Subdivision Surfaces

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3D Object Representations

- Raw data
  - Range image
  - Point cloud

- Surfaces
  - Polygonal mesh
  - Parametric
    - Subdivision
  - Implicit

- Solids
  - Voxels
  - BSP tree
  - CSG
  - Sweep

- High-level structures
  - Scene graph
  - Application specific
Subdivision Surfaces

• Alternative to parametric surfaces, overcoming:
  • Many patches
  • Difficult to mark sharp features
  • Irregularities after deformation

Woody’s hand (NURBS)  Geri’s hand (subdivision)

Stanford Graphics course notes
Geri’s Game

• “served as a demonstration of a new animation tool called subdivision surfaces” (Wikipedia)

• Subdivision used for head, hands & clothing

• Academy Award winner
Subdivision Surfaces

- Used in movie and game industries
- Supported by most 3D modeling software
Subdivision Surfaces

- What makes a good surface representation?
  - Accurate
  - Concise
  - Intuitive specification
  - Local support
  - Affine invariant
  - Arbitrary topology
  - Guaranteed continuity
  - Natural parameterization
  - Efficient display
  - Efficient intersections

Reif & Schroeder 2000
Review on Continuity

A curve / surface with $G^k$ continuity has a continuous k-th derivative, geometrically.

No continuity (G$^{-1}$?)

Similar to (but not the same as) $C^k$ continuity, which refers to continuity with respect to parameter
e.g.: $f_x(u) = r_x \cos(2u)$ (but we’re going to say $C^k$ from now on...)

$G^0$

$G^1$

$G^2$
Subdivision

• How do you make a curve with guaranteed continuity?
Subdivision

- How do you make a curve with guaranteed continuity? …
Subdivision

• How do you make a surface with guaranteed continuity?
Subdivision Surfaces

- Repeated application of
  1. Topology refinement (splitting faces)
  2. Geometry refinement (weighted averaging)

Zorin & Schroeder
SIGGRAPH 99
Course Notes
Subdivision Surfaces – Examples

- Base mesh
Subdivision Surfaces – Examples

- Topology refinement
Subdivision Surfaces – Examples

- Geometry refinement
Subdivision Surfaces – Examples

• Topology refinement
Subdivision Surfaces – Examples

• Geometry refinement
Subdivision Surfaces – Examples

- Topology refinement
Subdivision Surfaces – Examples

- Geometry refinement
Subdivision Surfaces – Examples

• Limit surface
Subdivision Surfaces – Examples

• Base mesh + limit surface
Design of Subdivision Rules

• What types of input?
  • Quad meshes, triangle meshes, etc.

• How to refine topology?
  • Simple implementations

• How to refine geometry?
  • Smoothness guarantees in limit surface
    » Continuity ($C^0$, $C^1$, $C^2$, …?)
  • Provable relationships between limit surface and original control mesh
    » Interpolation of vertices?
    » Surface within their convex hull?
Linear Subdivision

• Type of input
  • Quad mesh -- four-sided polygons (*quads*)

• Topology refinement rule
  • Split every quad into four at midpoints

• Geometry refinement rule
  • Average vertex positions

Note: simple example to demonstrate how such schemes work, but not the best scheme…
Linear Subdivision
Linear Subdivision

- Topology refinement
Linear Subdivision

- Geometry refinement
Linear Subdivision

LinearSubivision \((F_0, V_0, k)\)

for \(i = 1 \ldots k\) levels

\((F_i, V_i) = \text{RefineTopology}(F_{i-1}, V_{i-1})\)

RefineGeometry\((F_i, V_i)\)

return \((F_k, V_k)\)
Linear Subdivision

RefineTopology \((F, V)\)

new\(V = V\)

new\(F = {}\)

for each face \(F_i\)

Insert new vertex \(c\) at centroid of \(F_i\) into new\(V\)

return \((\text{new}F, \text{new}V)\)
Linear Subdivision

RefineTopology \((F, V)\)

\[ newV = V \]

\[ newF = \{ \} \]

for each face \(F_i\)

Insert new vertex \(c\) at centroid of \(F_i\) into \(newV\)

for \(j = 1\) to \(4\)

Insert in \(newV\) new vertex \(e_j\) at centroid of each edge \((F_{i,j}, F_{i,j+1})\)

return \((newF, newV)\)
Linear Subdivision

RefineTopology \( (F, V) \)

\[ \text{newV} = V \]

\[ \text{newF} = \{\} \]

for each face \( F_i \)

Insert new vertex \( c \) at centroid of \( F_i \) into \( \text{newV} \)

for \( j = 1 \) to \( 4 \)

Insert in \( \text{newV} \) new vertex \( e_j \) at centroid of each edge \( (F_{i,j}, F_{i,j+1}) \)

for \( j = 1 \) to \( 4 \)

Insert new face \( (F_{i,j}, e_j, c, e_{j-1}) \) into \( \text{newF} \)

return \( (\text{newF}, \text{newV}) \)
Linear Subdivision

RefineGeometry( \( F, V \) )

\[ newV = V \]
\[ newF = F \]

for each vertex \( V_i \) in \( newV \)

\[ weight = 0; \]
\[ newV[i] = (0,0,0) \]

return \( (newF, newV) \)
Linear Subdivision

RefineGeometry( F, V )

\[ newV = V \]
\[ newF = F \]

for each vertex \( V_i \) in newV

\[ \text{weight} = 0; \]
\[ newV[i] = (0,0,0) \]

for each face \( F_j \) connected to \( V_i \)

\[ newV[i] += \text{centroid of } F_j \]
\[ \text{weight} += 1.0; \]

\[ newV[i] /= \text{weight} \]

return \((newF, newV)\)
Linear Subdivision

- Example

Input mesh
Linear Subdivision

- Example

Topology refinement
Linear Subdivision

• Example
Linear Subdivision

• Example

Topology refinement
Linear Subdivision

• Example

Geometry refinement
Linear Subdivision

• Example

Topology refinement
Linear Subdivision

• Example

Geometry refinement
Linear Subdivision

- Example

Topology refinement

Scott Schaefer
Linear Subdivision

• Example

Final result
Subdivision Demo

https://threejs.org/examples/webgl_modifier_subdivision.html
Subdivision Schemes

- Common subdivision schemes
  - Catmull-Clark
  - Loop
  - Many others

- Differ in ...
  - Input topology
  - How refine topology
  - How refine geometry

... which makes differences in ...
- Provable properties
Catmull-Clark Subdivision

Scott Schaefer
Catmull-Clark Subdivision
Catmull-Clark Subdivision

Scott Schaefer
Catmull-Clark Subdivision

Scott Schaefer
Catmull-Clark Subdivision
Catmull-Clark Subdivision

New \cdot = ( \cdot - \cdot + (n-3) \cdot ) / n

n = \#faces a point belongs to.

Scott Schaefer
Catmull-Clark Subdivision

New \bullet = ( -1 \cdot \text{avg of } \bullet + (n-3) \cdot \bullet ) / n

\[ n = \# \text{faces a point belongs to}. \]
Catmull-Clark Subdivision

New \textcolor{green}{\bullet} = \left(4 \times \text{avg of } \textcolor{blue}{\bullet} - 1 \times \text{avg of } \textcolor{red}{\bullet} + (n-3) \times \textcolor{green}{\bullet}\right) / n

n = \#faces a point belongs to.

Scott Schaefer
Catmull-Clark Subdivision

Scott Schaefer
Catmull-Clark Subdivision
Catmull-Clark Subdivision

Scott Schaefer
Catmull-Clark Subdivision

Linear Subdivision  Catmull-Clark Subdivision

Scott Schaefer
Catmull-Clark Subdivision

Scott Schaefer
Catmull-Clark Subdivision
Catmull-Clark Subdivision

Scott Schaefer
Catmull-Clark Subdivision
Catmull-Clark Subdivision

- One round of subdivision produces all quads
- Smoothness of limit surface
  - $C^2$ almost everywhere
  - $C^1$ at vertices with valence $\neq 4$
- Relationship to control mesh
  - Does not interpolate input vertices
  - Within convex hull
- Most commonly used subdivision scheme in the movies…
Subdivision Schemes

• Common subdivision schemes
  • Catmull-Clark
  ➢ Loop
  • Many others

• Differ in ...
  • Input topology
  • How refine topology
  • How refine geometry

... which makes differences in ...
  • Provable properties
Loop Subdivision

- Operates on pure triangle meshes
- Subdivision rules
  - Linear subdivision
  - Averaging rules for “even / odd” (white / black) vertices

Zorin & Schroeder
SIGGRAPH 99
Course Notes
Loop Subdivision

- Operates on pure triangle meshes
- Subdivision rules
  - Linear subdivision
  - Averaging rules for “even / odd” (white / black) vertices
Loop Subdivision

Averaging rules

- Weights for “odd” and “even” vertices

Odd:

\[
\begin{align*}
\frac{1}{8} & \\
\frac{3}{8} & \\
\frac{3}{8} & \\
\frac{1}{8} &
\end{align*}
\]

… but what about vertices with valence ≠ 6?
Loop Subdivision

Averaging rules

- Weights for “odd” and “even” vertices

Odd:

$$\frac{3}{8} \quad \frac{1}{8}$$

Even:

$$\frac{3}{8} \quad \frac{1}{16} \quad \frac{10}{16} \quad \frac{1}{16}$$

… but what about vertices with valence ≠ 6?
Loop Subdivision

- Rules for *extraordinary vertices and boundaries*:

Odd:

\[
\begin{array}{c}
\frac{1}{8} \\
\frac{3}{8} \\
\frac{1}{8} \\
\frac{3}{8}
\end{array}
\]

\[
\begin{array}{c}
\frac{1}{2} \\
\frac{1}{2}
\end{array}
\]

*Interior*

*Crease and boundary*

\[a. \text{ Masks for odd vertices}\]
Loop Subdivision

• Rules for *extraordinary vertices and boundaries*:

Odd:

Even:

\[ \frac{1}{8} \]

\[ \frac{3}{8} \]

\[ \frac{1}{8} \]

\[ \frac{3}{8} \]

\[ \beta \]

\[ 1-k\beta \]

\[ \beta \]

\[ \beta \]

\[ \beta \]

\[ \beta \]

\[ \beta \]

\[ \frac{1}{8} \]

\[ \frac{3}{4} \]

\[ \frac{1}{8} \]

\[ \frac{1}{2} \]

\[ \frac{1}{2} \]

\[ a. \text{ Masks for odd vertices} \]

\[ b. \text{ Masks for even vertices} \]
Loop Subdivision

• How to choose $\beta$?
  • Analyze properties of limit surface
  • Interested in continuity of surface and smoothness

  » Original Loop

  $$\beta = \frac{1}{n} \left( \frac{5}{8} - \left( \frac{3}{8} + \frac{1}{4} \cos \frac{2\pi}{n} \right)^2 \right)$$

  » Warren

  $$\beta = \begin{cases} 
  \frac{3}{8n} & n > 3 \\
  \frac{3}{16} & n = 3 
  \end{cases}$$
Loop Subdivision

• Operates only on triangle meshes
• Smoothness of limit surface
  • $C^2$ almost everywhere
  • $C^1$ at vertices with valence $\neq 6$
• Relationship to control mesh
  • Does not interpolate input vertices
  • Within convex hull
Subdivision Schemes

- Common subdivision schemes
  - Catmull-Clark
  - Loop
  - Many others

- Differ in ...
  - Input topology
  - How refine topology
  - How refine geometry

… which makes differences in ...
  - Provable properties
# Subdivision Zoo

- **Other subdivision schemes**

<table>
<thead>
<tr>
<th>Primal (face split)</th>
<th>Quad Meshes</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Triangular meshes</strong></td>
<td><strong>Catmull-Clark(C^2)</strong></td>
</tr>
<tr>
<td>Approximating</td>
<td>Loop(C^2)</td>
</tr>
<tr>
<td>Interpolating</td>
<td>Mod. Butterfly (C^1)</td>
</tr>
<tr>
<td></td>
<td>Kobbelt (C^1)</td>
</tr>
</tbody>
</table>

**Dual (vertex split)**

- Doo-Sabin, Midedge(C^1)
- Biquartic (C^2)

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Zorin & Schroeder, SIGGRAPH 99, Course Notes
Other Subdivision Schemes

- Butterfly subdivision
Other Subdivision Schemes

• Butterfly subdivision
Other Subdivision Schemes

- Butterfly subdivision
Other Subdivision Schemes

- Vertex-split subdivision (Doo-Sabin, Midedge, Biquartic)

One step of Midedge subdivision
Other Subdivision Schemes

- Vertex-split subdivision (Doo-Sabin, Midedge, Biquartic)

Multiple steps of Midedge subdivision
Drawing Subdivision Surfaces

• Goal:
  • Draw best approximation of smooth limit surface
  • With limited triangle budget
Drawing Subdivision Surfaces

• Goal:
  • Draw best approximation of smooth limit surface
  • With limited triangle budget

• Solution:
  • Stop subdivision at different levels across the surface
  • Stop-criterion depending on quality measure

• Quality of approximation can be defined by
  • Projected (screen) area of final triangles
  • Local surface curvature
Adaptive Subdivision

10072 Triangles

228654 Triangles

[Kobbelt 2000]
Adaptive Subdivision

• Problem:
  • Different levels of subdivision may lead to gaps in the surface

[Kobbelt 2000]
Adaptive Subdivision

• Solution:
  • Replacing incompatible coarse triangles by *triangle fan*
  • Balanced subdivision: neighboring subdivision levels must not differ by more than one

[Unbalanced]  [Balanced]

[Kobbelt 2000]
Subdivision Surface Summary

• Advantages:
  • Simple method for describing complex surfaces
  • Relatively easy to implement
  • Arbitrary topology
  • Intuitive specification
  • Local support
  • Guaranteed continuity
  • Multiresolution

• Difficulties:
  • Parameterization
  • Intersections
Comparison

Parametric surfaces
- Provide parameterization
- More restriction on topology of control mesh
- Some require careful placement of control mesh vertices to guarantee continuity (e.g., Bezier)

Subdivision surfaces
- No parameterization
- Subdivision rules can be defined for arbitrary topologies
- Provable continuity for all placements of control mesh vertices
## Comparison

<table>
<thead>
<tr>
<th>Feature</th>
<th>Polygonal Mesh</th>
<th>Parametric Surface</th>
<th>Subdivision Surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accurate</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Concise</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Intuitive specification</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Local support</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Affine invariant</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Arbitrary topology</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Guaranteed continuity</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Natural parameterization</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Efficient display</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Efficient intersections</td>
<td>No</td>
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