)**.** *Select the candidate with the most first-place votes. Tie-break in favor of the lexicographically first candidate.*

Definition 2 (Borda)**.** *Every candidate gets one point every time a voter prefers it to another candidate. So, every voter gives $m - 1$ points to their first choice candidate, and in general, $m - i$ points to their i^{th} choice candidate. The candidate with the most points wins. Tie-break in favor of the lexicographically first candidate.*

For this problem, you *must* tie-break according to the above rule, i.e., the lexicographically first candidate wins in the event of a tie. Note that because of this, your proofs may also assume this tie-breaking rule to simplify analysis.

Because both rules are anonymous, for simplicity of notation in this problem we will just refer to a *set*[2] of votes $\{\succ_1, \ldots, \succ_n\}$, rather than a particular ordering. This problem will use the following definition:

Definition 3 (Cancels Out)**.** *We say a voting rule F* cancels out *with respect to a set of votes $p^* := \{\succ'_1, \ldots, \succ'_k\}$, if for every set $p = \{\succ_1, \ldots, \succ_n\}$ of votes, the outcome of the election is the same for p and $p \cup p^*$. That is to say,*for all p: $F(p) = F(p \cup p^*)$.

For example, if $m = 2$, and F is the Majority voting rule (tie-breaking for a over b), Majority cancels out with respect to the set of votes $\{a \succ'_1 b, b \succ'_2 a\}$. This is because no matter what p you start with, if you add the two votes $a \succ b$, and $b \succ a$, each candidate gets one additional vote, and therefore the majority is preserved. Similarly, Majority cancels out with respect to the set of votes $\{a \succ'_1 b, a \succ'_2 b, b \succ'_3 a, b \succ'_4 a\}$.

Majority *does not* cancel out with respect to the set of votes $p^* := \{b \succ'_1 a, b \succ'_2 a, b \succ'_3 a\}$. This is because if you start from a p where currently a has one more vote than b (for example, $p = \{a \succ_1 b\}$), then adding $\{b \succ'_1 a, b \succ'_2 a, b \succ'_3 a\}$ causes the majority to switch from a to b. This is an explicit choice of p such that $F(p) \neq F(p \cup p^*)$, and therefore Majority does not cancel out with respect to p^*.

Part a (definitions): Opposite votes

Define a pair of votes to be *opposing* if they rank the candidates in exactly the opposite order. Specifically, the preferences $\succ_i$ and $\succ_j$ are opposing if: $c_1 \succ_i c_2 \succ_i \ldots \succ_i c_m$ and $c_m \succ_j c_{m-1} \succ_j \ldots \succ_j c_1$.

We say a voting rule is *opposite-cancelling* if it cancels out with respect to **every** pair of opposite votes. That is, a rule is opposite-cancelling if for every profile, adding/removing any pair of opposing votes preserves the outcome of the election.

[2] For those who like to be mathematically extra-careful, this is actually a *multi-set*, because we allow the possibility that the same vote is repeated twice. If you are not familiar with this langugae, and you understand that a “set” of votes might have two voters with the same preference, you can ignore this footnote.

Part a (question): Opposite votes (10 points)

For each of the two rules defined above (Plurality and Borda), prove that it is opposite-cancelling, or provide a counterexample. If you provide a counterexample, include a brief explanation of why it is a counterexample.

Part b (definition): Cyclic votes

Define a set of (exactly) m votes to be *cyclic* if they rank the candidates in cyclic order. Specifically, preferences $\succ_1, \ldots, \succ_m$ are cyclic if:

- Voter 1 votes like this: $c_1 \succ_1 c_2 \succ_{v_1} \ldots \succ_1 c_m$,
- Voter 2 votes like this: $c_2 \succ_2 c_3 \succ_{v_2} \ldots \succ_2 c_1$,
- ...
- Voter j votes like this: $c_j \succ_j c_{j+1} \succ_{v_j} \ldots \succ_j c_{j-1}$,
- ...
- Voter m votes like this: $c_m \succ_m c_1 \succ_m \ldots \succ_m c_{m-1}$.

For example, if there were 4 candidates (say, a, b, c, d) then the set of voters Alice, Bob, Charlie, Danielle would be cyclic if, for instance,

- $a \succ_{Alice} b \succ_{Alice} c \succ_{Alice} d$,
- $b \succ_{Bob} c \succ_{Bob} d \succ_{Bob} a$,
- $c \succ_{Charlie} d \succ_{Charlie} a \succ_{Charlie} b$,
- $d \succ_{Danielle} a \succ_{Danielle} b \succ_{Danielle} c$.

We say a voting rule is *cycle-cancelling* if it cancels out with respect to **every** set of cyclic votes. That is, a rule is cycle-cancelling if for every set of votes, adding/removing any set of cyclic votes preserves the outcome of the election.

Part b (question): Cyclic votes (10 points)

For each of the two rules defined above (Plurality and Borda), prove that it is cycle-cancelling, or provide a counterexample. If you provide a counterexample, include a brief explanation of why it is a counterexample.

Problem 2: Two Candidates, One Rule (40 points)

For this problem, there are n voters and $m = 2$ candidates, and $n \geq 3$ is odd. Therefore, for a voting rule to have property X, it only needs to have property X when $m = 2$ and $n \geq 3$ is odd. For a voting rule to not have property X, there must exist a counterexample with $m = 2$ and $n \geq 3$ odd. Recall also the following definitions:

Definition 4 (Unanimous)**.** *F is* unanimous *if whenever everyone puts x as their first choice, F selects x.*

Definition 5 (Anonymous)**.** *Let σ be any permutation from $[n]$ to $[n]$. If F is anonymous, then $F(\succ_1, \ldots, \succ_n) = F(\succ_{\sigma(1)}, \ldots, \succ_{\sigma(n)})$.*

Definition 6 (Neutral)**.** *Let σ be any permutation from $[m]$ to $[m]$. If F is neutral, then $F(\sigma(\succ_1), \ldots, \sigma(\succ_n)) = \sigma(F(\succ_1, \ldots, \succ_n))$. Here, we have abused notation and let $\sigma(\succ_i)$ denote the preference $\succ$ where $\sigma(a) \succ \sigma(b)$ if and only if $a \succ_i b$.*

Definition 7 (Strategy-Proof)**.** *A voting rule F is strategy-proof if for all players i, all true preferences of that player $\succ_i$, and all preferences $\succ_i'$ they might report instead, and all possible preferences $\vec{\succ}_{-i}$ of the other players, Player i weakly prefers telling the truth than misreporting. In other words, $F(\succ_i; \vec{\succ}_{-i}) \succeq_i F(\succ_i'; \vec{\succ}_{-i})$.*

Part a (5 points)

Design a voting rule which **is not** unanimous, **is not** neutral, and **is** strategyproof (and briefly prove that it **is not** unanimous, **is not** neutral, and **is** strategyproof).

Part b (5 points)

Design a voting rule which **is** unanimous, **is** neutral, and **is not** strategyproof (and briefly prove that it **is** unanimous, **is** neutral, and **is not** strategyproof).

Part c (5 points)

Design a voting rule which **is** unanimous, **is not** neutral, and **is** strategyproof (and briefly prove that it **is** unanimous, **is not** neutral, and **is** strategyproof).

Part d (5 points)

Design a voting rule which **is** neutral, **is** anonymous, and **is** strategyproof (and briefly prove that it **is** neutral, **is** anonymous, and **is** strategyproof).

Part e (20 points)

Prove that every voting rule which is neutral and strategyproof is also unanimous.

Problem 3: "Never Worst" Preference Sets (40 points)

In this problem, there are $m \geq 3$ candidates. Any claim you prove must hold for all $m \geq 3$. Any counterexample you provide can pick a particular $m \geq 3$ of your choice.

In Lecture 6, we saw that if we *know* voter preferences are *single-peaked*, then we can design voting rules with interesting properties.

Definition 8 (Single-Peaked)**.** *A set $V \neq \emptyset$ of voter preferences is* single-peaked *if there exists an ordering of the alternatives $a_1 \ldots, a_m$ such that for all a_i and for all $\succ \in V$ whose favorite alternative is a_i, it holds that:*

- *If $j < k \leq i$, then $a_k \succ a_j$.*
- *If $j > k \geq i$, then $a_k \succ a_j$.*

One of the properties we saw in Lecture 6 concerned the concept of a Condorcet winner:

Definition 9 (Condorcet Winner)**.** *An alternative $a \in A$ is a* Condorcet winner *if for every other candidate $b \in A \setminus \{a\}$, a wins a strict majority of votes in a head-to-head against b. Note that a Condorcet winner doesn't necessarily exist.*

This problem will explore a generalization of single-peaked preferences, called *Never-Worst*.

Definition 10. *A set $V \neq \emptyset$ of voter preferences are* Never-Worst *if for all sets of three candidates $\{a, b, c\}$, there exists an $x \in \{a, b, c\}$ such that* every single $\succ \in V$ *has $x \succ y$ for* some $y \in \{a, b, c\}$. *That is, for all sets of three candidates, there is some candidate x in that set, such that* all voters agree that x is not the worst in that set.

Part a (10 points)

Prove that if V is single-peaked, then V is Never-Worst.

Part b (10 points)

Prove that being single-peaked is not the *only* way to be Never-Worst. That is, find a set V of preferences that is *not* single-peaked, but *is* Never-Worst, and (briefly) prove that it is Never-Worst, and (briefly) prove that it is not single-peaked.

Observe that to prove a set of preferences is not single-peaked, you must show that *no* ordering on the candidates results in single-peaks.

You may not use code to find your example. When you find your example, your proof should be readable by a human, and should not simply exhaust every possible ordering to confirm it is not single-peaked. There exists an example with four candidates and two preferences, and a short non-exhaustive proof (but you are allowed to use another comparably-sized example with a comparably-short proof).

Part c (20 points)

Prove that if V is Never-Worst, and there are an odd number of voters with preferences in V, then there is a Condorcet winner.

Hint: You may want to first try to prove the claim when there are only three candidates.

Extra Credit: Why is nothing Strategyproof?

Recall that extra credit is not directly added to your PSet scores, but will contribute to your participation. Some extra credits are **quite** challenging. We do not suggest attempting the extra credit problems for the sake of your grade, but only to engage deeper with the course material. If you are interested in pursuing an IW/thesis in CS theory, the extra credits will give you a taste of what that might be like.[3]

For this problem, you *may* collaborate with any students and office hours. You *may not* consult course resources or external resources. In this problem we will guide you through the proof of a well-known result, so you should *not* copy the proof from one of the course texts (nor should you try to find a proof from external sources). You must follow the guide below (and not provide an alternative proof).

A *Full-Ranking-Function (FRF)* F is given a set of alternatives A and a profile of preferences over n voters, p, (just as with voting rules) except that now it must output a full ranking over A instead of a single winner.

Here are some desirable properties of FRFs:

- Unanimous: a FRF F is *unanimous* if whenever $a \succ_i b$ for all i, $\succ = F(\succ_1, \ldots, \succ_n)$ has $a \succ b$ (whenever *everyone* likes a better than b, the final ranking has a above b).
- Independence of Irrelevant Alternatives: consider two profiles $p = (\succ_1, \succ_2, ..., \succ_n)$, $p' = (\succ'_1, \succ'_2, ..., \succ'_n)$ and let $\succ = F(p), \succ' = F(p')$.

  A FRF F satisfies *Independence of Irrelevant Alternatives* if for any two alternatives $a, b \in A$, if $a \succ_i b \iff a \succ'_i b, \forall i$ (every voter has the same preference between a and b) then $a \succ b \iff a \succ' b$ (the ordering output by F ranks a vs. b the same). Intuitively this property suggests that our preferences for c should not interfere with the ranking of a and b, and is related to strategyproof-ness.

Here is an undesirable property of FRFs:

- Dictatorship: voter i is a dictator in a in FRF F if for all $p = (\succ_1, \succ_2, ..., \succ_n)$, $\succ_i = F(p)$. That is to say, no matter what everyone else submits, the FRF chooses the ordering of the dictator. F is a dictatorship if some voter is a dictator.

It would be nice to produce FRFs that are unanimous with Independence of Irrelevant Alternatives, and are *not* dictatorships. Unfortunately, the following theorem says that for $|A| \geq 3$, this is not possible:

Theorem 11. *Every unanimous FRF F satisfying Independence of Irrelevant Alternatives over a set of more than 2 alternatives is a dictatorship.*

[3] Keep in mind, of course, that you will do an IW/thesis across an entire semester/year, and you are doing the extra credit in a week. Whether or not you make progress on the extra credit in a week is not the important part — it's whether or not you enjoy the process of tackling an extremely open-ended problem with little idea of where to get started.

Part a

Prove the following lemma:

Lemma 12. *Let* $p = (\succ_1, \succ_2, ..., \succ_n)$*,* $p' = (\succ'_1, \succ'_2, ..., \succ'_n)$ *be two profiles such that for every player* i*,* $a \succ_i b \iff c \succ'_i d$*. Then if* F *is unanimous with Independence of Irrelevant Alternatives,* $a \succ b \iff c \succ' d$*, where* $\succ = F(p), \succ' = F(p')$ *(when there are* > 2 *alternatives).*

Note that a complete proof needs to consider all of the following cases:

1. $a = c, b = d$.
2. $a = c, b \neq d$.
3. $a \notin \{c, d\}, b = c$.
4. $a \notin \{c, d\}, b = d$.
5. $a \notin \{c, d\}, b \notin \{c, d\}$.
6. $a = d, b = c$.
7. $a = d, b \neq c$.

You do not need to provide a proof for all 7 cases, as many are similar. Provide a proof for cases One, Two, and Five.

Part b

Take any $a \neq b \in A$, and for every $0 \leq i \leq n$ let π^i be some preference profile in which exactly the first i voters rank a above b, and the remaining voters rank b above a.

Prove that, if F is unanimous with Independence of Irrelevant Alternatives, there must be *some* $i^* \in [1, n]$ such that in $F(\pi^{i^*-1})$ we have $b \succ a$, but in $F(\pi^{i^*})$ we have $a \succ b$ (at this point, i^* might not be unique).

Part c

Use the lemma from part a to show the following lemma and corollary.

Lemma 13. *Let* a *and* b *be any two candidates in* A*. For all* i*, let* π^i *be some preference profile in which exactly the first* i *voters rank* a *above* b*, and the remaining voters rank* b *above* a*. Let* F *be unanimous with Independence of Irrelevant Alternatives, and let* i^* *be such that in* $F(\pi^{i^*-1})$ *we have* $b \succ a$*, but in* $F(\pi^{i^*})$ *we have* $a \succ b$*.*

Then, for any $c \neq d \in A$*, and* any preference profile*: if* $c \succ_{i^*} d$ *then* F *ranks* c *above* d*. Conclude that* i^* *is a dictator for* F*.*