
https://algs4.cs.princeton.edu

# 4. Graphs and Digraphs II

- *breadth-first search (in digraphs)*
- *breadth-first search (in graphs)*
- *topological sort*
- *challenges*

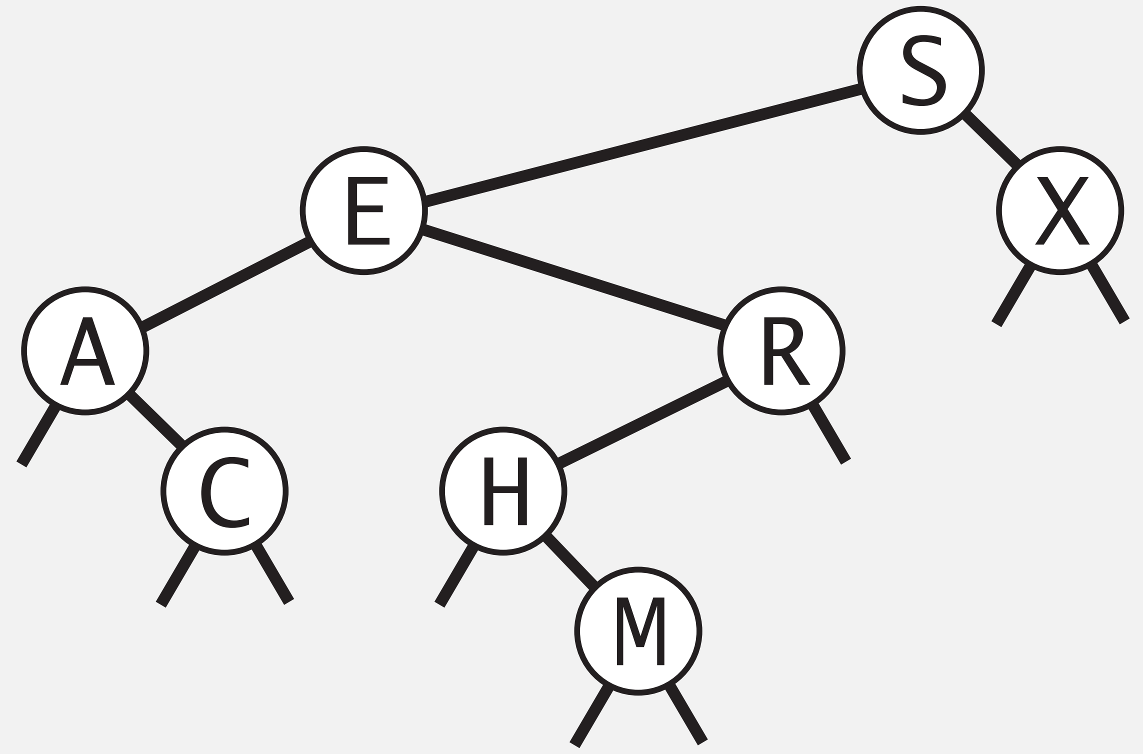
# Graph search

Tree traversal. Many ways to explore a binary tree.

- Inorder: A C E H M R S X
- Preorder: S E A C R H M X
- Postorder: C A M H R E X S
- Level-order: S E X A R C H M

stack/recursion

queue




Graph search. Many ways to explore a graph.

- DFS preorder: vertices in order of calls to `dfs(G, v)`.
- DFS postorder: vertices in order of returns from `dfs(G, v)`.
- Breadth-first: vertices in increasing order of distance from `s`.

stack/recursion

queue

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

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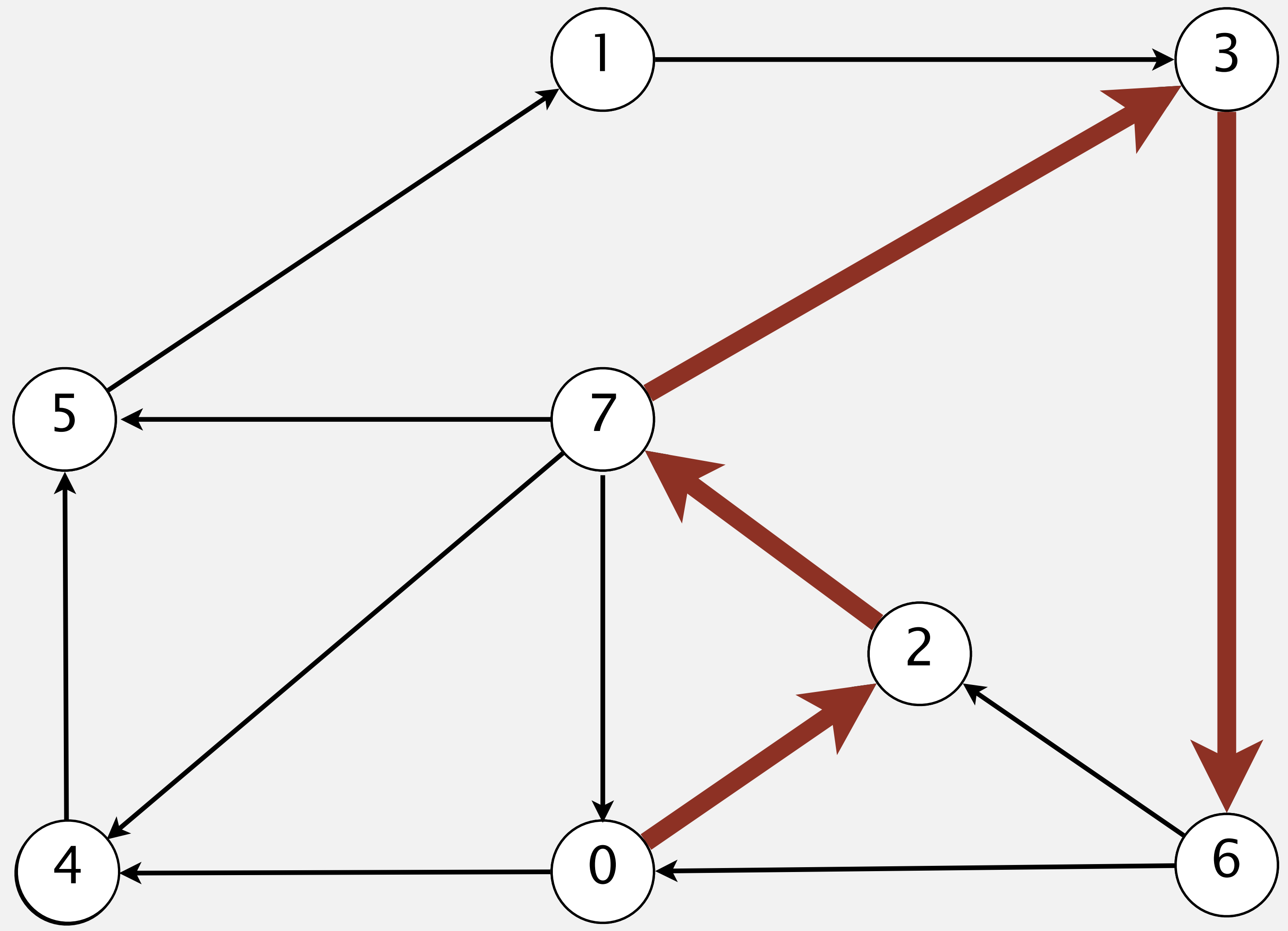
# 4. GRAPHS AND DIGRAPHS II

- *breadth-first search (in digraphs)*
- *breadth-first search (in graphs)*
- *topological sort*
- *challenges*

# Shortest paths in a digraph

Problem. Find directed path from $s$ to each other vertex that uses the fewest edges.




**directed paths from 0 to 6**

$0 \to 2 \to 7 \to 4 \to 5 \to 1 \to 3 \to 6$

$0 \to 4 \to 5 \to 1 \to 3 \to 6$

$0 \to 2 \to 7 \to 3 \to 6$

$0 \to 2 \to 7 \to 0 \to 2 \to 7 \to 3 \to 6$

Note: shortest paths must be simple
(no repeated vertices)

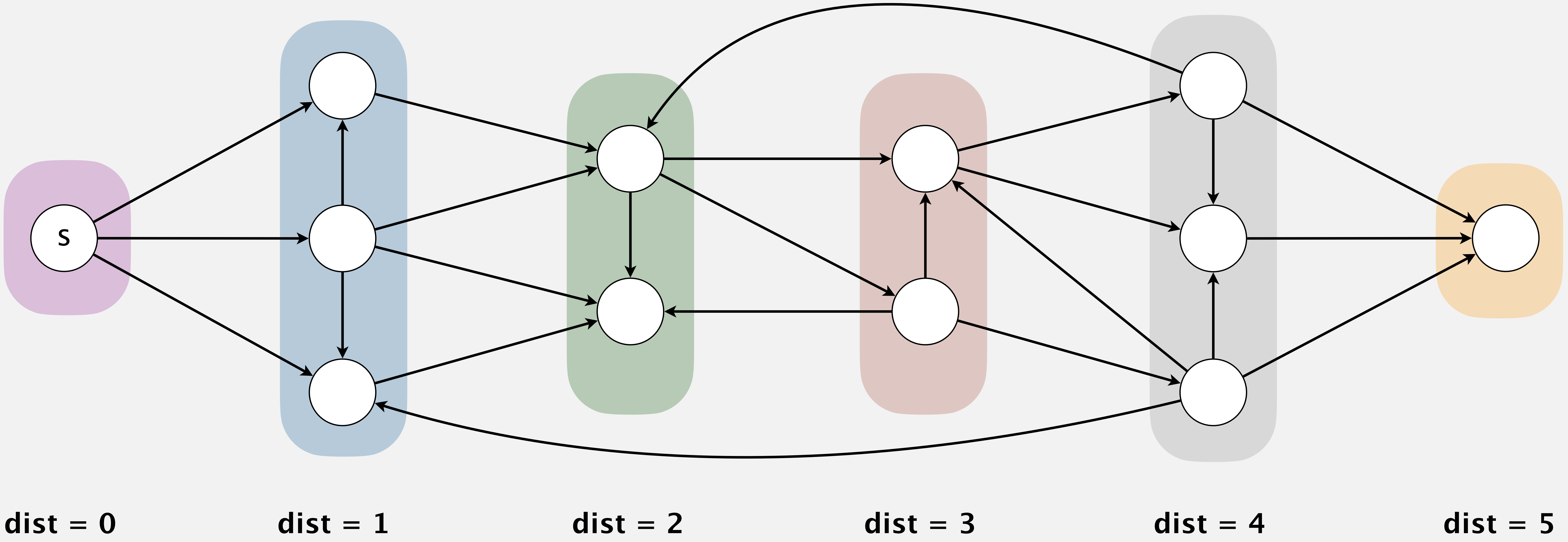
**shortest path from 0 to 6 (length = 4)**

$0 \to 2 \to 7 \to 3 \to 6$

# Shortest paths in a digraph

Problem. Find directed path from $s$ to each other vertex that uses the fewest edges.

Key idea. Visit vertices in increasing order of distance from $s$.

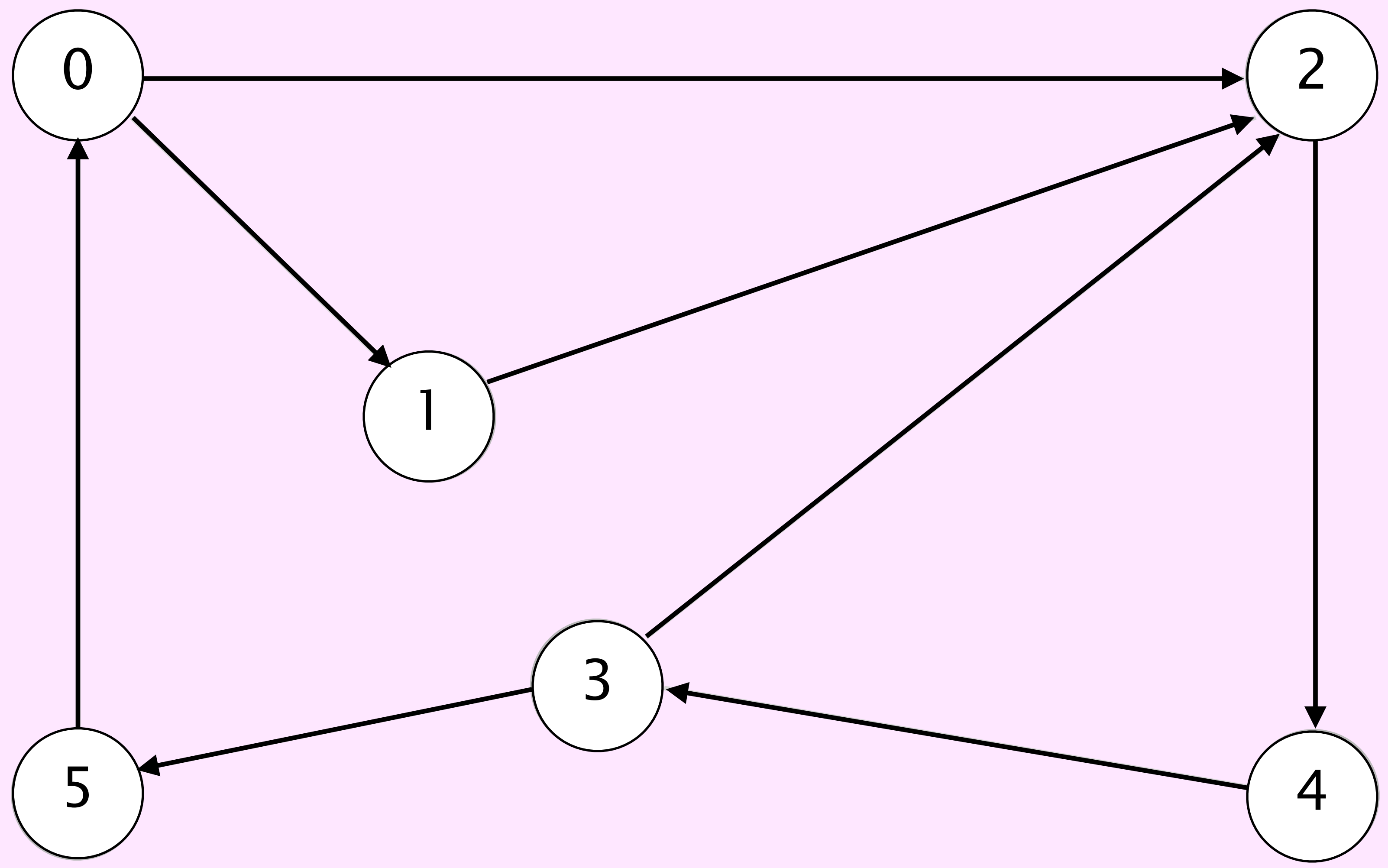



Key data structure. Queue of vertices to visit.

# Breadth-first search demo

Repeat until queue is empty:

- Remove vertex $v$ from queue.
- Add to queue all unmarked vertices adjacent from $v$ and mark them.




graph G

`tinyDG2.txt`

V

6

E

8

5 0

2 4

3 2

1 2

0 1

4 3

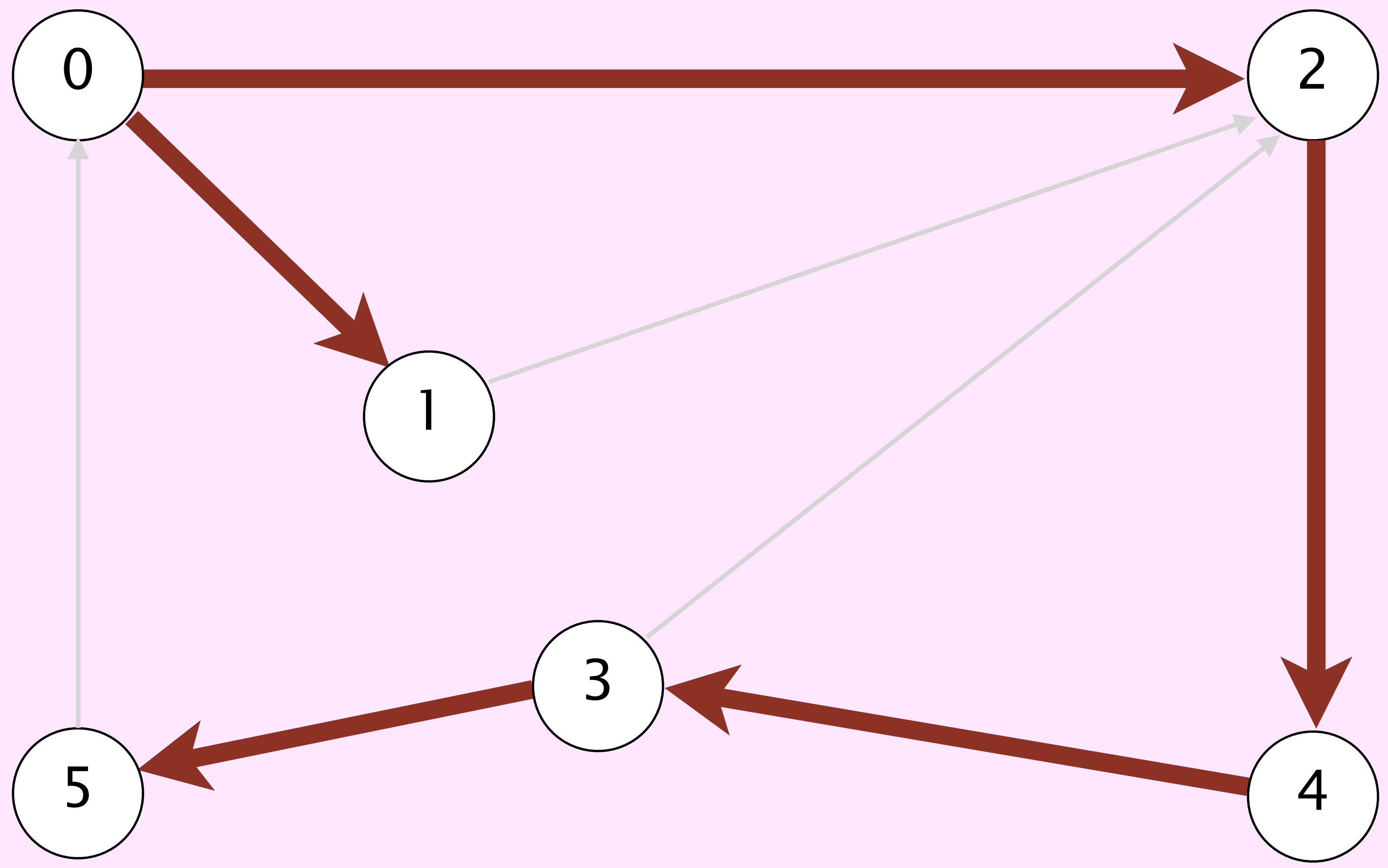
3 5

0 2

# Breadth-first search demo

Repeat until queue is empty:

- Remove vertex $v$ from queue.
- Add to queue all unmarked vertices adjacent from $v$ and mark them.




| v | edgeTo[] | marked[] | distTo[] |
|---|---|---|---|
| 0 | – | T | 0 |
| 1 | 0 | T | 1 |
| 2 | 0 | T | 1 |
| 3 | 4 | T | 3 |
| 4 | 2 | T | 2 |
| 5 | 3 | T | 4 |

**vertices reachable from 0**
**(and shortest directed paths)**

# Breadth-first search

Repeat until queue is empty:

- Remove vertex $v$ from queue.
- Add to queue all unmarked vertices adjacent from $v$ and mark them.

visit vertex $v$

**BFS (from source vertex s)**

**Add s to FIFO queue and mark s.**

**Repeat until the queue is empty:**

- **remove the least recently added vertex v**
- **for each unmarked vertex w adjacent from v:**
  **add w to queue and mark w.**

# Breadth-first search:  Java implementation

```java
public class BreadthFirstDirectedPaths
{
   private boolean[] marked;
   private int[] edgeTo;
   private int[] distTo;
   …

   private void bfs(Digraph G, int s) {
      Queue<Integer> queue = new Queue<>();
      queue.enqueue(s);
      marked[s] = true;
      distTo[s] = 0;

      while (!queue.isEmpty()) {
         int v = queue.dequeue();
         for (int w : G.adj(v)) {
            if (!marked[w]) {
               queue.enqueue(w);
               marked[w] = true;
               edgeTo[w] = v;
               distTo[w] = distTo[v] + 1;
            }
         }
      }
   }
}
```

initialize FIFO queue of vertices to explore

found new vertex *w* via edge $v \to w$

https://algs4.cs.princeton.edu/42digraph/BreadthFirstDirectedPaths.java.html

# Breadth-first search properties

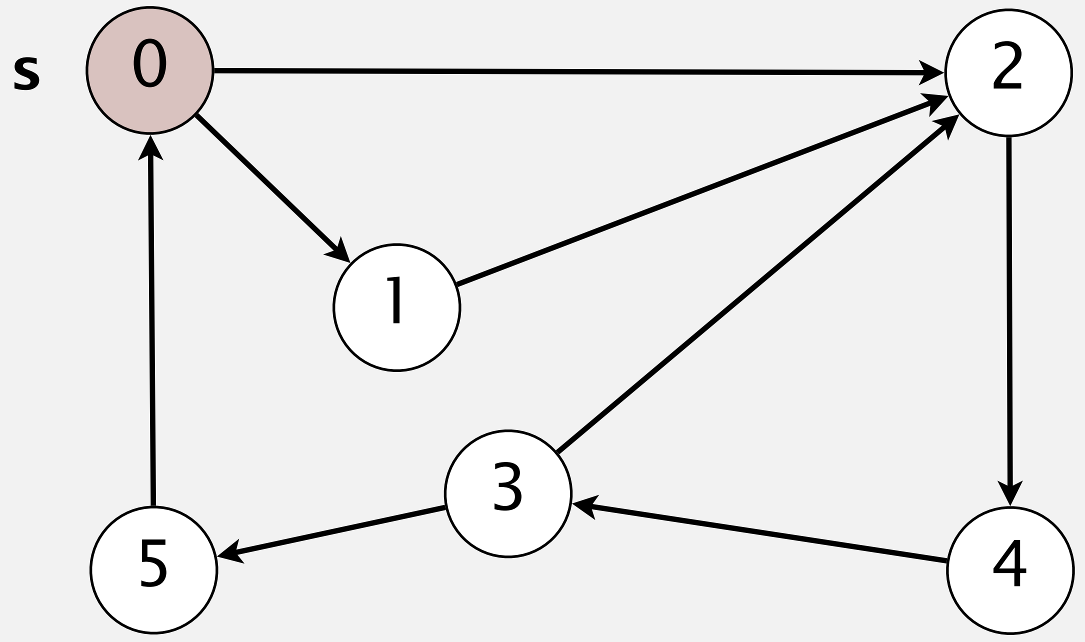
Proposition. In the worst case, BFS takes $\Theta(E + V)$ time.

Pf. Each vertex reachable from $s$ is visited once.

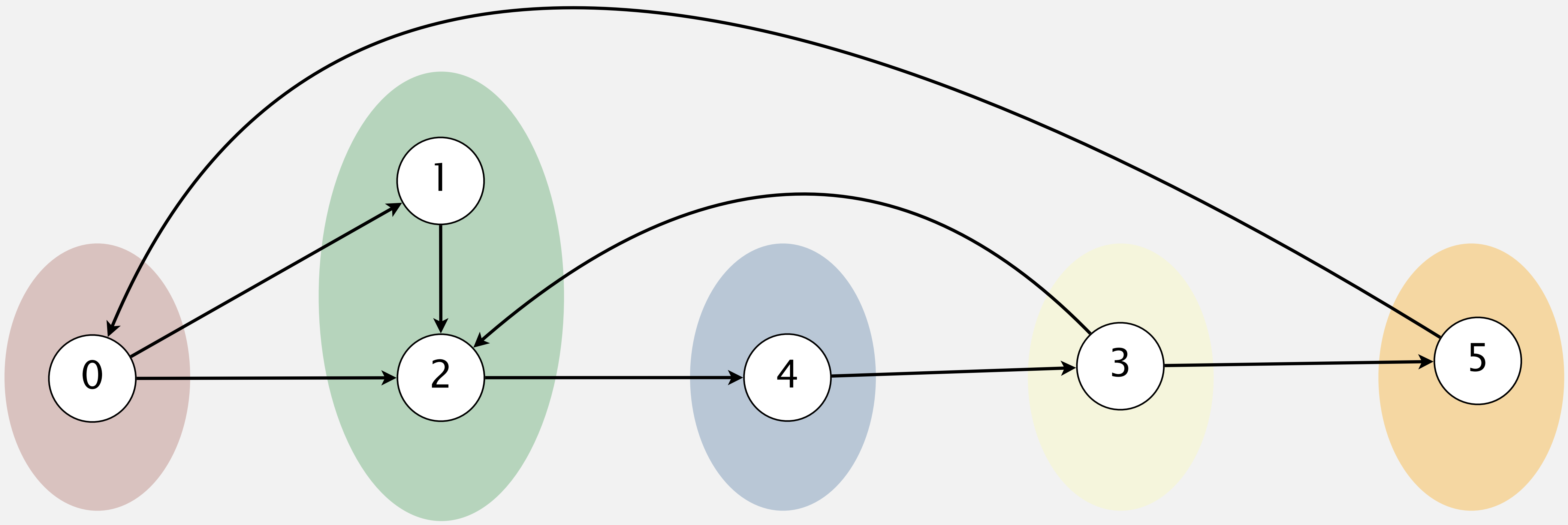
Proposition. BFS computes shortest paths from $s$.

Pf idea. BFS examines vertices in increasing distance (number of edges) from $s$.

invariant: queue contains vertices of distance $k$ from $s$,
followed by $\geq 0$ vertices of distance $k$+1 (and no other vertices)




digraph G




dist = 0 dist = 1 dist = 2 dist = 3 dist = 4

**What could happen if we mark a vertex when it is dequeued (instead of enqueued)?**

- **A.** Not guaranteed to find shortest paths.
- **B.** Takes exponential time.
- **C.** Both A and B.
- **D.** Neither A nor B.

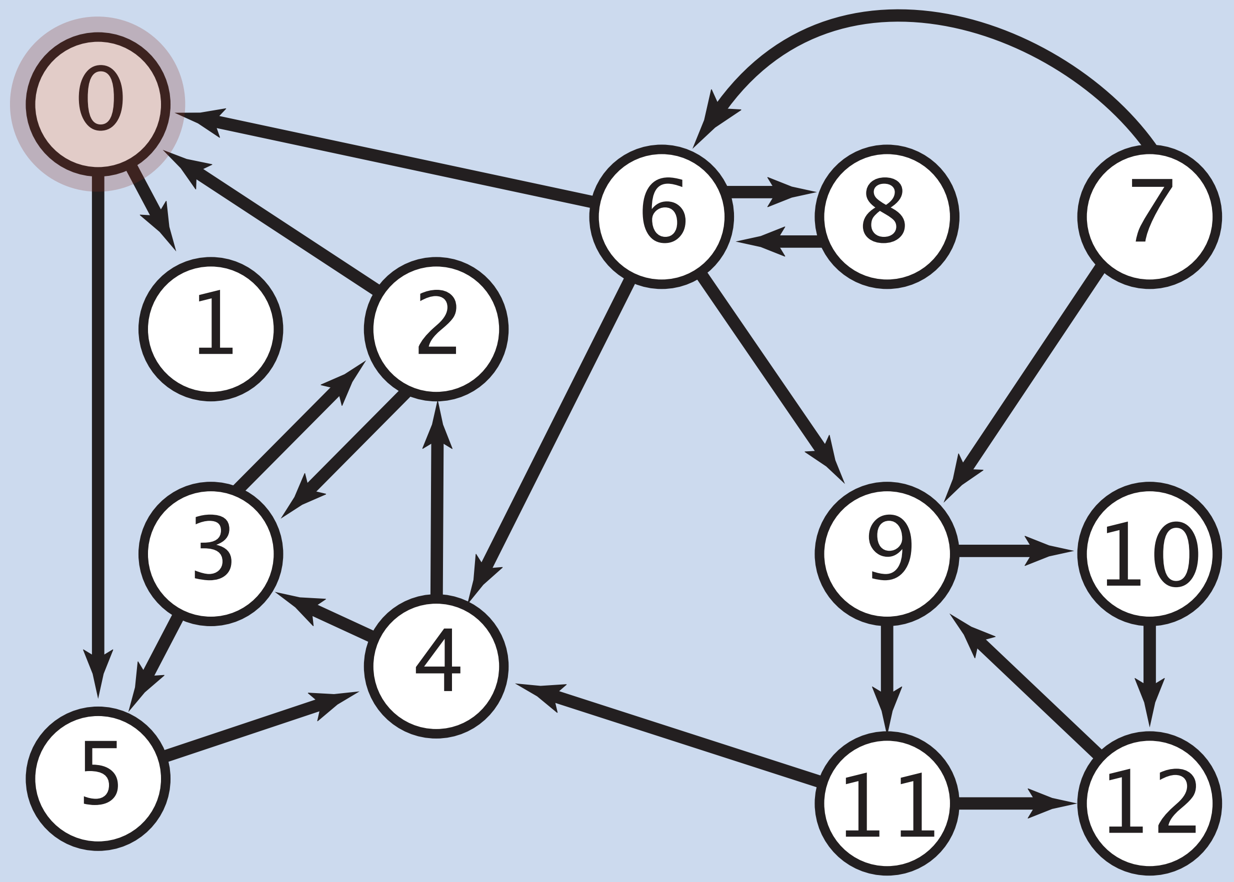
```
while (!queue.isEmpty()) {
   int v = queue.dequeue();
   for (int w : G.adj(v)) {
      if (!marked[w]) {
         q.enqueue(w);
         marked[w] = true;
         edgeTo[w] = v;
         distTo[w] = distTo[v] + 1;
      }
   }
}
```

# Single-sink shortest paths

Given a digraph and a target vertex $t$, find shortest path from every vertex to $t$.

Ex. $t = 0$

- Shortest path from $7$ is $7 \to 6 \to 0$.
- Shortest path from $5$ is $5 \to 4 \to 2 \to 0$.
- Shortest path from $12$ is $12 \to 9 \to 11 \to 4 \to 2 \to 0$.



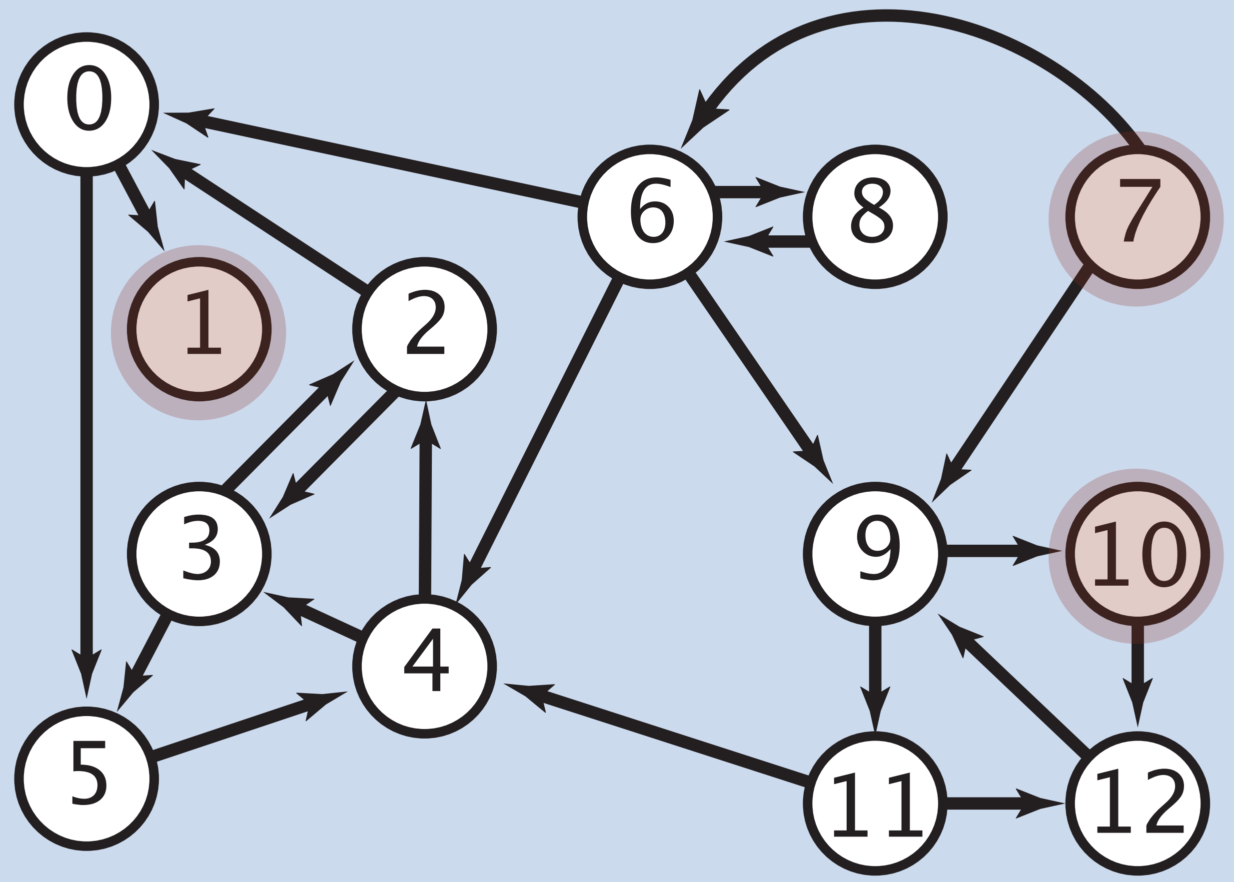

Q. How to implement single-target shortest paths algorithm?

# Multiple-source shortest paths

Given a digraph and a set of source vertices, find shortest path from any vertex in the set to every other vertex.

Ex. $S = \{ 1, 7, 10 \}$.

- Shortest path to $4$ is $7 \to 6 \to 4$.
- Shortest path to $5$ is $7 \to 6 \to 0 \to 5$.
- Shortest path to $12$ is $10 \to 12$.




needed for WordNet assignment

Q. How to implement multi-source shortest paths algorithm?

# Graphs and digraphs: quiz 2

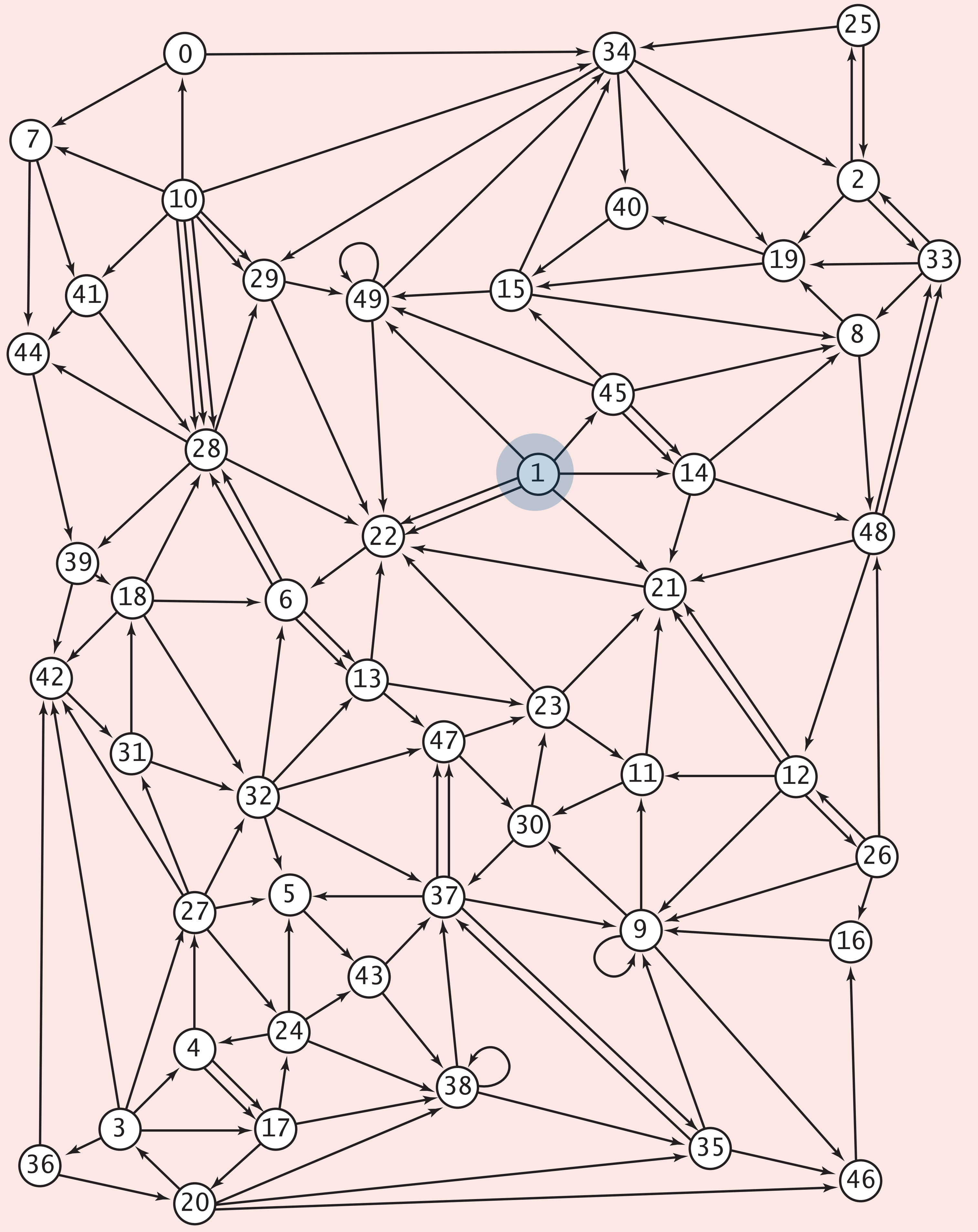
**Suppose that you want to design a web crawler. Which algorithm should you use?**

**A.** Depth-first search.

**B.** Breadth-first search.

**C.** Either A or B.

**D.** Neither A nor B.



# Web crawler output

**BFS crawl**

```
http://www.princeton.edu
http://www.w3.org
http://ogp.me
http://giving.princeton.edu
http://www.princetonartmuseum.org
http://www.goprincetontigers.com
http://library.princeton.edu
http://helpdesk.princeton.edu
http://tigernet.princeton.edu
http://alumni.princeton.edu
http://gradschool.princeton.edu
http://vimeo.com
http://princetonusg.com
http://artmuseum.princeton.edu
http://jobs.princeton.edu
http://odoc.princeton.edu
http://blogs.princeton.edu
http://www.facebook.com
http://twitter.com
http://www.youtube.com
http://deimos.apple.com
http://qeprize.org
http://en.wikipedia.org
...
```

**DFS crawl**

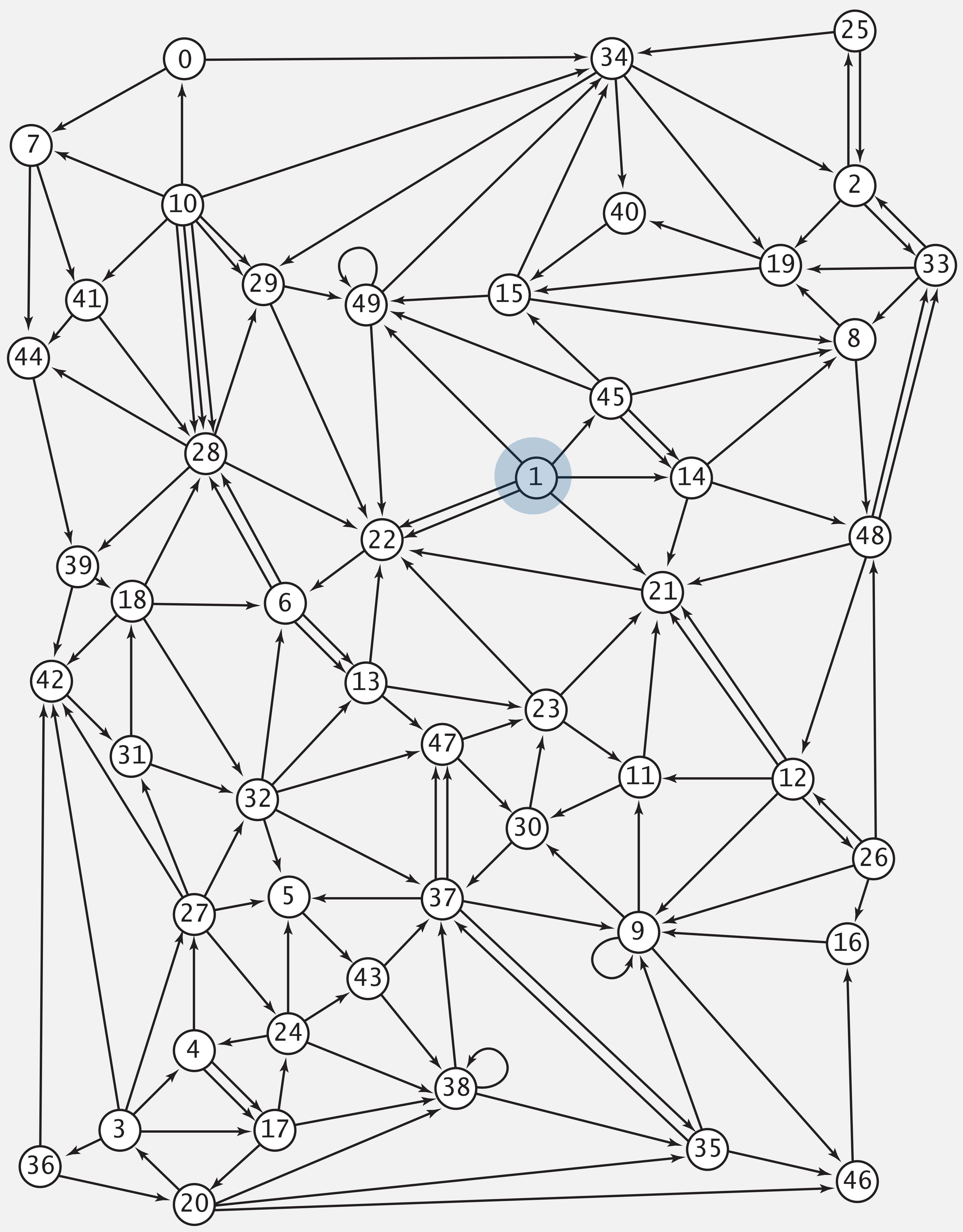
```
http://www.princeton.edu
http://deimos.apple.com
http://www.youtube.com
http://www.google.com
http://news.google.com
http://csi.gstatic.com
http://googlenewsblog.blogspot.com
http://labs.google.com
http://groups.google.com
http://img1.blogblog.com
http://feeds.feedburner.com
http:/buttons.googlesyndication.com
http://fusion.google.com
http://insidesearch.blogspot.com
http://agoogleaday.com
http://static.googleusercontent.com
http://searchresearch1.blogspot.com
http://feedburner.google.com
http://www.dot.ca.gov
http://www.TahoeRoads.com
http://www.LakeTahoeTransit.com
http://www.laketahoe.com
http://ethel.tahoeguide.com
...
```

# Breadth-first search application: web crawler

Goal. Crawl web, starting from some root web page, say `http://www.princeton.edu`.

Solution. [BFS with implicit digraph]

- Choose root web page as source $s$.
- Maintain a queue of websites to explore.
- Maintain a set of marked websites.
- Dequeue the next website and enqueue any unmarked websites to which it links.




Remark. Industrial-strength web crawlers use more sophisticated algorithms.

# Bare-bones web crawler: Java implementation

```
Queue<String> queue = new Queue<>();
SET<String> marked = new SET<>();

String root = "http://www.princeton.edu";
queue.enqueue(root);
marked.add(root);

while (!queue.isEmpty())
{
   String v = queue.dequeue();
   StdOut.println(v);
   In in = new In(v);
   String input = in.readAll();

   String regexp = "http://(\\w+\\.)+(\\w+)";
   Pattern pattern = Pattern.compile(regexp);
   Matcher matcher = pattern.matcher(input);

   while (matcher.find())
   {
      String w = matcher.group();

      if (!marked.contains(w))
      {
          marked.add(w);
          queue.enqueue(w);
      }
   }
}
```

queue of websites to crawl

set of marked websites

start crawling from root website

read in raw HTML from next website in queue

use regular expression to find all URLs in website of form `http://xxx.yyy.zzz`
[crude pattern misses relative URLs]

if unmarked, mark and enqueue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# 4. Graphs and Digraphs II

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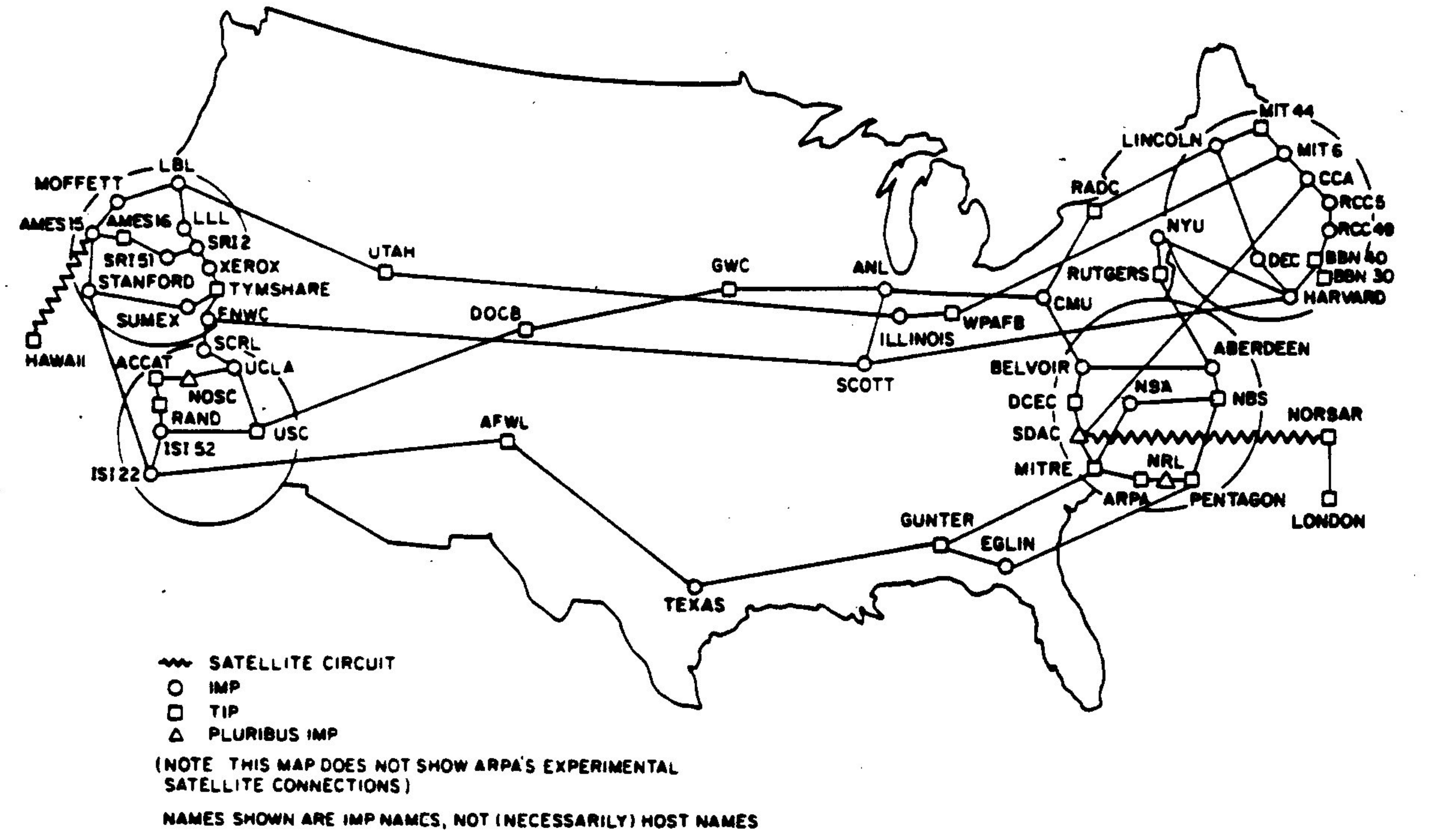
- *breadth-first search (in undirected graphs)*
- *topological sort*
- *challenges*

# Breadth-first search application: routing

Fewest number of hops in a communication network.




ARPANET, July 1977

# Breadth-first search in undirected graphs

Problem. Find path between $s$ and each other vertex that uses fewest edges.

Solution. Treat as a digraph, replacing each undirected edge with two antiparallel edges.

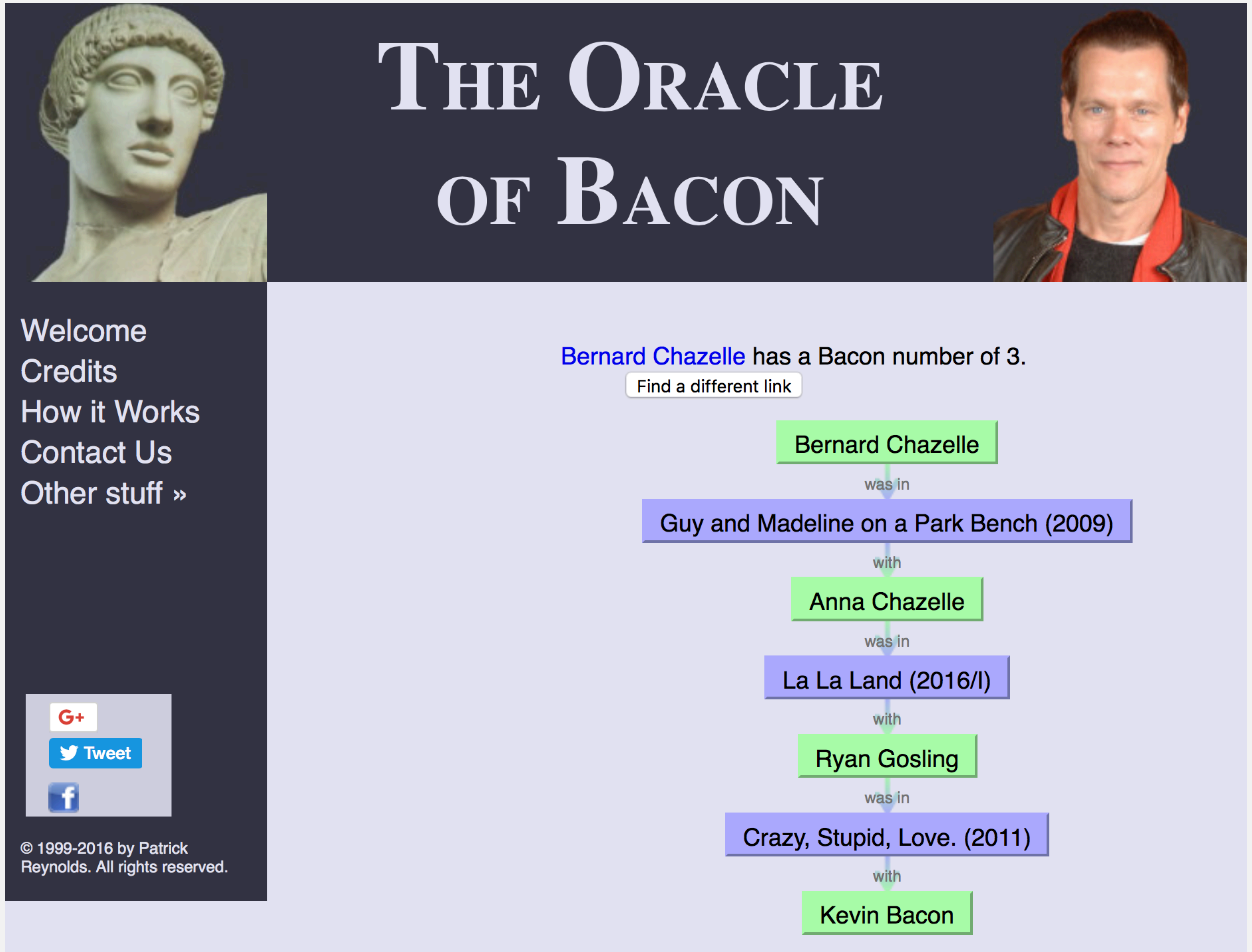
**BFS (from source vertex s)**

**Add s to FIFO queue and mark s.**

**Repeat until the queue is empty:**

- **remove the least recently added vertex v**
- **for each unmarked vertex w adjacent to v:**
  **add w to queue and mark w.**

# Breadth-first search application:  Kevin Bacon numbers



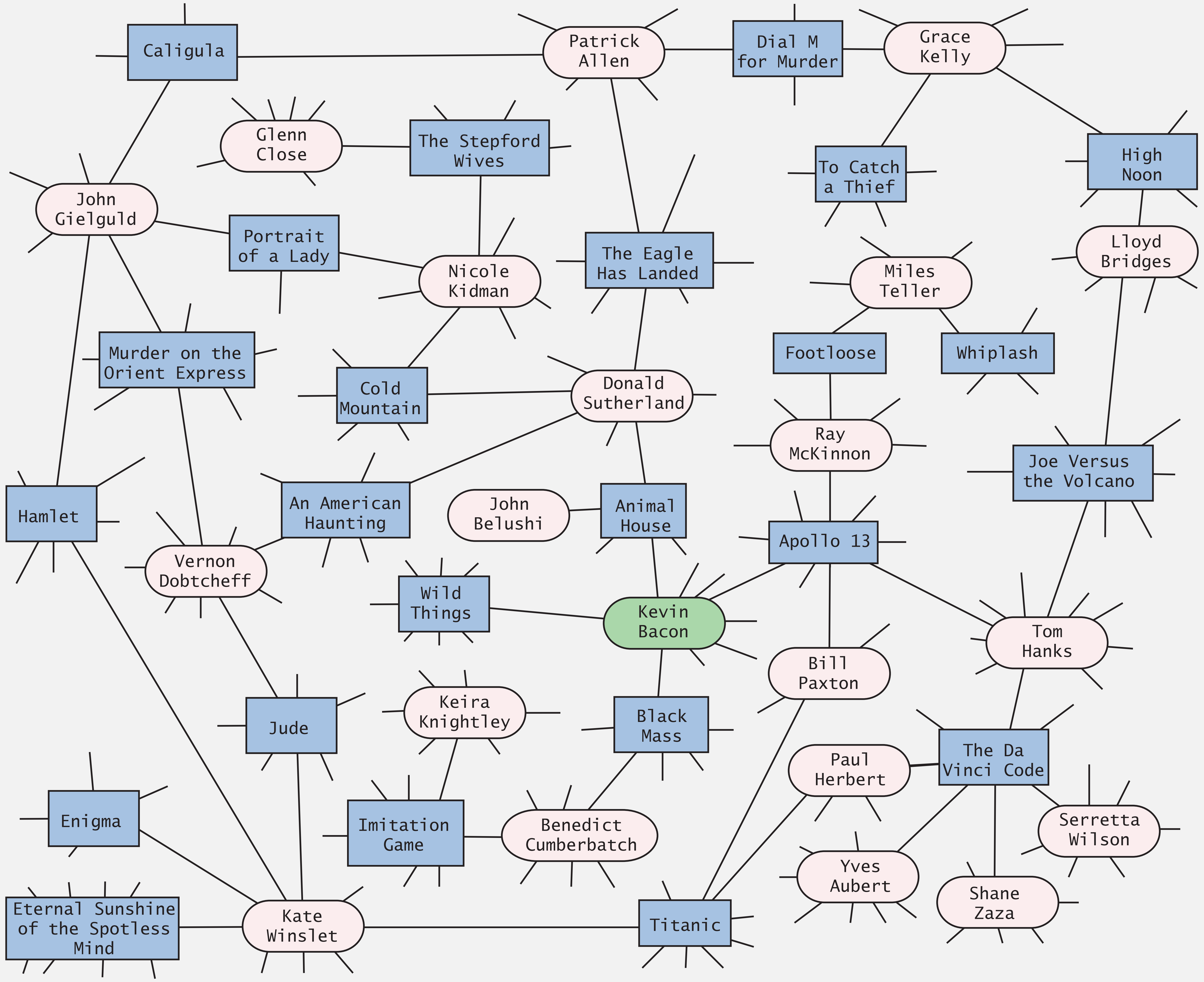

https://oracleofbacon.org


Endless Games board game


SixDegrees iPhone App

# Kevin Bacon graph

- Include one vertex for each performer and one for each movie.
- Connect a movie to all performers that appear in that movie.
- Compute shortest paths between $s$ = Kevin Bacon and every other performer.



Algorithms

Robert Sedgewick | Kevin Wayne

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# 4. Graphs and Digraphs II

- *breadth-first search (in digraphs)*
- *breadth-first search (in undirected graphs)*
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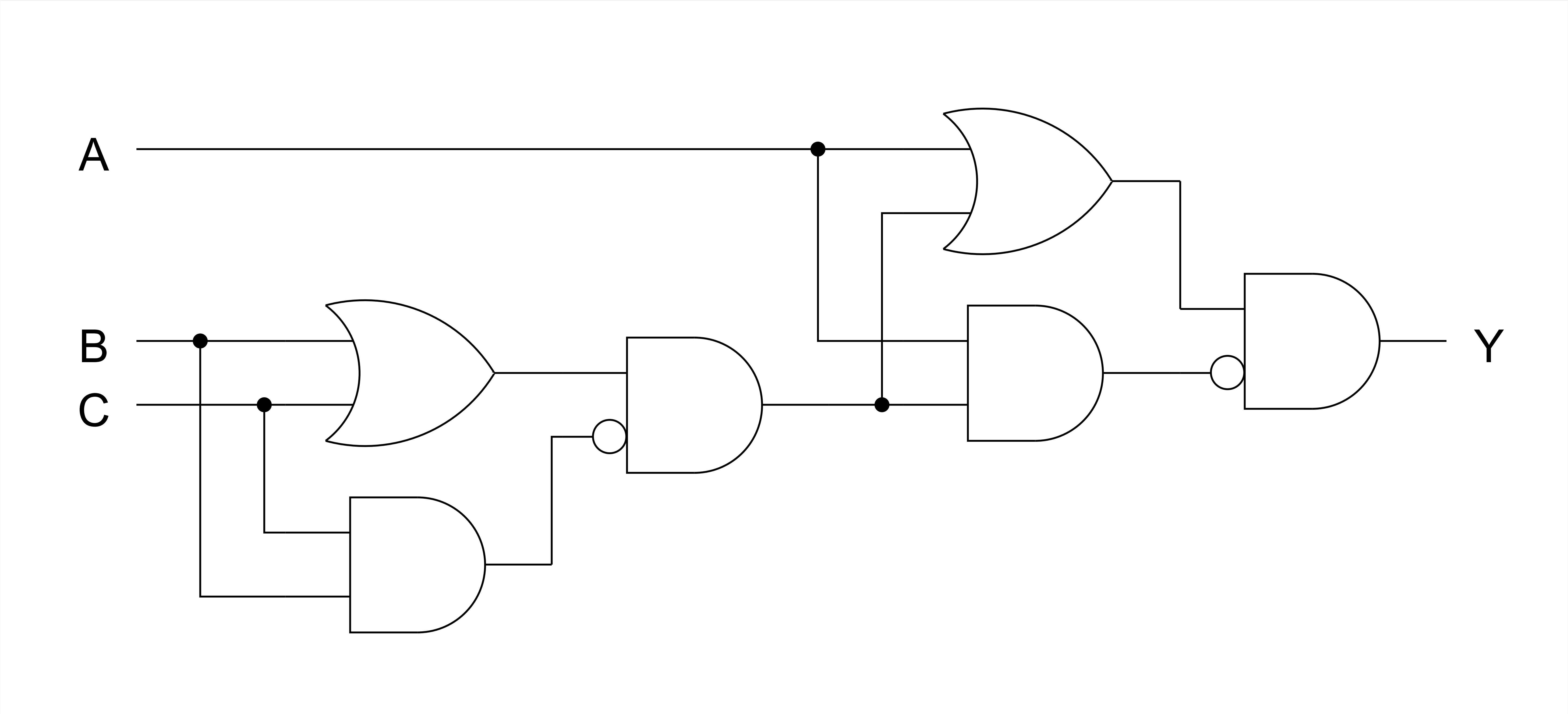
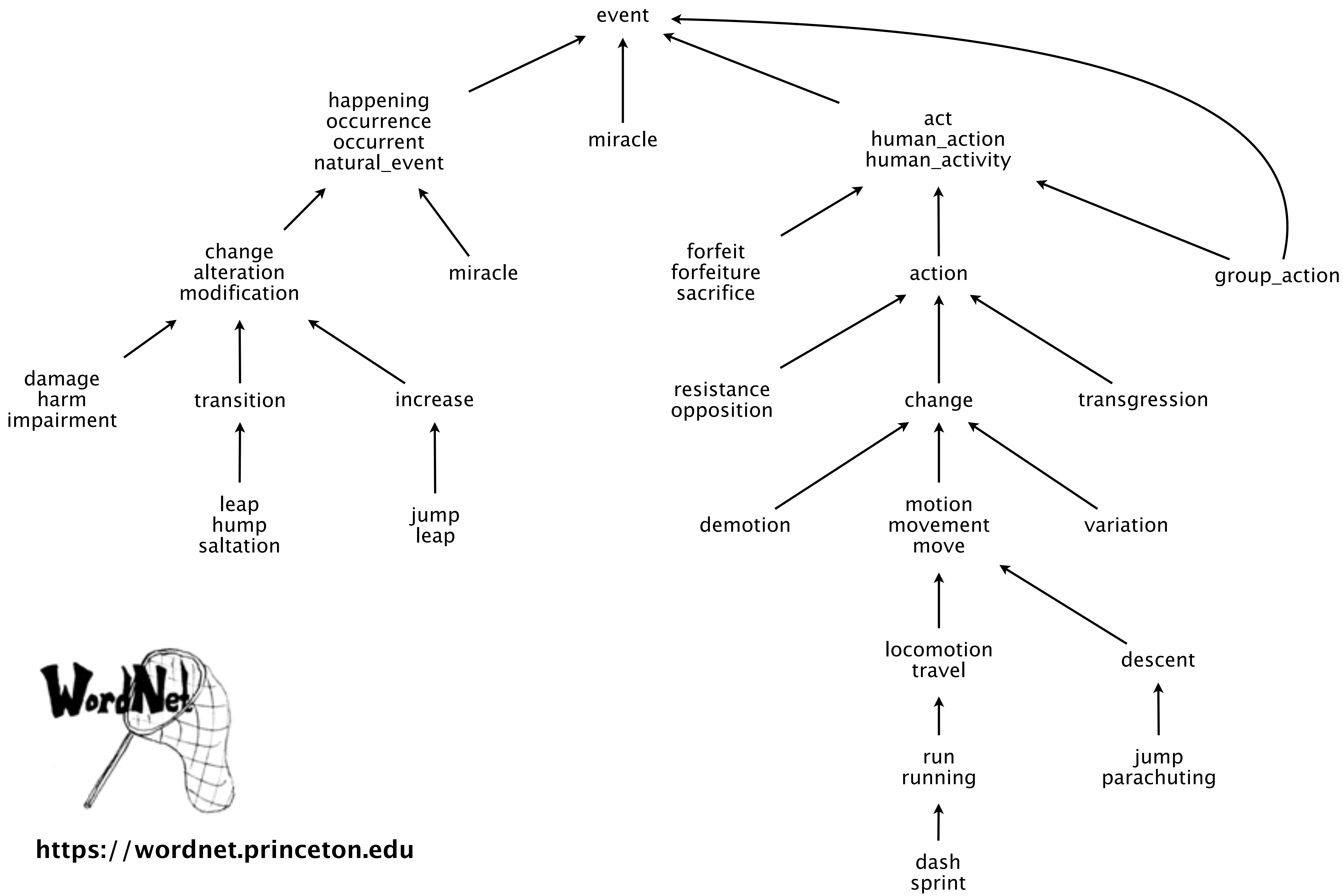
# Combinational circuit

Vertex = logical gate; edge = wire.



# WordNet digraph

Vertex = synset; edge = hypernym relationship.




**https://wordnet.princeton.edu**

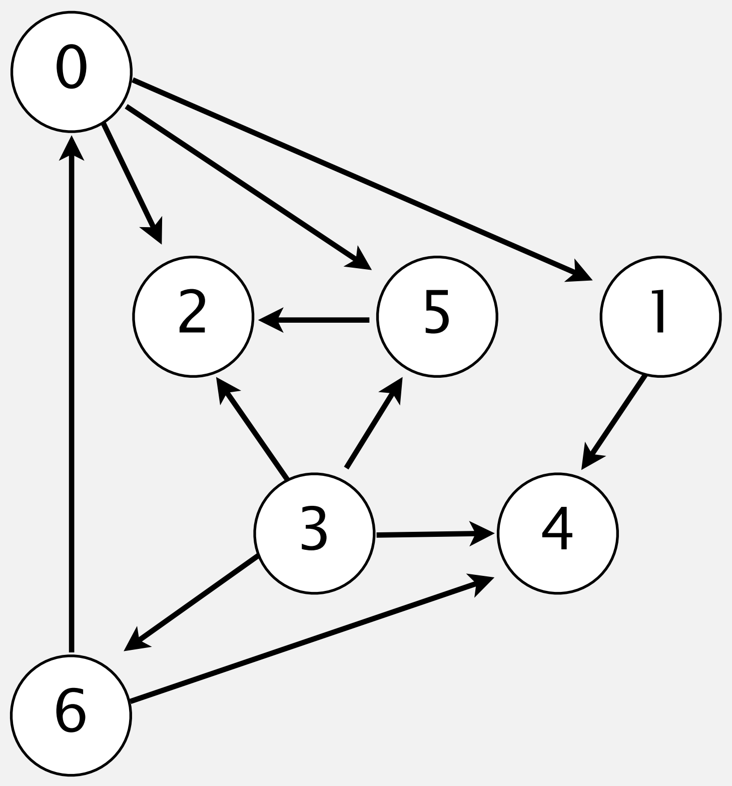
# Precedence scheduling

Goal. Given a set of tasks to be completed with precedence constraints, in which order should we schedule the tasks?

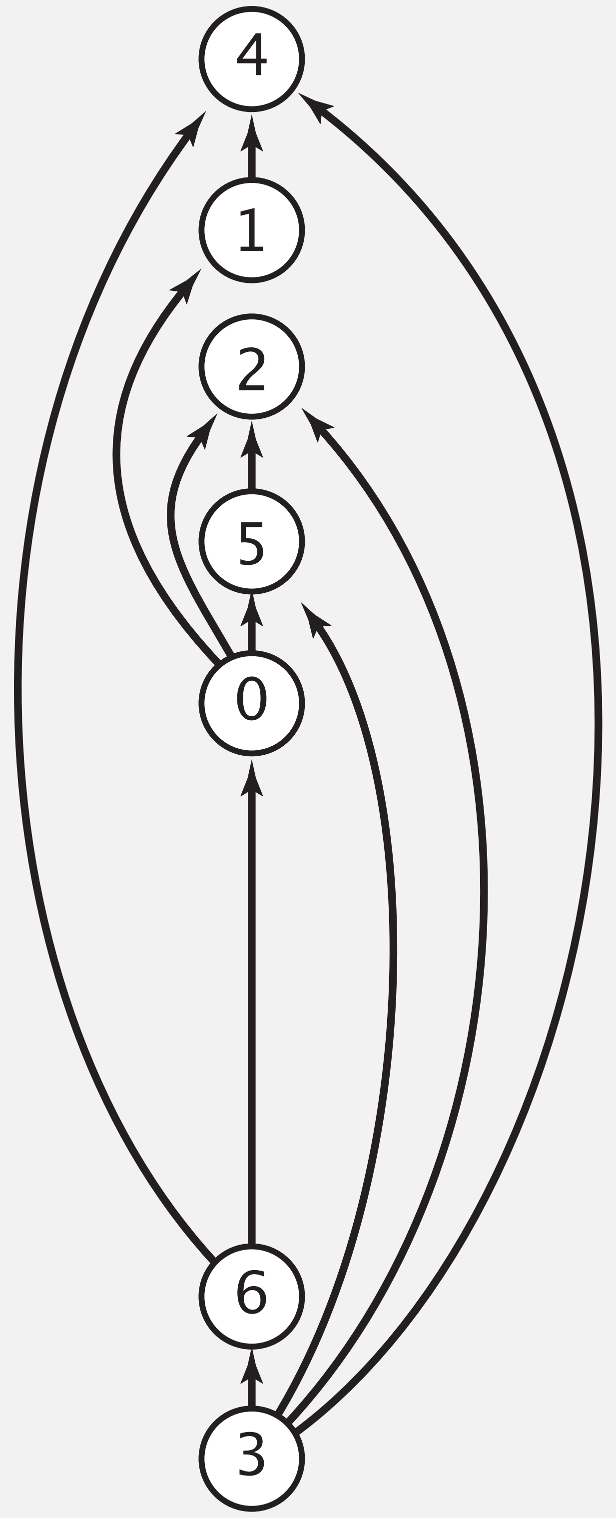
Digraph model. vertex = task; edge = precedence constraint.

0. Math for CS
1. Complexity Theory
2. Machine Learning
3. Intro to CS
4. Cryptography
5. Scientific Computing
6. Algorithms

**tasks**




**precedence constraint graph**




**feasible schedule**

# Topological sort

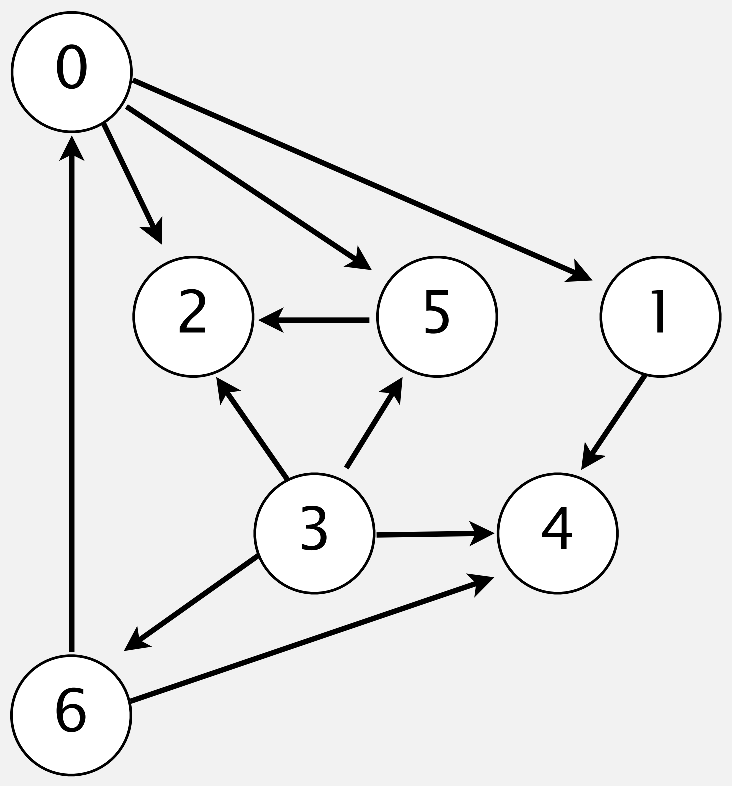
DAG. Directed acyclic graph.

Topological sort. Redraw DAG so all edges point upwards.

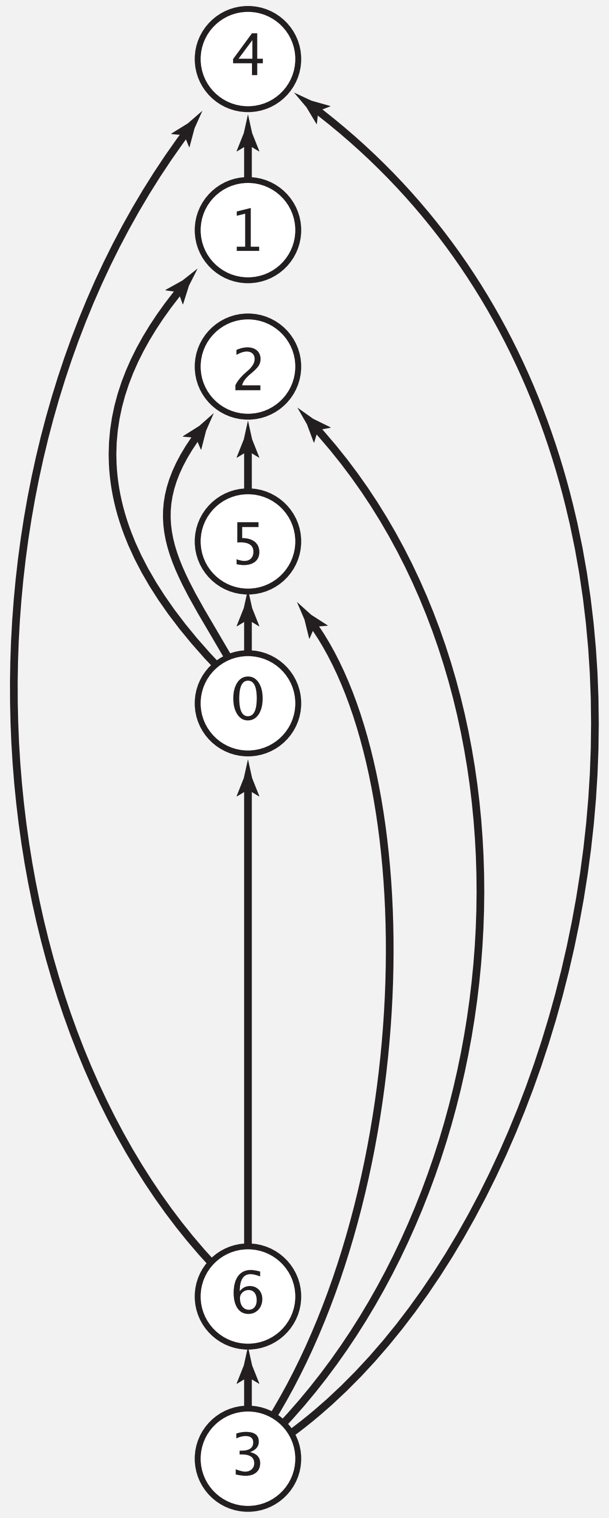
edges in DAG define a "partial order" for vertices

| | |
|---|---|
| $0 \to 5$ | $0 \to 2$ |
| $0 \to 1$ | $3 \to 6$ |
| $3 \to 5$ | $3 \to 4$ |
| $5 \to 2$ | $6 \to 4$ |
| $6 \to 0$ | $3 \to 2$ |
| $1 \to 4$ | |

**directed edges**




**DAG**




**topological order**

## Directed graphs: quiz 4

**Suppose that you want to topologically sort the vertices in a DAG.**

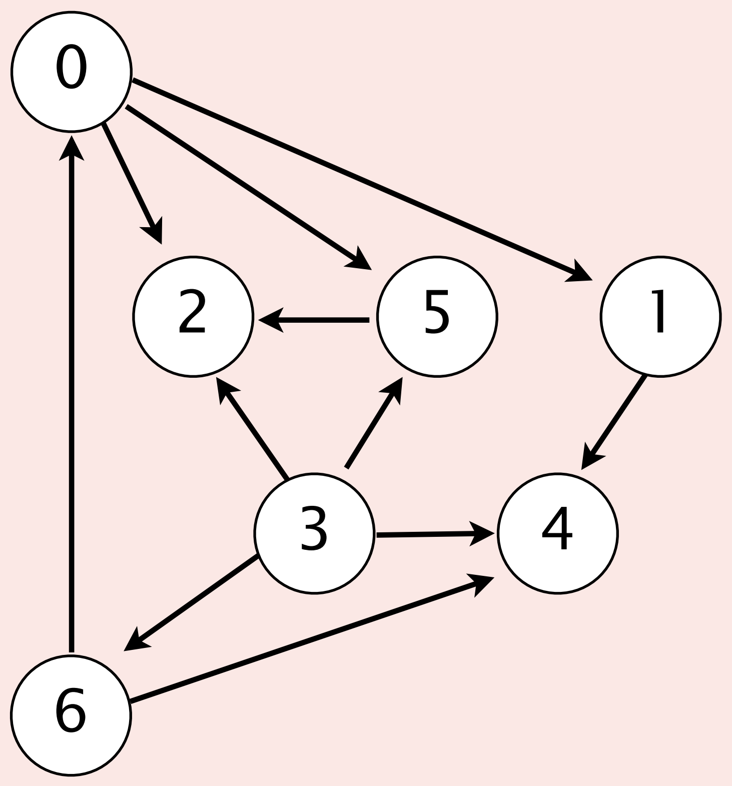
**Which graph-search algorithm should you use?**

**A.** Depth-first search.

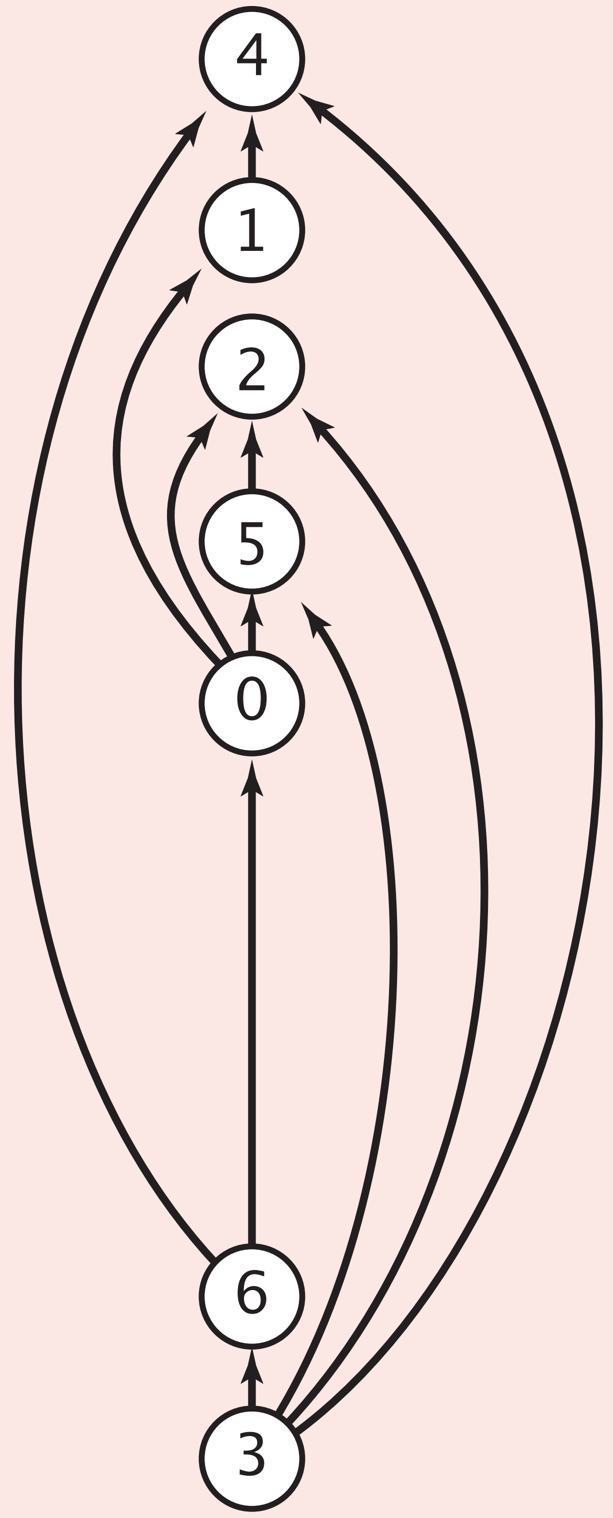
**B.** Breadth-first search.

**C.** Either A or B.

**D.** Neither A nor B.




**DAG**

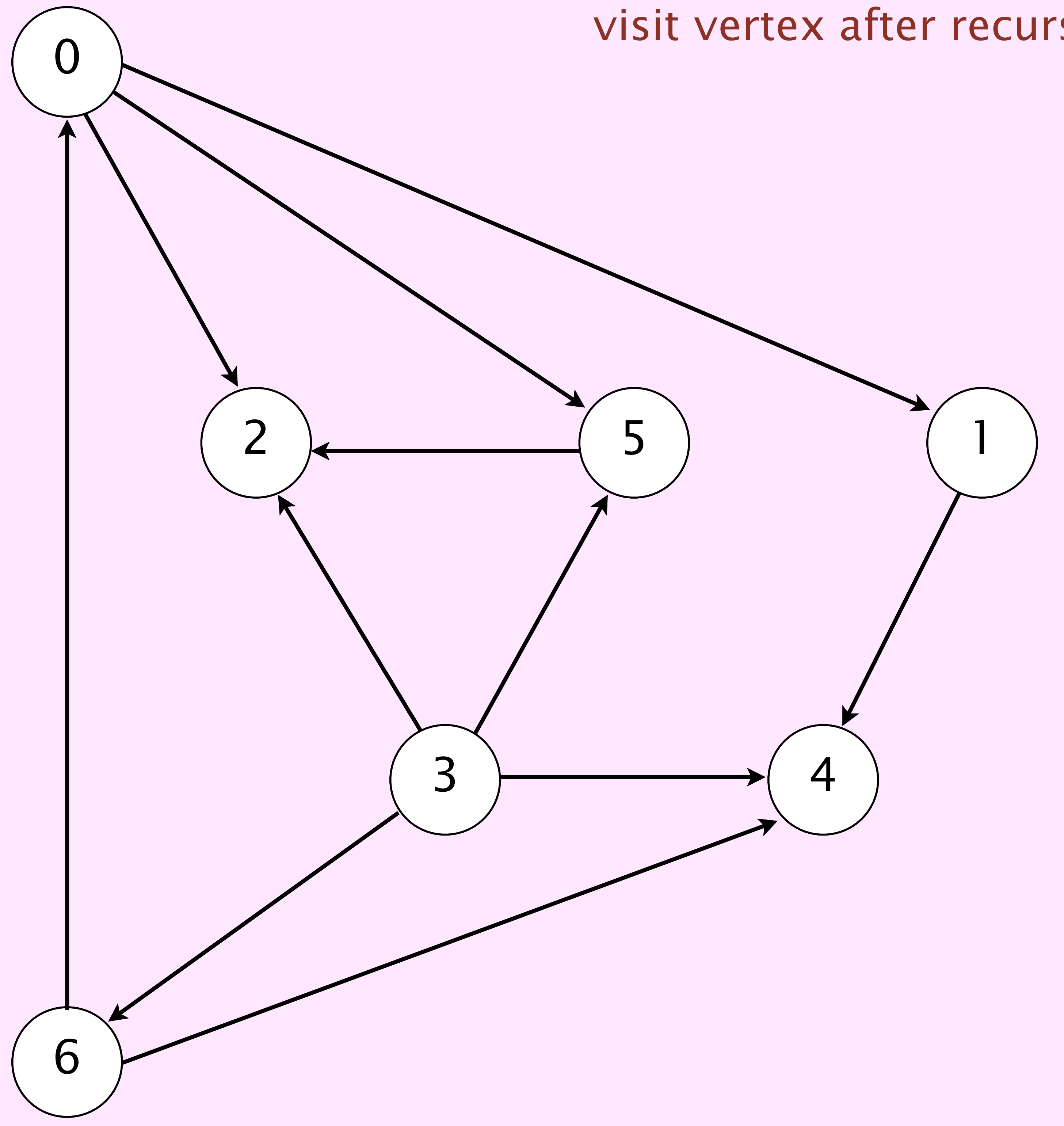



**topological order**

# Topological sort demo

- Run depth-first search.
- Return vertices in reverse DFS postorder.

visit vertex after recursive calls



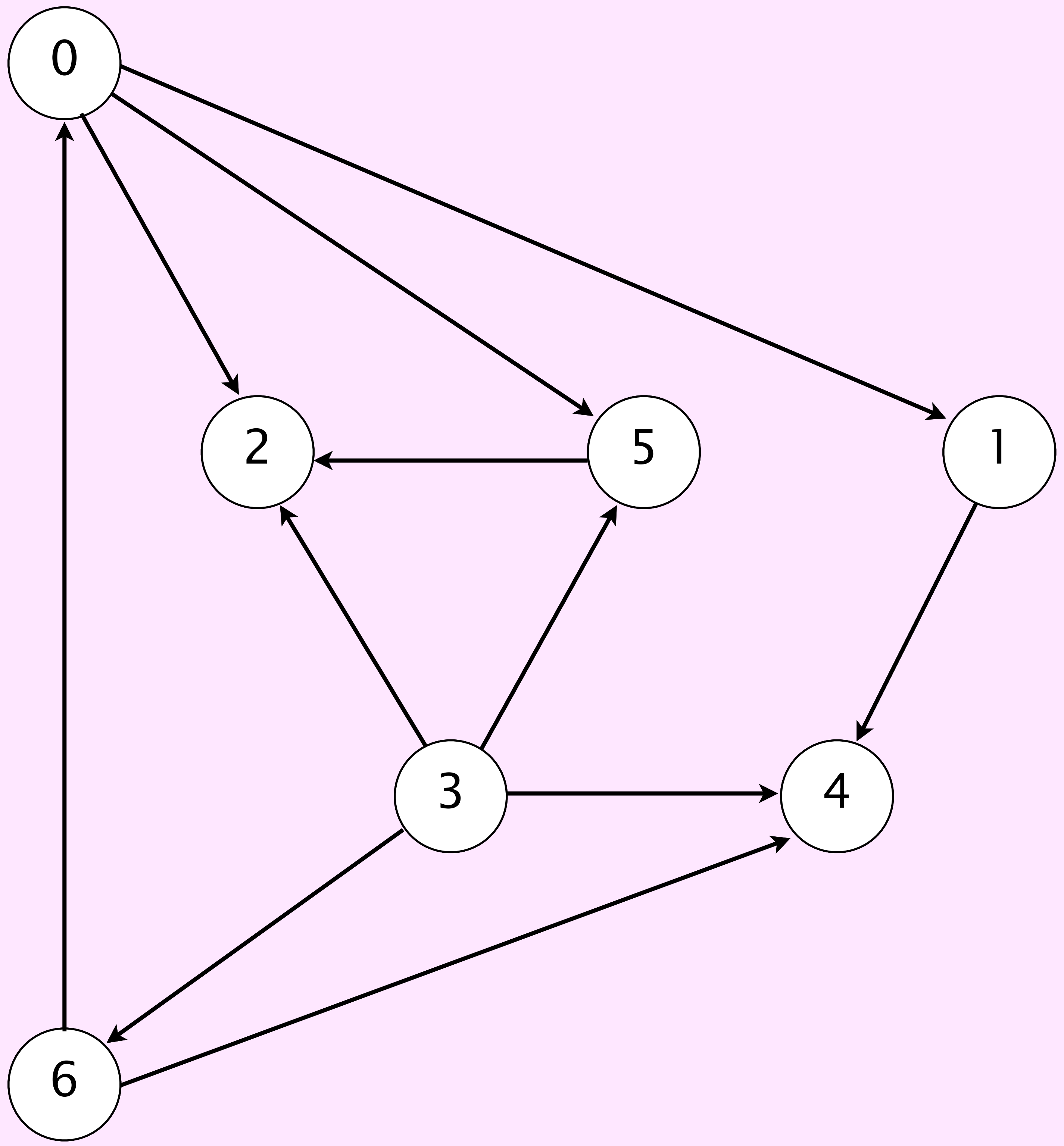

a directed acyclic graph

**tinyDAG7.txt**

```
7
11
 0  5
 0  2
 0  1
 3  6
 3  5
 3  4
 5  2
 6  4
 6  0
 3  2
```

# Topological sort demo

- Run depth-first search.
- Return vertices in reverse DFS postorder.




**DFS postorder**

4 1 2 5 0 6 3

**topological order**
**(reverse DFS postorder)**

3 6 0 5 2 1 4

**done**

# Depth-first search: reverse postorder

```
public class DepthFirstOrder
{
   private boolean[] marked;
   private Stack<Integer> reversePostorder;

   public DepthFirstOrder(Digraph G)
   {
      reversePostorder = new Stack<>();
      marked = new boolean[G.V()];
      for (int v = 0; v < G.V(); v++)
         if (!marked[v]) dfs(G, v);
   }

   private void dfs(Digraph G, int v)
   {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w);
      reversePostorder.push(v);
   }

   public Iterable<Integer> reversePostorder()
   {  return reversePostorder;  }

}
```

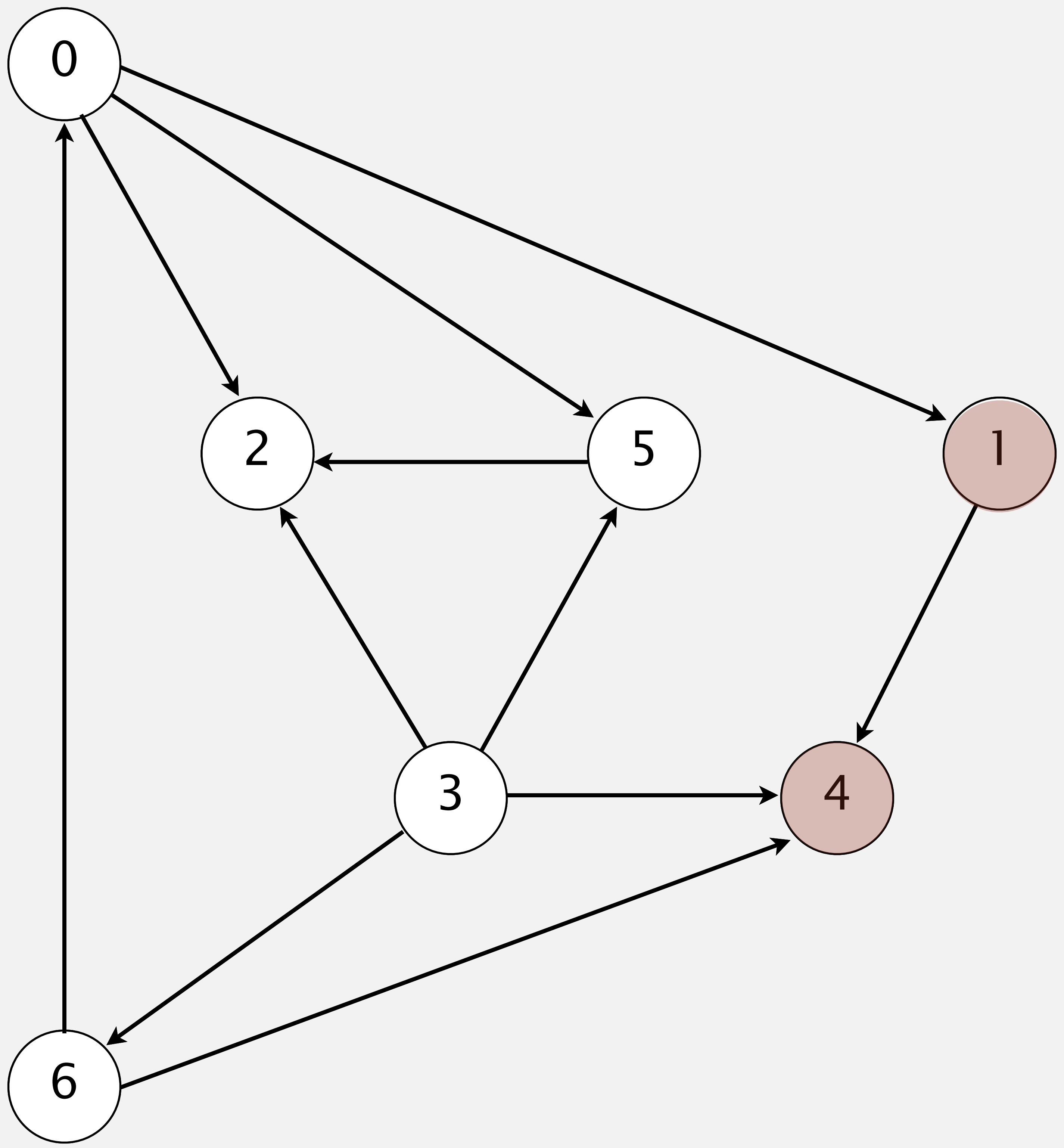
run DFS from all vertices

returns all vertices in "reverse DFS postorder"

# Topological sort in a DAG: intuition

Why is the reverse DFS postorder a topological order?

- First vertex in DFS postorder has outdegree 0.
- Second vertex in DFS postorder can point only to first vertex.
- ...




**DFS postorder**

4 1 2 5 0 6 3

**topological order**
**(reverse DFS postorder)**

3 6 0 5 2 1 4

# Topological sort in a DAG: correctness proof

Proposition. Reverse DFS postorder of a DAG is a topological order.

Pf. Consider any edge $v \rightarrow w$. When `dfs(v)` is called:

- Case 1: `dfs(w)` has already been called and returned.
  - thus, $w$ appears before $v$ in DFS postorder
- Case 2: `dfs(w)` has not yet been called.
  - `dfs(w)` will get called directly or indirectly by `dfs(v)`
  - so, `dfs(w)` will return before `dfs(v)` returns
  - thus, $w$ appears before $v$ in DFS postorder
- Case 3: `dfs(w)` has already been called, but has not yet returned.
  - function-call stack contains directed path from $w$ to $v$
  - edge $v \rightarrow w$ would complete a directed cycle
  - contradiction (it's a DAG)

```
dfs(0)
   dfs(1)
      dfs(4)
      4 done
   1 done
   dfs(2)
   2 done
   dfs(5)
      check 2
   5 done
0 done
check 1
check 2
dfs(3)
   check 2
   check 4
   check 5
   dfs(6)
      check 0
      check 4
   6 done
3 done
check 4
check 5
check 6
done
```

$v = 3$ → `dfs(3)`

case 1 ($w = 2, 4, 5$) → `check 2`, `check 4`, `check 5`

case 2 ($w = 6$) → `dfs(6)`, `6 done`

# Topological sort in a DAG: running time

Proposition. For any DAG, the DFS algorithm computes a topological order in $\Theta(E + V)$ time.

Pf. For every vertex $v$, there is exactly one call to `dfs(v)`.

critical that vertices are marked
(and never unmarked)

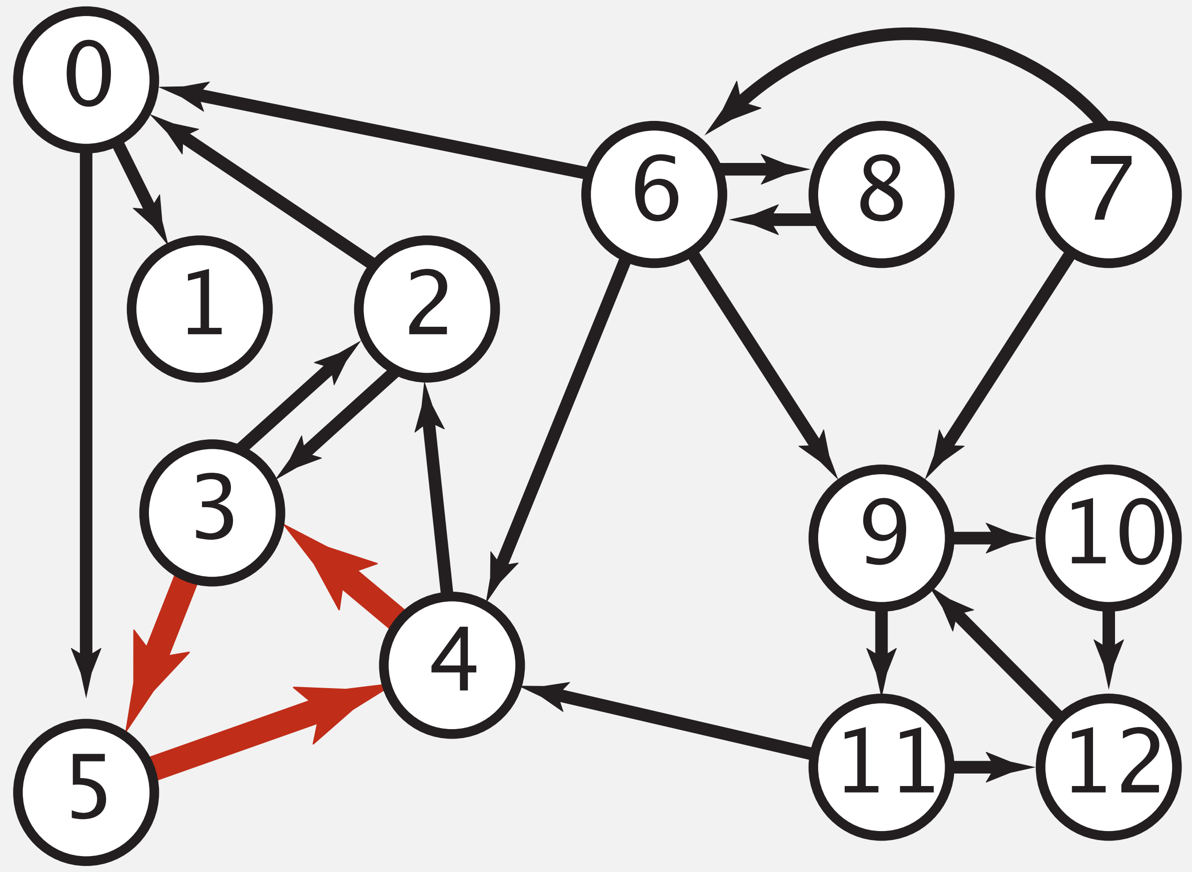
Q. What if we run algorithm on a digraph that is not a DAG?

# Directed cycle detection

Proposition. A digraph has a topological order if and only if contains no directed cycle.

Pf.

- If directed cycle, topological order impossible.
- If no directed cycle, DFS-based algorithm finds a topological order.



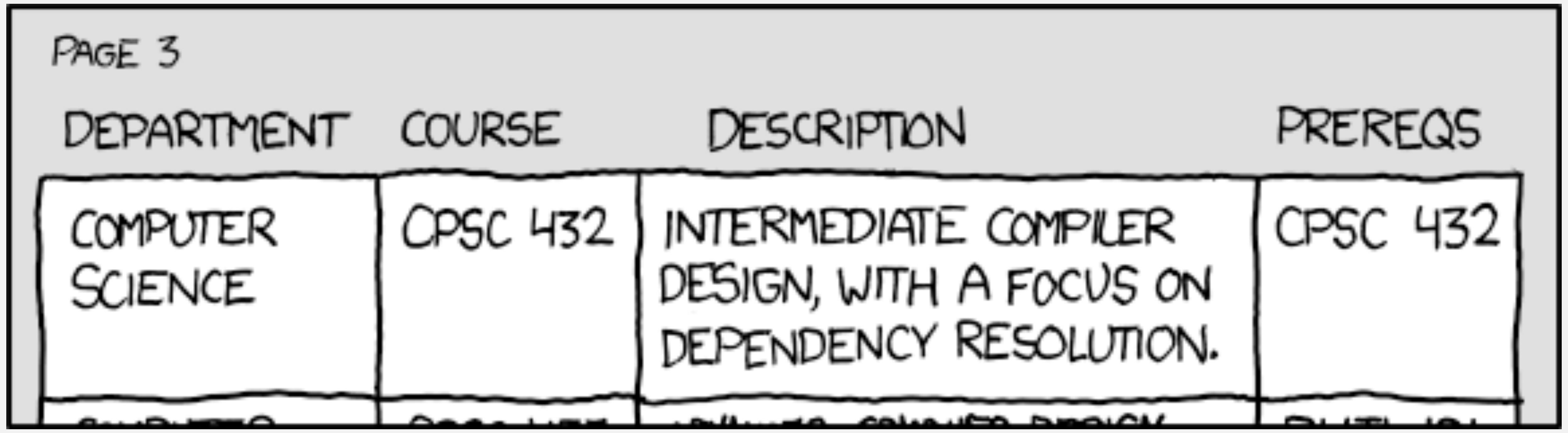

a digraph with a directed cycle

Goal. Given a digraph, find a directed cycle.

Solution. DFS. What else? See textbook/precept.

# Directed cycle detection application: precedence scheduling

Scheduling. Given a set of tasks to be completed with precedence constraints, in which order should we schedule the tasks?




https://xkcd.com/754

Remark. A directed cycle implies scheduling problem is infeasible.

# Directed cycle detection application: cyclic inheritance

The Java compiler does directed cycle detection.

```java
public class A extends B
{
   ...
}
```

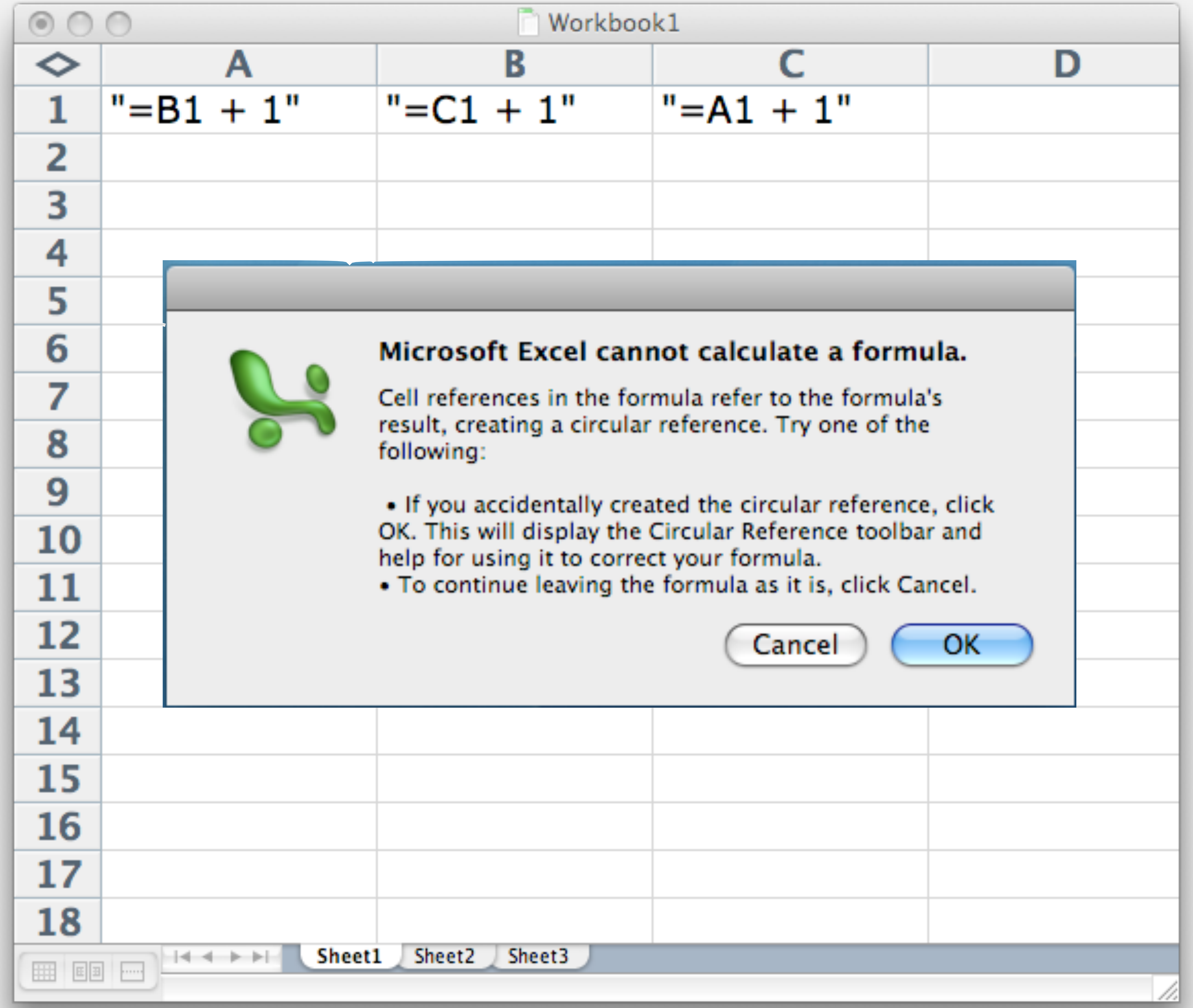
```java
public class B extends C
{
   ...
}
```

```java
public class C extends A
{
   ...
}
```

```
~/Desktop/graph> javac A.java
A.java:1: cyclic inheritance involving A
public class A extends B { }
       ^
1 error
```

# Directed cycle detection application: spreadsheet recalculation

Microsoft Excel does directed cycle detection.



Algorithms

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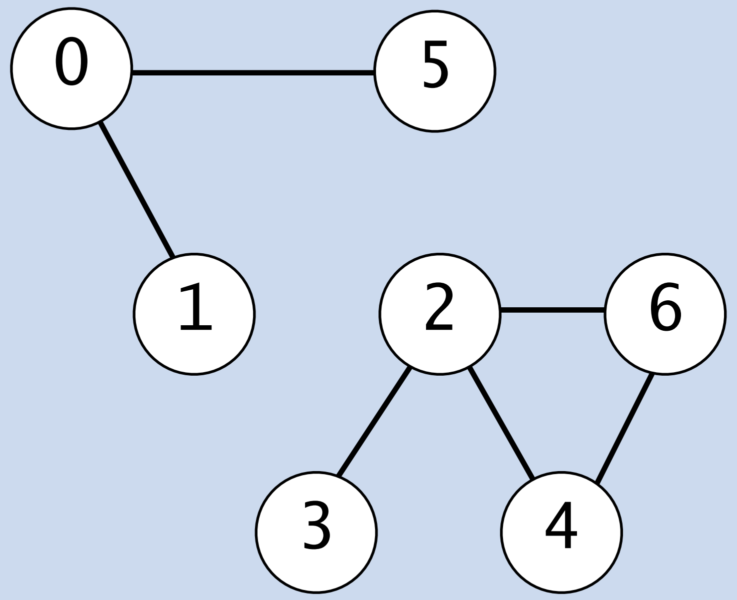
# 4. Graphs and Digraphs II

- breadth-first search (in digraphs)
- breadth-first search (in undirected graphs)
- topological sort
- *challenges*

# Graph-processing challenge 1

Problem. Identify connected components.




0–1
0–5
2–6
2–3
2–4
4–6

**How difficult?**

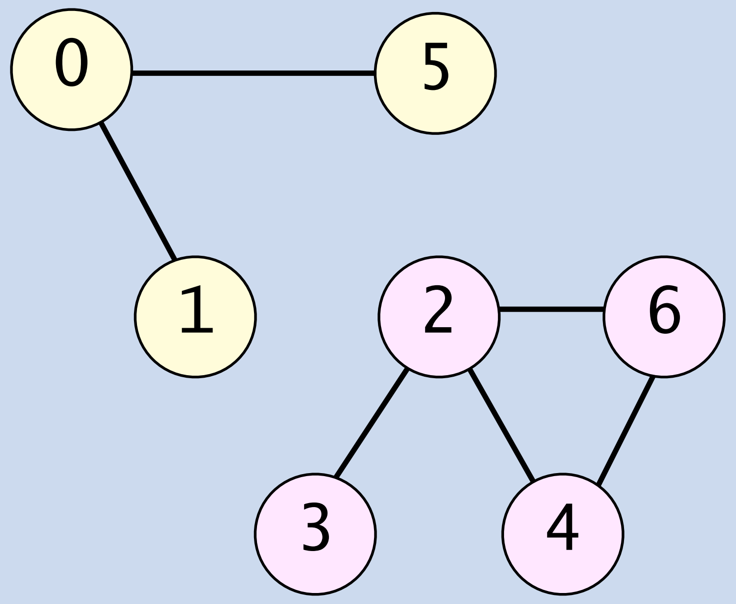
**A.** Any programmer could do it.

**B.** Diligent algorithms student could do it.

**C.** Hire an expert.

**D.** Intractable.

**E.** No one knows.



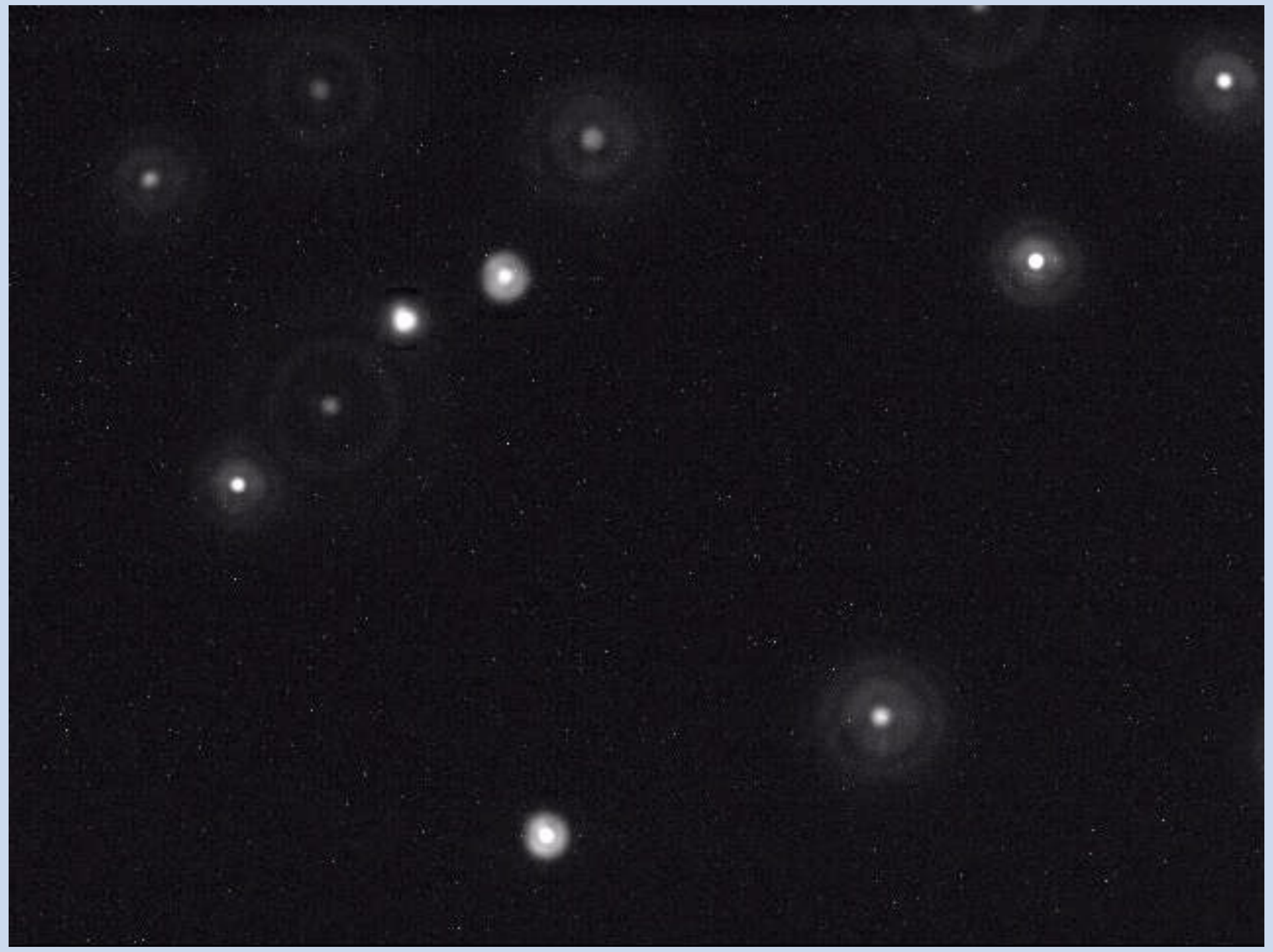

| v | id[] |
|---|---|
| 0 | 0 |
| 1 | 0 |
| 2 | 1 |
| 3 | 1 |
| 4 | 1 |
| 5 | 0 |
| 6 | 1 |

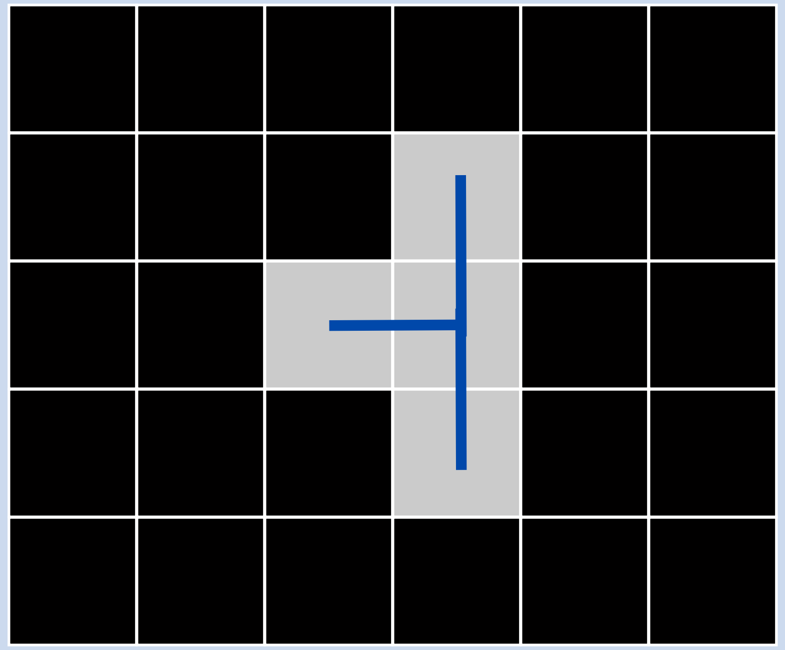
# Graph-processing challenge 1

Problem. Identify connected components.

Particle detection. Given grayscale image of particles, identify "blobs."

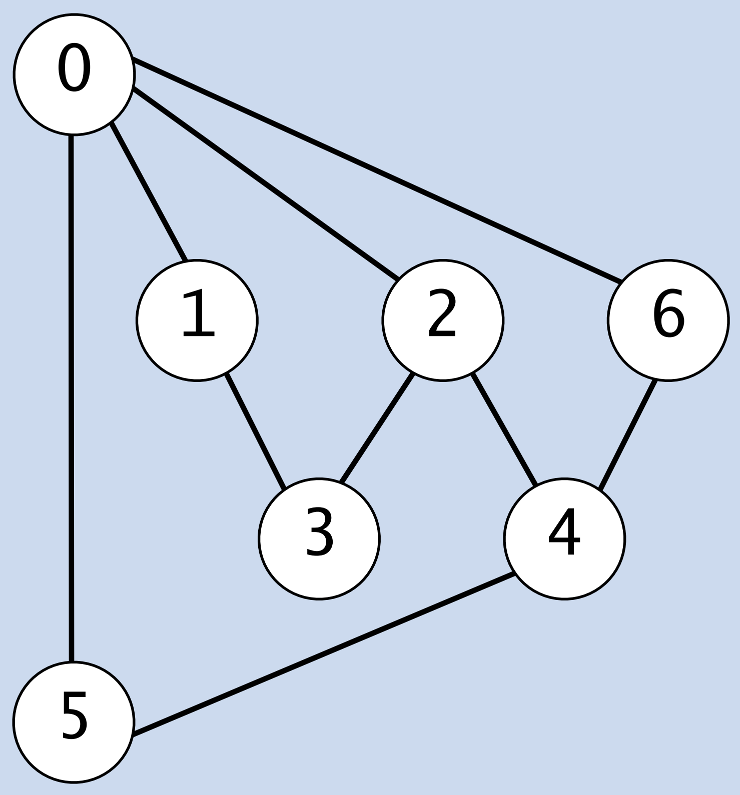
- Vertex: pixel.
- Edge: between two adjacent pixels with grayscale value ≥ 70.
- Blob: connected component of 20–30 pixels.





# Graph-processing challenge 2

Problem. Is a graph bipartite?




0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6

**How difficult?**

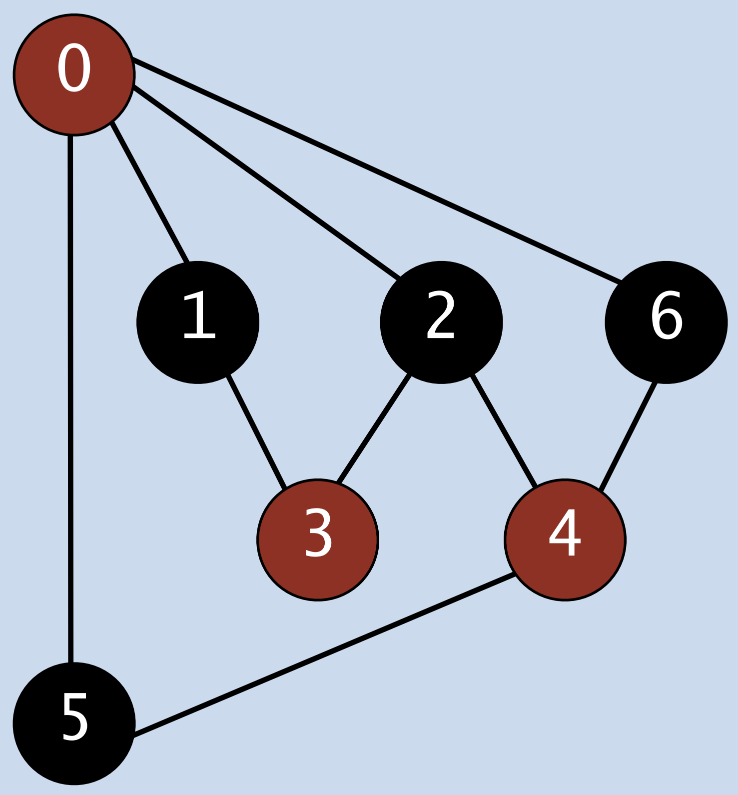
**A.** Any programmer could do it.

**B.** Diligent algorithms student could do it.

**C.** Hire an expert.

**D.** Intractable.

**E.** No one knows.

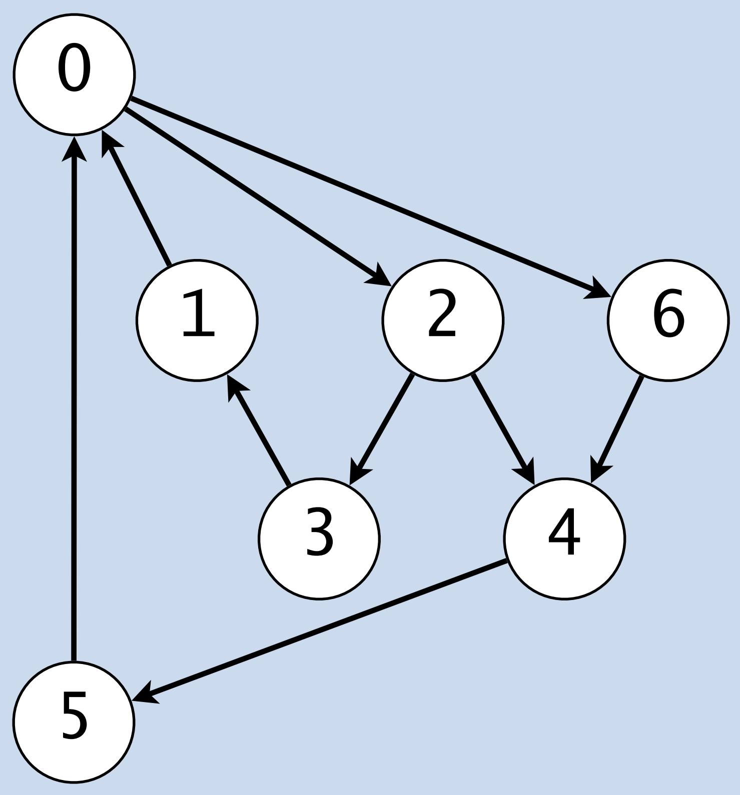



0-1
0-2
0-5
0-6
1-3
2-3
2-4
4-5
4-6

{ 0, 3, 4 }

# Graph-processing challenge 3

**Problem.** Find the girth of a digraph (length of a shortest directed cycle).




0→2
0→6
1→0
2→3
3→1
2→4
4→5
5→0
6→4

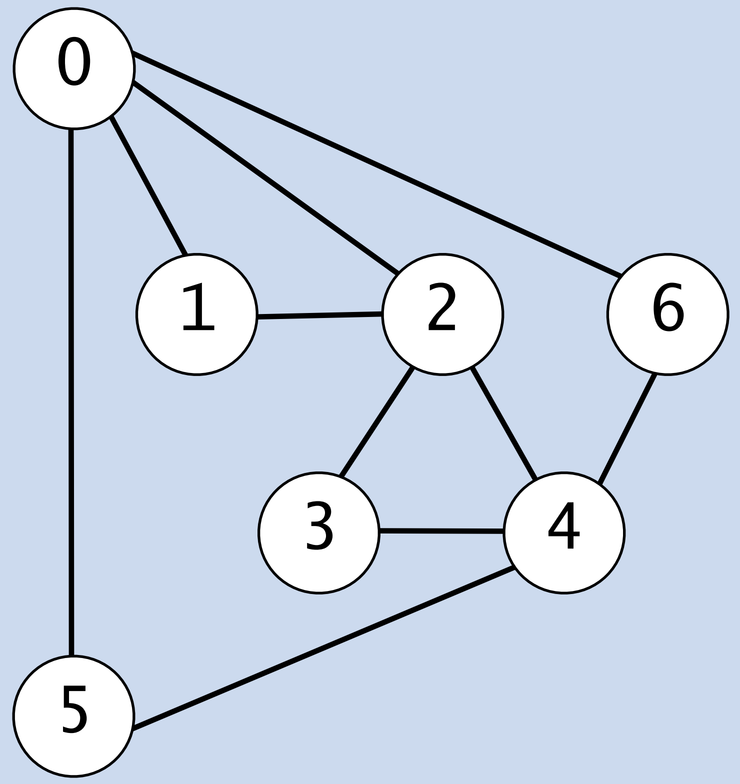
0→2→3→1→0

**How difficult?**

- **A.** Any programmer could do it.
- **B.** Diligent algorithms student could do it.
- **C.** Hire an expert.
- **D.** Intractable.
- **E.** No one knows.

# Graph-processing challenge 4

Problem. Is there a (non-simple) cycle that uses every edge exactly once?




0–1
0–2
0–5
0–6
1–2
2–3
2–4
3–4
4–5
4–6

0–1–2–3–4–2–0–6–4–5–0

**How difficult?**

**A.** Any programmer could do it.

**B.** Diligent algorithms student could do it.

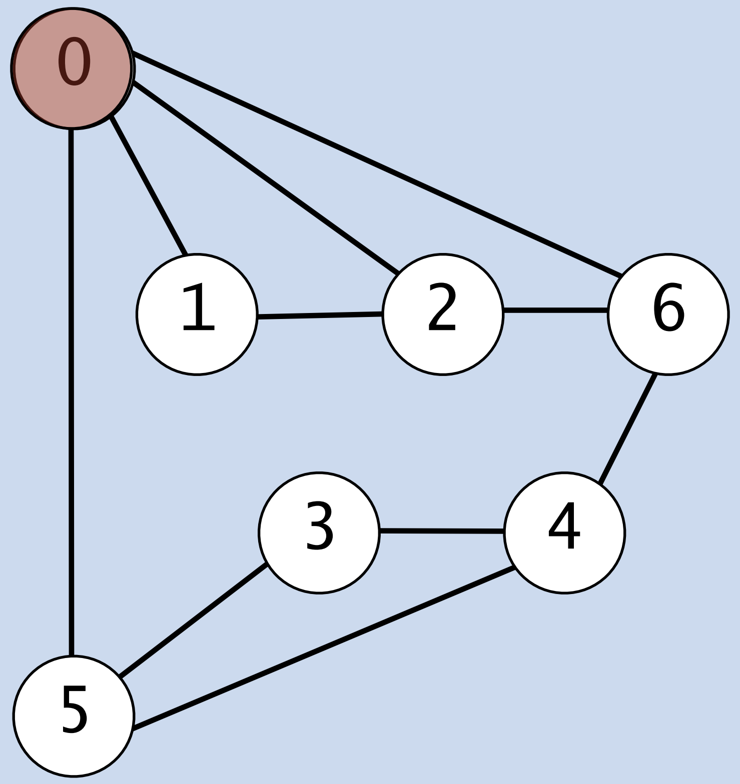
**C.** Hire an expert.

**D.** Intractable.

**E.** No one knows.

## Graph-processing challenge 5

Problem. Is there a cycle that uses every vertex exactly once?




0–1
0–2
0–5
0–6
1–2
2–6
3–4
3–5
4–5
4–6

0-5-3-4-6-2-1-0

**How difficult?**

**A.** Any programmer could do it.

**B.** Diligent algorithms student could do it.

**C.** Hire an expert.

**D.** Intractable.

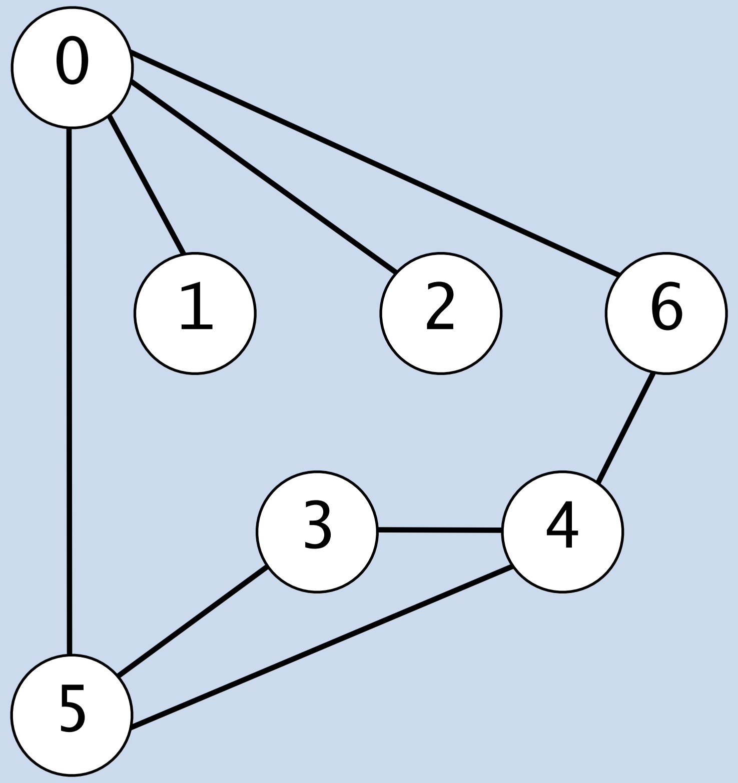
**E.** No one knows.

# Graph-processing challenge 6

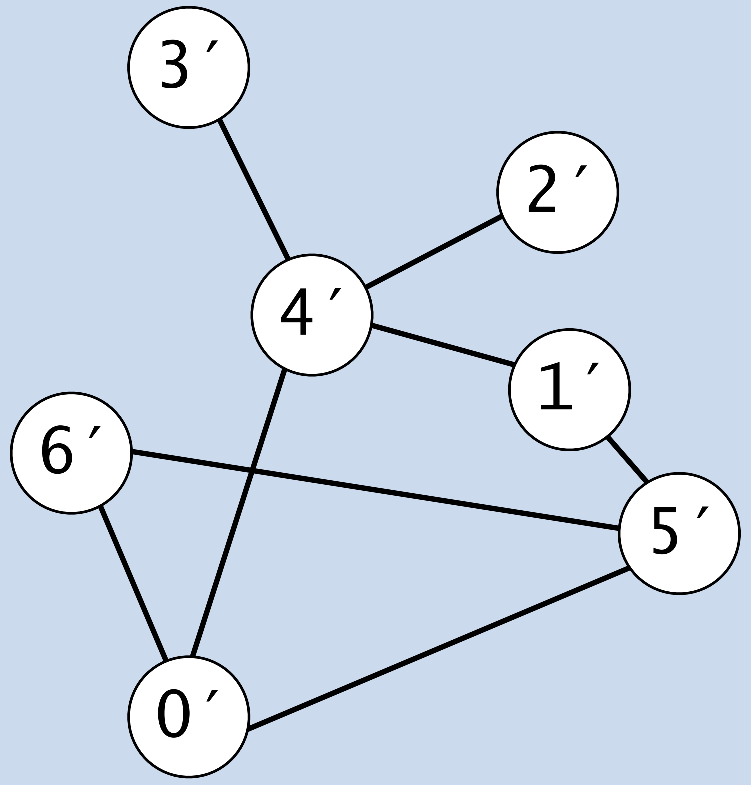
Problem. Are two graphs identical except for vertex names?

**How difficult?**

- **A.** Any programmer could do it.
- **B.** Diligent algorithms student could do it.
- **C.** Hire an expert.
- **D.** Intractable.
- **E.** No one knows.




0-1
0-2
0-5
0-6
3-4
3-5
4-5
4-6



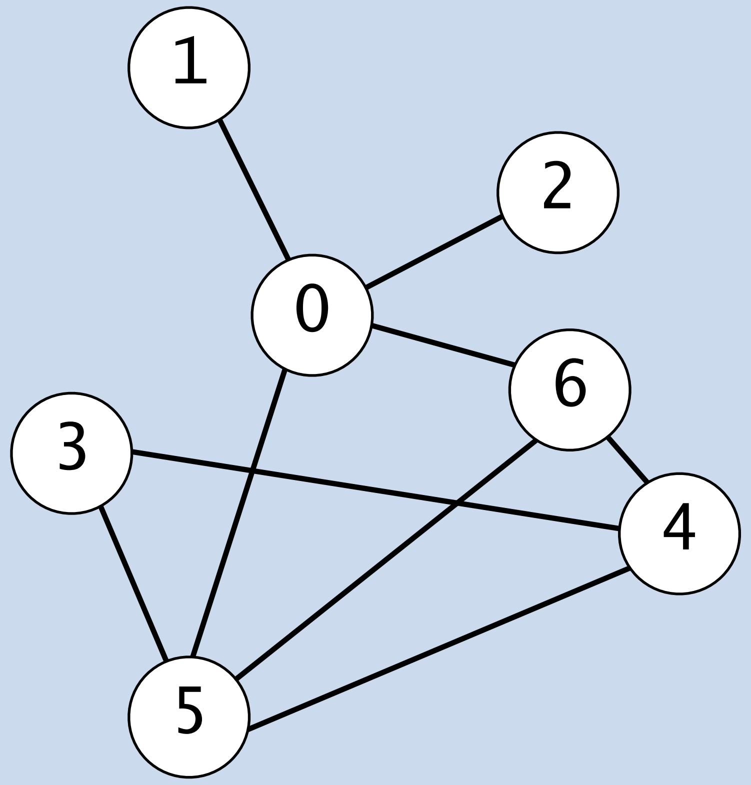

0′-4′
0′-5′
0′-6′
1′-4′
1′-5′
2′-4′
3′-4′
5′-6′

0⇔4′, 1⇔3′, 2⇔2′, 3⇔6′, 4⇔5′, 5⇔0′, 6⇔1′

# Graph-processing challenge 7

Problem. Can you draw a graph in the plane with no crossing edges?

try it yourself at https://www.jasondavies.com/planarity/




0–1
0–2
0–5
0–6
3–4
3–5
4–5
4–6
5–6

**How difficult?**

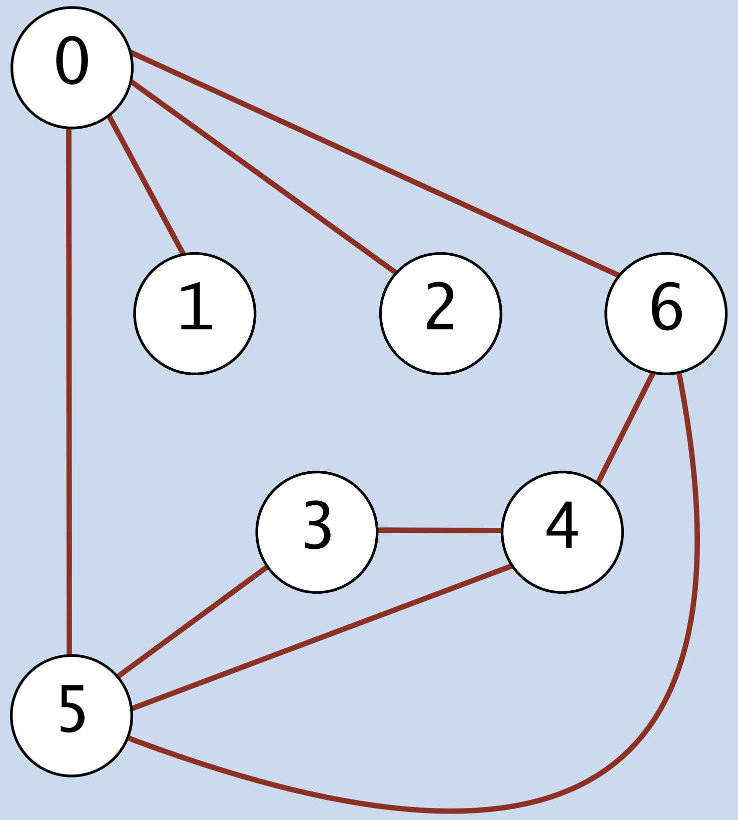
**A.** Any programmer could do it.

**B.** Diligent algorithms student could do it.

**C.** Hire an expert.

**D.** Intractable.

**E.** No one knows.

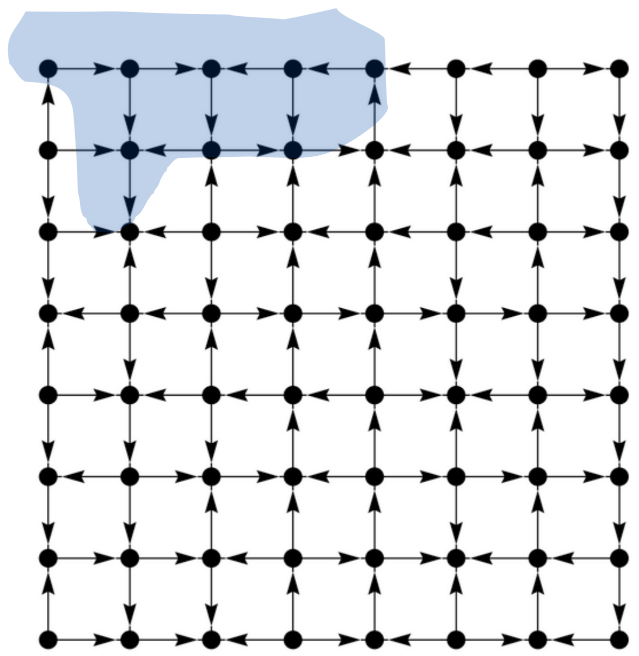
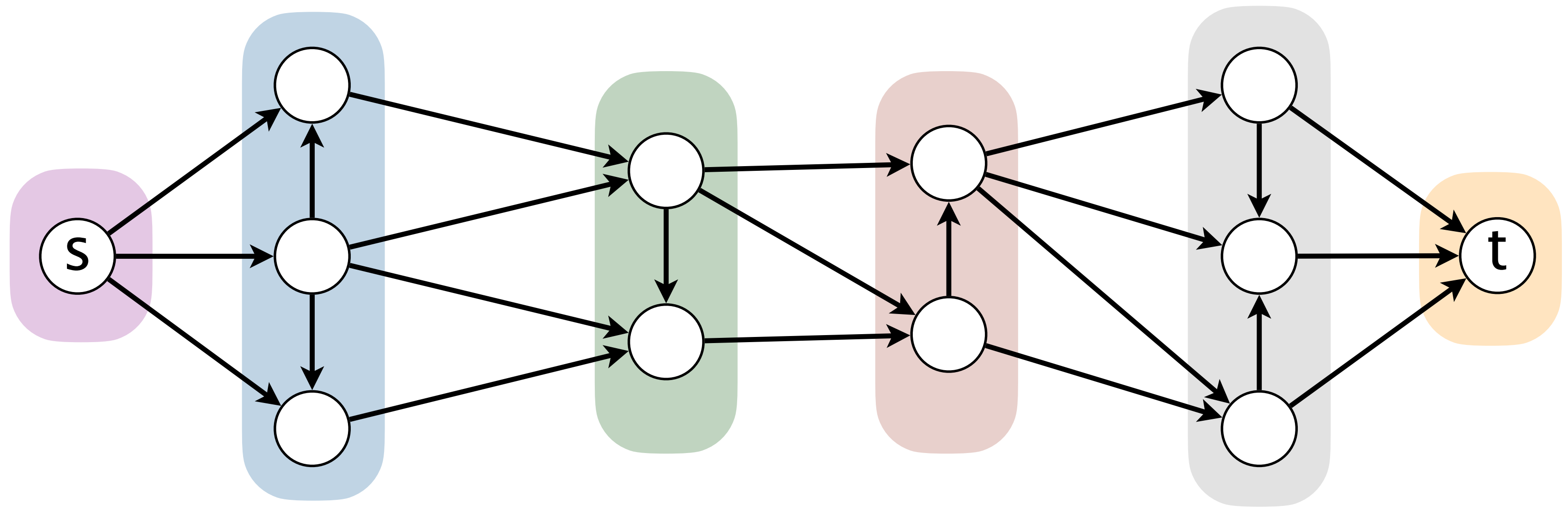
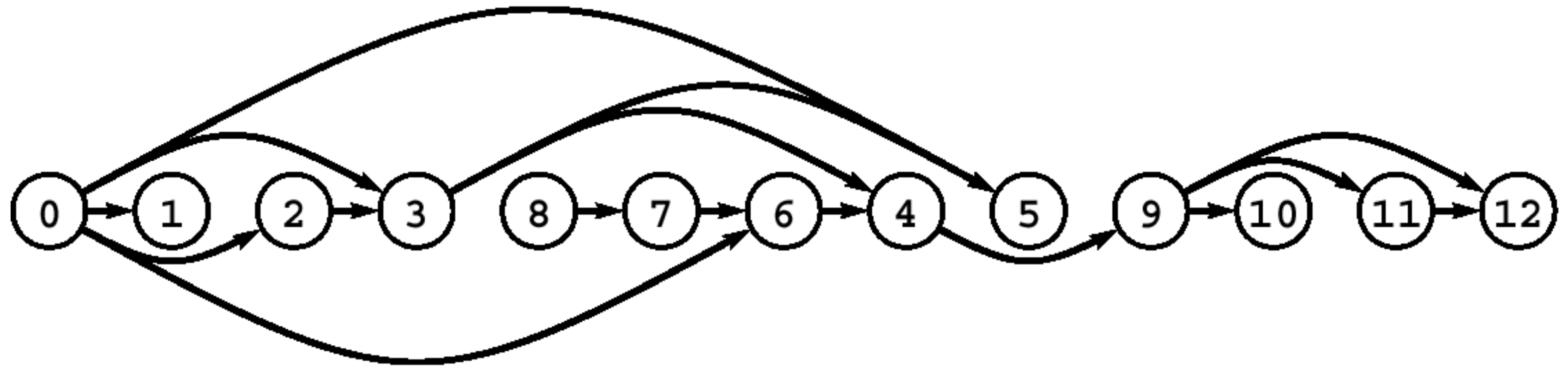


# Graph traversal summary

BFS and DFS enables efficient solution of many (but not all) graph and digraph problems.

| graph problem | BFS | DFS | time |
|---|---|---|---|
| **s-t path** | ✔ | ✔ | $E + V$ |
| **shortest s-t path** | ✔ | | $E + V$ |
| **shortest directed cycle (girth)** | ✔ | | $E\,V$ |
| **Euler cycle** | | ✔ | $E + V$ |
| **Hamilton cycle** | | | $2^{1.657\,V}$ |
| **bipartiteness (odd cycle)** | ✔ | ✔ | $E + V$ |
| **connected components** | ✔ | ✔ | $E + V$ |
| **strong components** | | ✔ | $E + V$ |
| **planarity** | | ✔ | $E + V$ |
| **graph isomorphism** | | | $2^{c \ln^3 V}$ |

# Graph-processing summary: algorithms of the week

| | | |
|---|---|---|
| **single-source reachability** |  | DFS/BFS |
| **shortest paths** |  | BFS |
| **topological sort** |  | DFS |