COS 445 - Midterm

Due online Monday, March 11th at 11:59 pm

- All problems on this exam are *no collaboration* problems.
- You *may not* discuss any aspect of any problems with *anyone except for the course staff.*
- You *may not* consult any external resources, the Internet, etc.
- You *may* consult the course lecture notes on Ed, any of the five course readings, past Ed discussion, or any notes directly linked on the course webpage (e.g. the cheatsheet, or notes on linear programming).
- You *may* discuss the test with the course staff, but **we will only answer clarification questions and will not give any guidance or hints**. You should feel free to ask any questions and let us judge whether or not to answer, but just know that we may choose to politely decline to answer. We may choose to answer questions with a response of "I'm sorry, but I'm not comfortable answering that question," or "it is within the scope of the exam for you to answer that question yourself" (or some variant of these).
- If you choose to ask a question on Ed, *ask it privately*. We will maintain a pinned FAQ for questions that are asked multiple times (please also reference this FAQ).
- Please upload each problem as a separate file via codePost, as usual.
- You *may not* use late days on the exam. You must upload your solution by March 11th at 11:59pm. If you are working down to the wire, upload your partial progress in advance. **There is no grace period for the exam**. In case of a true emergency where you cannot upload, email smweinberg@princeton.edu and dallagnol@princeton.edu your solutions asap.
- If you miss the codePost deadline, we will not completely ignore your submission, but we will apply a **substantial** deduction before grading it. Please make sure you have something submitted by the deadline, and take into account that the server may be overloaded or sluggish near the end.
- There are *no exceptions, extensions, etc.* to the exam policy (again, in case of a *truly exceptional* circumstance, you should reach out to your residential dean and have them contact us).

Problem 1: Matching, Voting, Games, and LPs (40 points)

For each of the 4 problems below: **unless otherwise specified**[1] you do not need to show any work and can just state the answer. However, if you simply state an incorrect answer with no justification, we cannot award partial credit. You are encouraged to provide a **very brief** outline/justification in order to receive partial credit in the event of a tiny mistake. For example, we will award very significant partial credit if you clearly execute the correct outline, but make a mistake in implementation.

Part a: Stable Matching (10 points)

Find the matching output by student-proposing deferred acceptance in the following example. A reminder of the Deferred Acceptance algorithm is the Lecture Stable Matchings I.

Alice: Princeton $\succ$ Harvard $\succ$ Yale
Bob: Yale $\succ$ Harvard $\succ$ Princeton
Charlie: Princeton $\succ$ Yale $\succ$ Harvard

Princeton: Alice $\succ$ Bob $\succ$ Charlie
Harvard: Bob $\succ$ Charlie $\succ$ Alice
Yale: Alice $>$ Charlie $>$ Bob

Part b: Voting (10 points)

Definition 1 (Copeland Rule)**.** *For every pair of candidates a, b, give one point to whichever candidate a majority of voters prefer. Output the candidate with the most points. Break all ties for the lexicographically-first candidate.*

Find the candidate (the candidates are the foods — Alice, Bob, and Charlie are voting to pick a restaurant) **output by the Copeland rule** in the following example.

Alice: Pizza $\succ$ Steak $\succ$ Fish
Bob: Steak $\succ$ Fish $\succ$ Pizza
Charlie: Fish $\succ$ Steak $\succ$ Pizza

Part c: Game Theory (10 points)

Find a Nash equilibrium of the following game and state the expected payoff for both players. A definition of Nash equilibrium can be found in Lecture Game Theory II.

Player X, the row player, chooses between actions x_1 and x_2. Player Y, the column player, chooses between actions y_1 and y_2. The first number in each box denotes the payoff to X, and the second number is the payoff to Y. For example, if X plays action x_1 and the column player plays action y_1, then X gets payoff -2 and Y gets payoff -4.

	y_1	y_2
x_1	(-2,-4)	(6,8)
x_2	(0,8)	(4,4)

[1] If otherwise specified, you should follow the otherwise specifications.

Part d: Linear Programming (10 points)

Write the dual of the following LP. **You do not need to solve the LP. You only need to write the dual.** A reminder of LP duality is in Lecture Linear Programming.

Maximize $3x + 4y$, such that:

- $2x + 7y \leq 10$.
- $9x + 4y \leq 3$.
- $x, y \geq 0$.

Problem 2: Stable Matchings for Students (45 points)

Consider an instance of the stable matching problem. To be extra clear, there are n students and n universities. Each university has capacity for one student. Every student has strict preferences over the n universities, and every university has strict preferences over the n students. Let M denote the matching output by student-proposing deferred acceptance.

Part a: A helper Lemma (10 points)

Prove the following claim: when student-proposing deferred acceptance terminates, there is at least one university u that received only a single proposal.

Part b: Very weakly Pareto-optimal for students (20 points)

Prove that there **does not exist** any matching M' (**stable or not**) such that *every* student *strictly prefers* their match in M' to M.

Part c: But not pareto-optimal for students (15 points)

Provide an example of preferences for students and universities such that M is not pareto-optimal for students. That is, there exists another matching M' (**not necessarily stable**) such that every student is at least as happy with their match in M' versus M, and at least one student is strictly happier.[2] For full credit, you should briefly walk through the execution of student-proposing DA, and state which students are happier in M' versus M.

You may not use code to find your example. When you find your example, your analysis should be readable by a human. Specifically, it should be easy for the grader to confirm that you have computed M correctly, and that indeed some students are better off in M' and all students are at least as happy. There is an example with three students and three universities, although you are free to use a larger example as long as you explain its analysis well.

[2] Note that this sentence is exactly what the problem asks for: you do not need to show that universities are more or less happy in M'. You just need to show that all students are at least as happy, and that some students are strictly happier.

Problem 3: Nash Equilibria (45 points)

Consider the following game. The first number in each square denotes the left player's payoff, and the second number in each square denotes the top player's payoff. For instance, if the left player plays A, and the top player plays D, then the left player gets payoff 5 and the top player gets payoff 1.

Table 1: A 2-player 3-action game.

	D	E	F
A	(5,1)	(1,2)	(2,3)
B	(1,8)	(2,2)	(3,0)
C	(4,0)	(7,4)	(1,7)

Part a: Find a Nash (15 points)

Find a Nash equilibrium of this game. Prove that it is a Nash equilibrium (this proof should be brief), and state the expected payoff received by each player when this Nash equilibrium is played.

Hint: First try to find a player i and action a such that player i cannot possibly use action a in any Nash equilibrium.

Part b: Uniqueness (30 points)

Prove that the Nash equilibrium you found in part a is unique (that is, there is no other Nash equilibrium). Recall that for full credit, your proof should be easy for a grader to evaluate (if you choose to make use of the hint, your proof is likely to be *much* simpler than a massive list of case analysis).

Hint: First try to find a player i and action a and prove that player i cannot possibly use action a in any Nash equilibrium.

Problem 4: Equivalency, Unanimity, Non-Dictatorship Don't Mix (70 points)

In a previous iteration of this class, I asked the class on a midterm to provide a voting rule that was Equivalent (defined below), and unanimous, and not a dictatorship (and then some) for $m \geq 3$ candidates. Unfortunately, this is impossible.[3] This problem guides you through part of the proof. **For all parts of this problem there are** $m \geq 3$ **candidates, but only** $n = 2$ **voters**. That is, your proofs must work for any number $m \geq 3$ of candidates, but should assume there are only $n = 2$ voters.

Definition 2 (Unanimous)**.** *A voting rule f is unanimous if whenever all voters have the same favorite candidate, f selects that candidate.*

Definition 3 (Equivalent)**.** *Two preferences $\succ, \succ'$ over candidates are a-*equivalent *if for all candidates $b \neq a$, we have $a \succ b \Leftrightarrow a \succ' b$. That is, $\succ$ and $\succ'$ are a-equivalent if exactly the same candidates are above a (and below a) in both preferences (but the candidates above a might be arbitrarily moved around, staying above a. Separately, the candidates below a might be arbitrarily moved around, staying below a).*

A voting rule is Equivalent *if whenever $f(\succ_1, \ldots, \succ_n) = a$, and $\succ_i, \succ'_i$ are a-equivalent for all i, then $f(\succ'_1, \ldots, \succ'_n) = a$ as well.*

Suggestion: Before beginning the problem, I strongly suggest making sure you are comfortable with this definition. In particular, I strongly suggest feeling that you can write two preferences $\succ$ and $\succ'$ that are a-equivalent (and understand why), and two preferences $\succ$ and $\succ'$ that are not a-equivalent (and understand why). If you would find it helpful to see a similar definition briefly considered in lecture, here is an anaology: Equivalent:a-equivalent::Strongly Montone (Lecture 5):a-improves (Lecture 5). Rereading Lecture 5 may help you understand the definition of Equivalent (but it is expected that you can fully solve this problem without reading Lecture 5 at all, and rereading Lecture 5 is unlikely to help you solve any part of this problem beyond understanding this definition — you will have to be creative to get your solutions!). Unfortunately, the course staff will not answer questions to help you understand the definition — this is considered part of the exam.

Part a (10 points)

Prove the following lemma, and its corollary.

Lemma 4. *Let f be unanimous and Equivalent. Then for all candidates c, if there exists another candidate d such that $d \succ_1 c$ and $d \succ_2 c$, then $f(\succ_1, \succ_2) \neq c$. That is, if f is unanimous and Equivalent, f cannot output a candidate c such that there exists a d that* both *voters prefer to c.*

Definition 5 ((a, b)-first)**.** *We say that a preference $\succ$ is (a, b)-*first *if a is the favorite candidate, and b is the second-favorite candidate (that is, $a \succ b$, and for all other $c \notin \{a, b\}, b \succ c$).*

Corollary 6. *Let f be unanimous and Equivalent. Let also $\succ_1$ be (a, b)-first, and $\succ_2$ be (b, a)-first. Prove that $f(\succ_1, \succ_2) \in \{a, b\}$. That is, prove that because f is unanimous and Equivalent, that $f(\succ_1, \succ_2)$ must output either a or b whenever one voter is (a, b)-first, and the other is (b, a)-first.*

[3] Fortunately, we caught the mistake early in the exam period before it burned too many students' time. But it was quite a frantic two-day period catching the mistake and figuring out what to do about it. It's a good mindset to be skeptical of any claim until you can prove it yourself, but I promise that every problem on this midterm is solvable.

Part b (15 points)

Let f be unanimous and Equivalent. Let $\succ_1$ be (a, b)-first, and $\succ_2$ be (b, a)-first. Prove that if $f(\succ_1, \succ_2) = a$, then $f(\succ_1, \succ_2') = a$, for any $\succ_2'$ satisfying the following condition:

- (Informally, identical to the bullet below): $\succ_2'$ can be obtained from $\succ_2$ by only moving a, while keeping all other candidates in place.
- (Formally, identical to the bullet above): For all $c \neq a, d \neq a$, $c \succ_2 d \Leftrightarrow c \succ_2' d$.

Part c (20 points)

Prove the following lemma.

Lemma 7. *Let f be unanimous and Equivalent. Let $\succ_1$ be (a, b)-first and $\succ_2$ be (b, a)-first. If $f(\succ_1, \succ_2) = a$, then $f(\succ_1, \succ_2') = a$* ***for all*** *$\succ_2'$.*

Part d (5 points)

Prove the following lemma.

Definition 8 (Dictator for a specific candidate)**.** *Voter i is a dictator under voting rule f for candidate j if whenever voter i ranks j first in their ordering, f selects candidate j.*

Lemma 9. *Let f be unanimous and Equivalent. If there exists a $\succ_1$ that is (a,b)-first, and a $\succ_2$ that is (b,a)-first, where $f(\succ_1, \succ_2) = a$, then Voter 1 is a dictator under f for candidate a.*

In subsequent parts of this problem, you may use the following lemma instead (its proof is identical to that of Lemma 9, but requires more notation. You do not need to repeat the proof with additional notation, and can simply use Lemma 10 instead).

Lemma 10. *Let f be unanimous and Equivalent. For any distinct candidates $x \neq y$, let also $\succ_1$ be (x, y)-first and $\succ_2$ be (y, x)-first. Then $f(\succ_1, \succ_2) \in \{x, y\}$. Moreover, if $f(\succ_1, \succ_2) = x$, then voter one is a dictator for x. If instead $f(\succ_1, \succ_2) = y$, then voter two is a dictator for y.*

Part e (5 points)

Prove the following for all $a \neq b$, and all voting rules f (even f that are not unanimous or Equivalent): if voter 1 is a dictator under rule f for candidate a, then voter 2 is *not* a dictator under rule f for candidate b.

Part f (15 points)

Let f be unanimous and Equivalent. Prove that f is a Dictatorship (recall that $n = 2$ and $m \geq 3$). Recall below the definition of a Dictatorship:

Definition 11 (Dictatorship)**.** *A voting rule f is a Dictatorship if there exists a voter i such that f always outputs i's favorite candidate.*