Parametric Surfaces

COS 426, Spring 2022
Felix Heide
Princeton University

3D Object Representations

- Points
  - Range image
  - Point cloud
- Surfaces
  - Polygonal mesh
  - Parametric
  - Subdivision
  - Implicit
- Solids
  - Voxels
  - BSP tree
  - CSG
  - Sweep
- High-level structures
  - Scene graph
  - Application specific

Parametric Surfaces

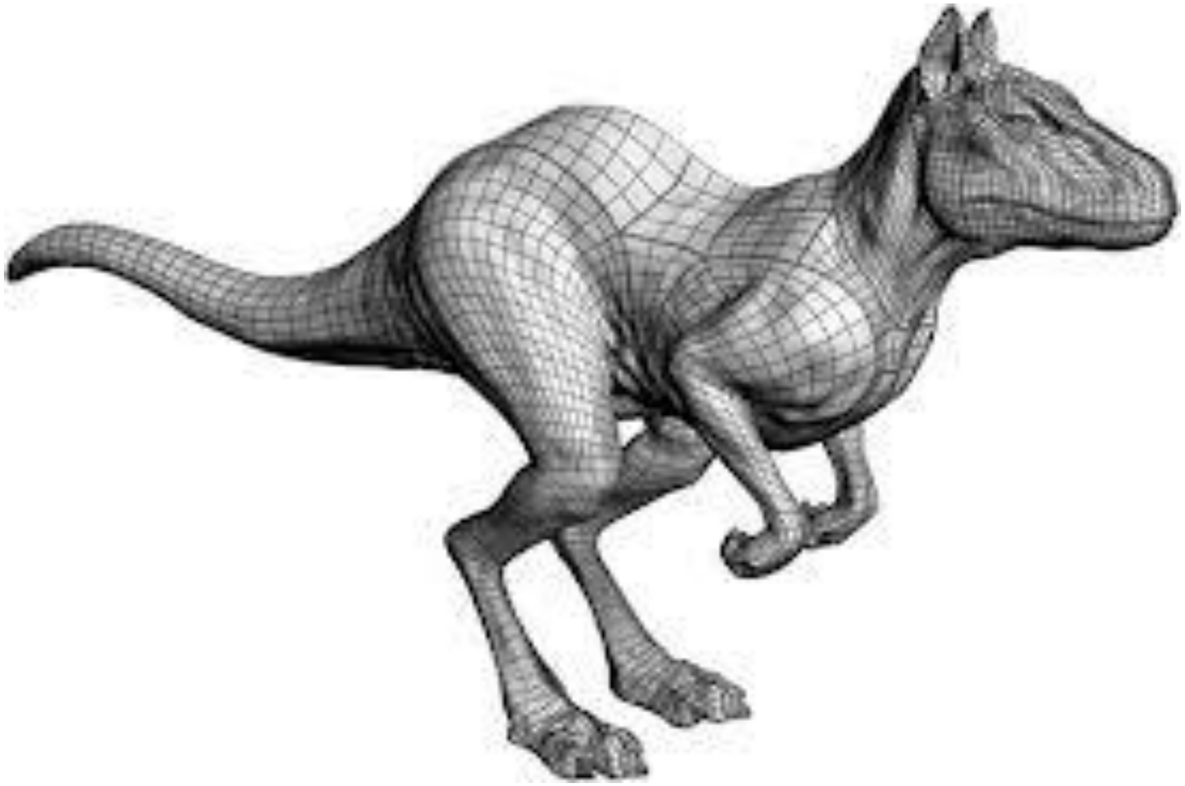
- Applications
  - Design of smooth surfaces in cars, ships, etc.

Parametric Surfaces

- Applications
  - Partitioning into surface patches.

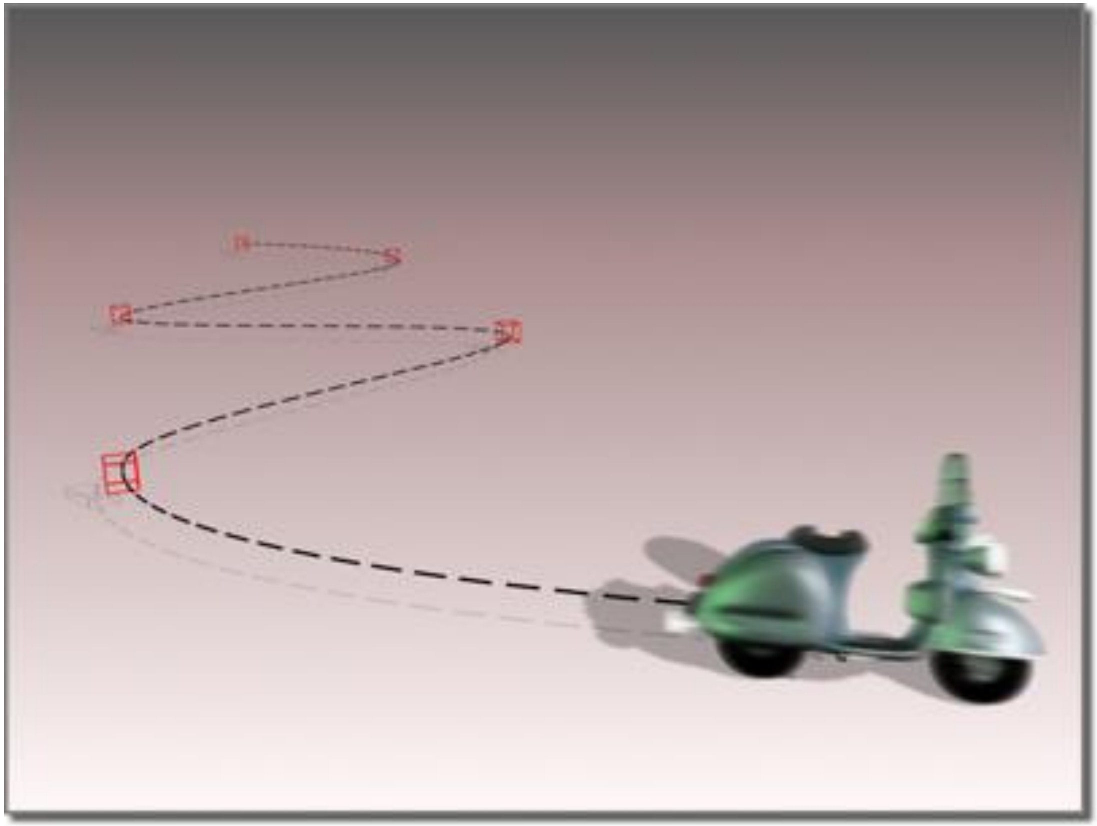
Parametric Surfaces

- Applications
  - Design of smooth surfaces in cars, ships, etc.
  - Creating characters or scenes for movies



Parametric Curves

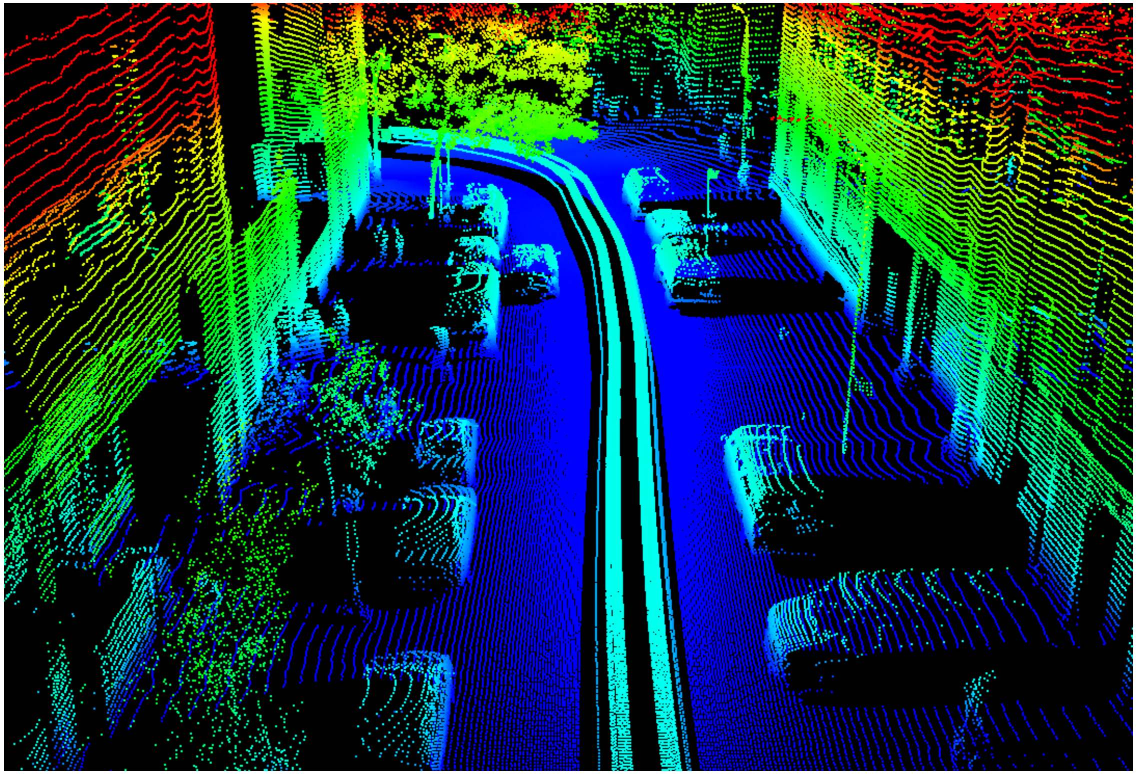
- Applications
  - Defining motion trajectories for objects or cameras



AutoDesk

Parametric Curves

- Applications
  - Defining motion trajectories for objects or cameras
  - Defining smooth interpolations of sparse data



Google Street View

Outline

- Parametric curves
  - Cubic B-Spline
  - Cubic Bézier
- Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

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- Parametric curves
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- Parametric surfaces
  - Bi-cubic B-Spline
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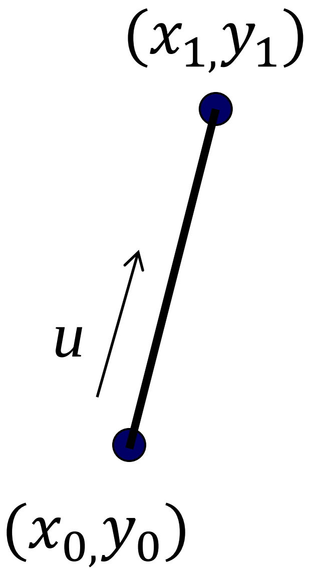
Parametric Curves

- Defined by parametric functions:
  - $x = f_x(u)$
  - $y = f_y(u)$
- Example: line segment

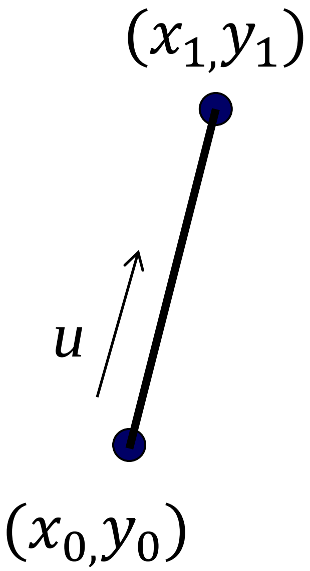


H&B Figure 10.10

Parametric Curves

- Defined by parametric functions:
  - $x = f_x(u)$
  - $y = f_y(u)$
- Example: line segment

$$f_x(u) = (1-u)x_0 + ux_1$$
$$f_y(u) = (1-u)y_0 + uy_1$$
$$u \in [0..1]$$



H&B Figure 10.10

Parametric Curves

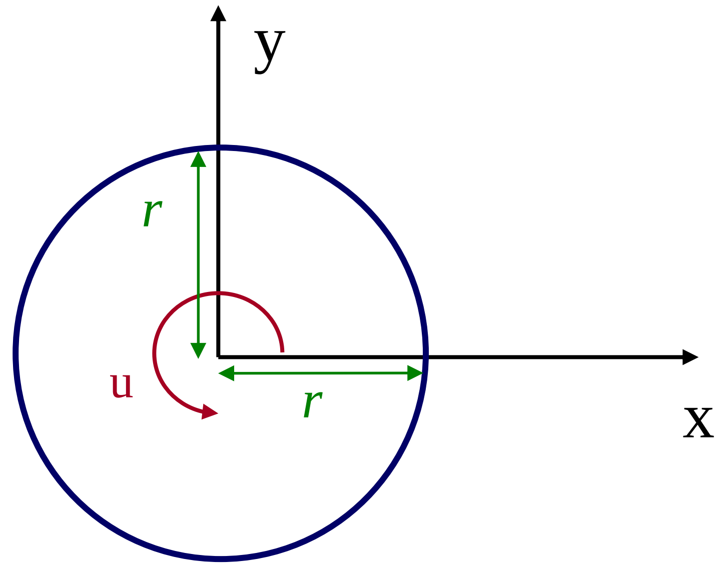
- Defined by parametric functions:
  - $x = f_x(u)$
  - $y = f_y(u)$

- Example: circle

$$f_x(u) = r\ \cos(2\pi u)$$

$$f_y(u) = r\ \sin(2\pi u)$$

$$u \in [0..1]$$



H&B Figure 10.10

Parametric Curves

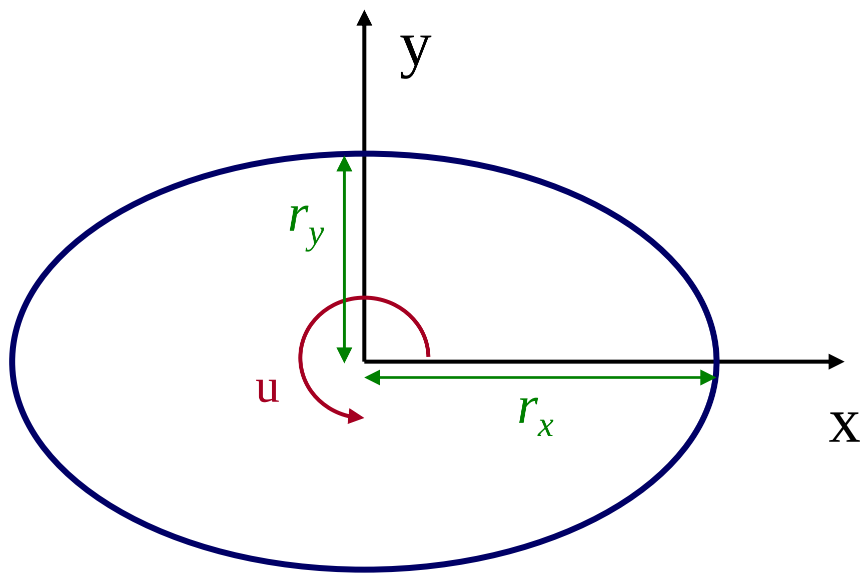
- Defined by parametric functions:
  - $x = f_x(u)$
  - $y = f_y(u)$

- Example: ellipse

$$f_x(u) = r_x \cos(2\pi u)$$

$$f_y(u) = r_y \sin(2\pi u)$$

$$u \in [0..1]$$



H&B Figure 10.10

Parametric curves

How to easily define arbitrary curves?

$$x = f_x(u)$$
$$y = f_y(u)$$

Parametric curves

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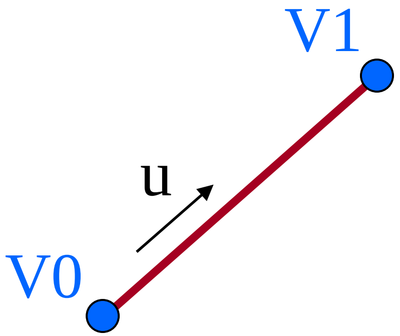

Parametric curves

How to easily define arbitrary curves?

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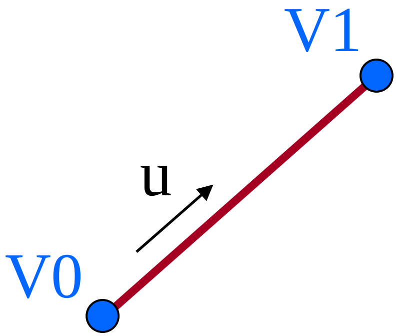
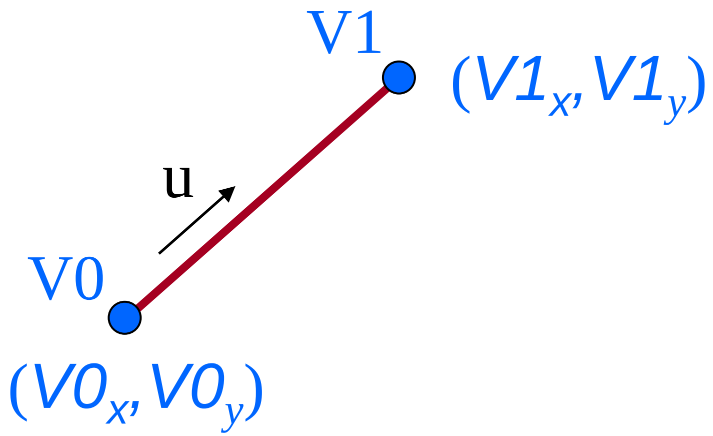


Slightly different notation than before:



Parametric curves

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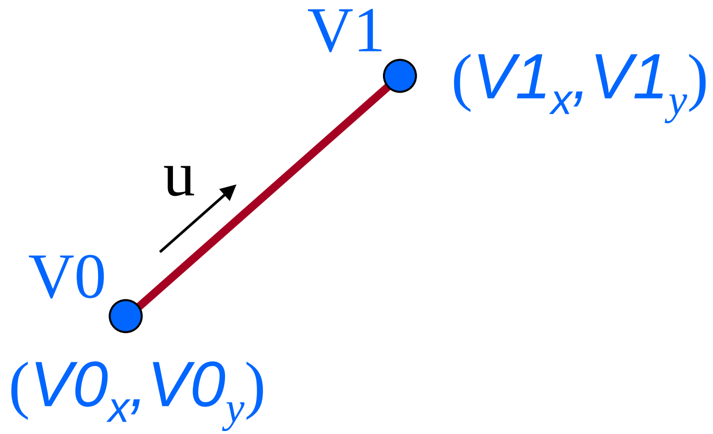


Use functions that “blend” control points

$$x = f_x(u) = V0_x*(1 - u) + V1_x*u$$
$$y = f_y(u) = V0_y*(1 - u) + V1_y*u$$

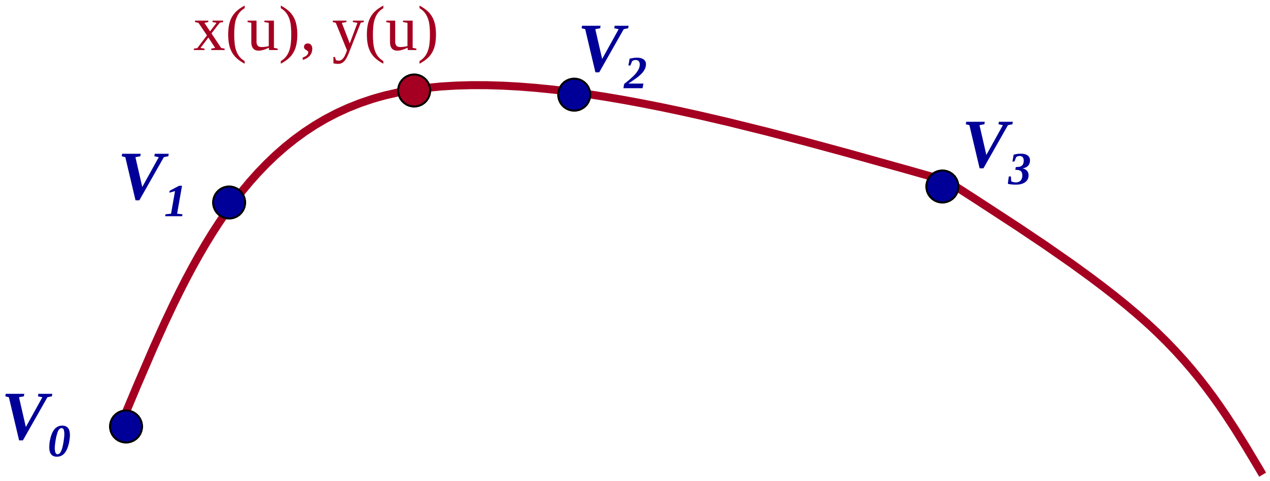
Simple functions of u

Parametric curves

More generally:

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * Vi_y$$



Parametric curves

More generally:

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

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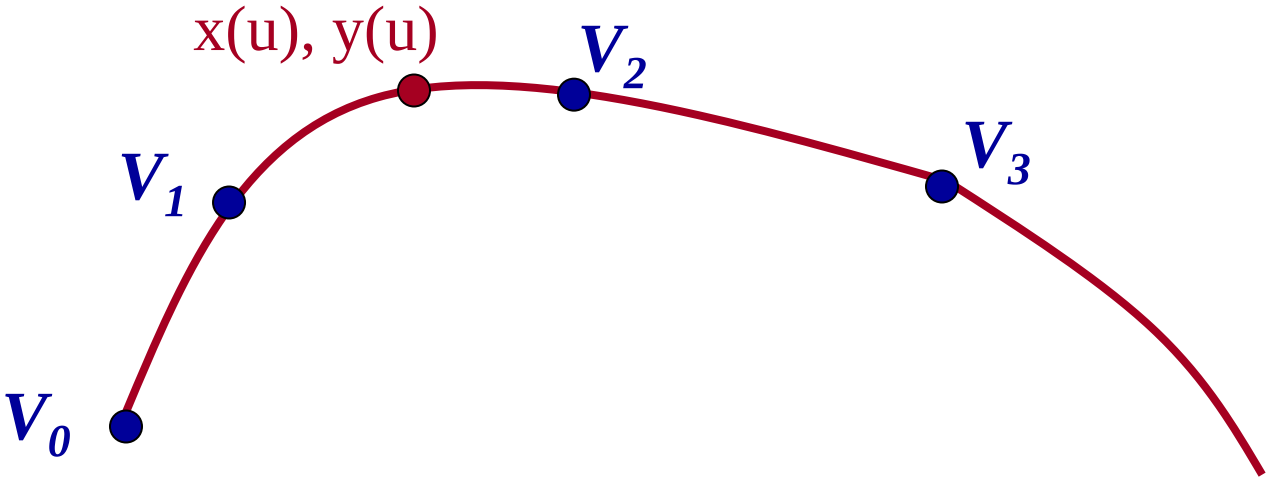
May use ***all*** points!



Parametric curves

What B(u) functions should we use?

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

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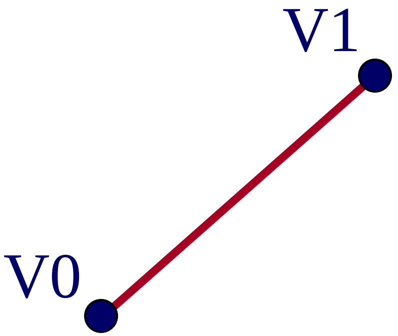

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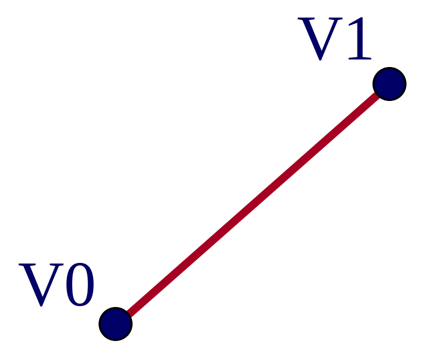

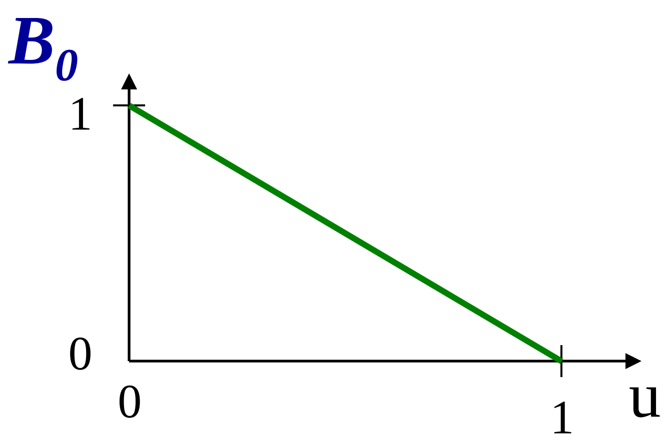

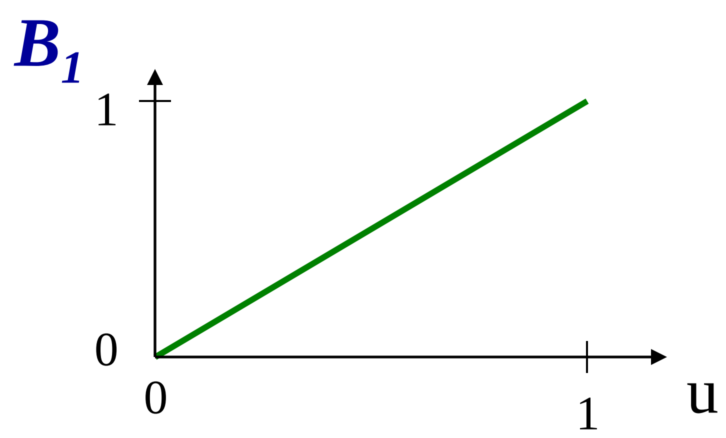

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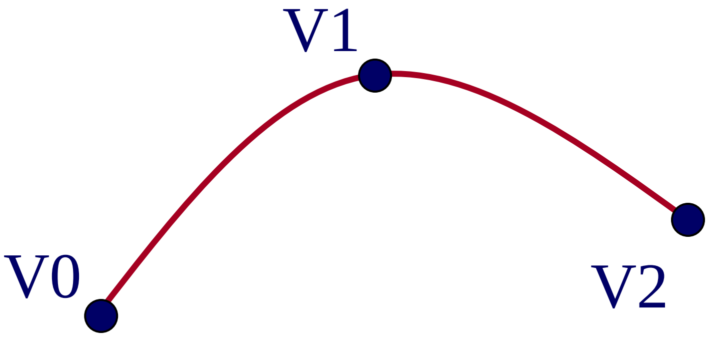

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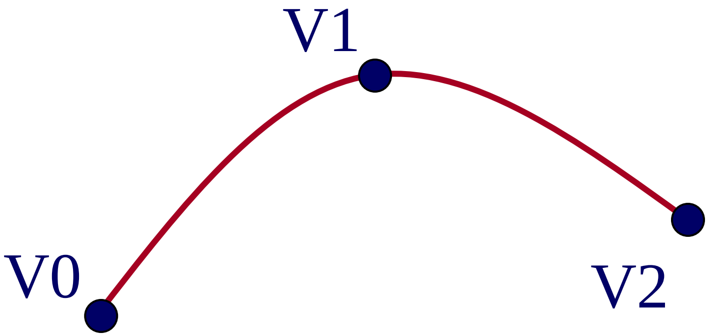
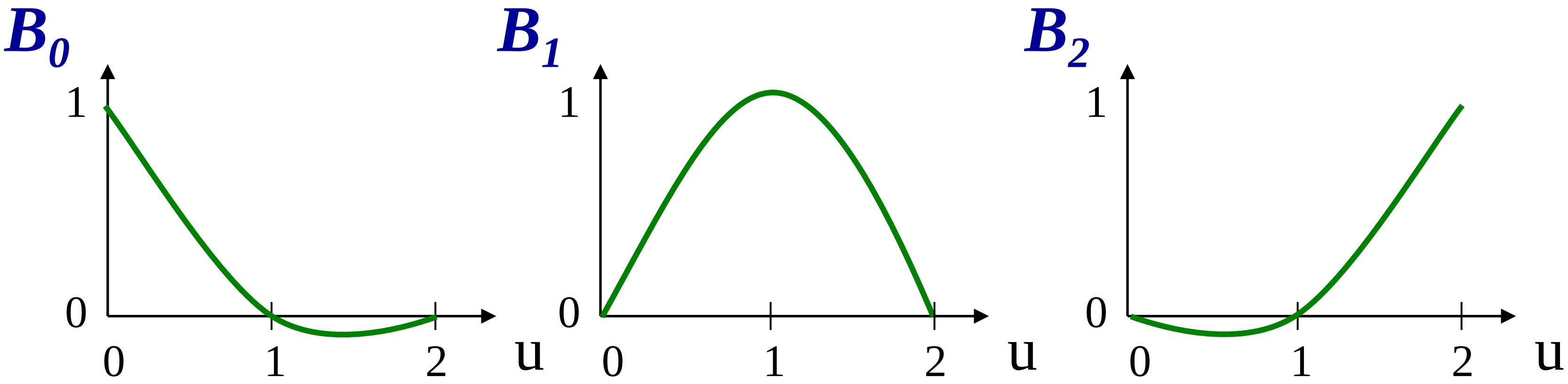


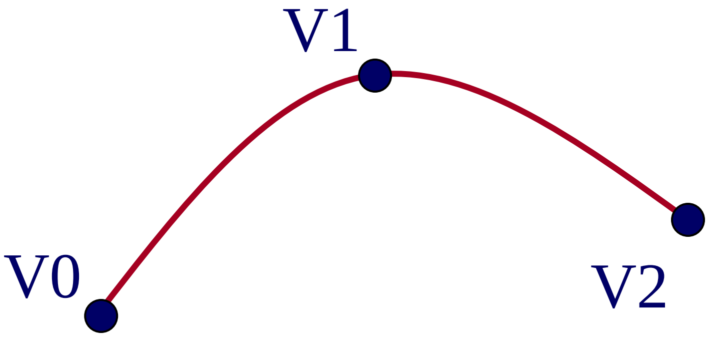


Blending Functions B

- Polynomial blending functions:

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * Vi_y$$



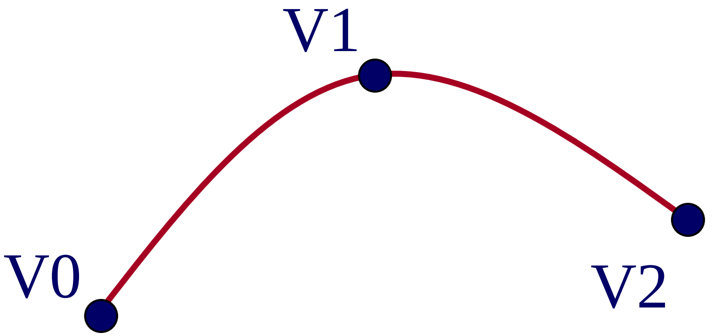
Blending Functions B

- Polynomial blending functions:

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * Vi_y$$



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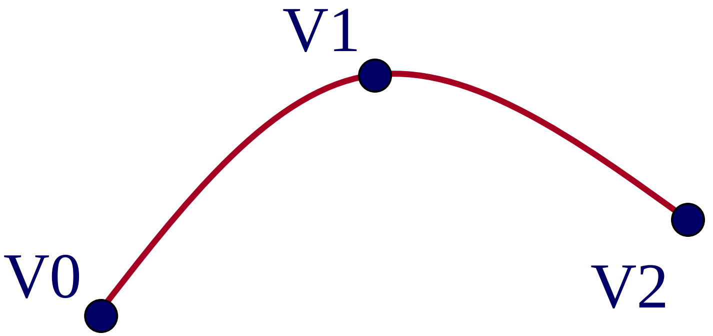
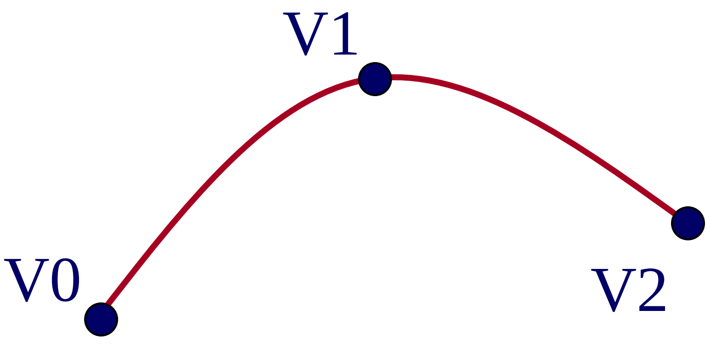


- Advantages of polynomials
    - Easy to compute
    - Easy to derive curve properties

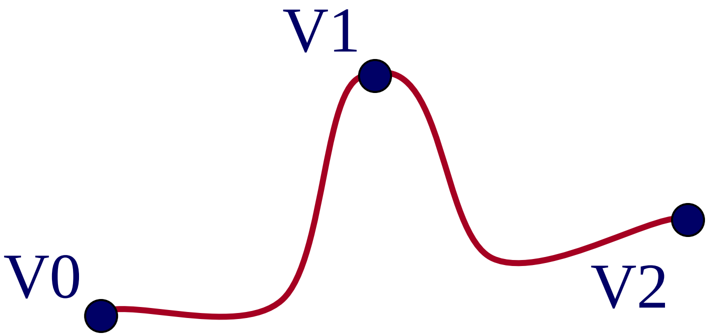
Parametric Polynomial Curves

- Polynomial blending functions:

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

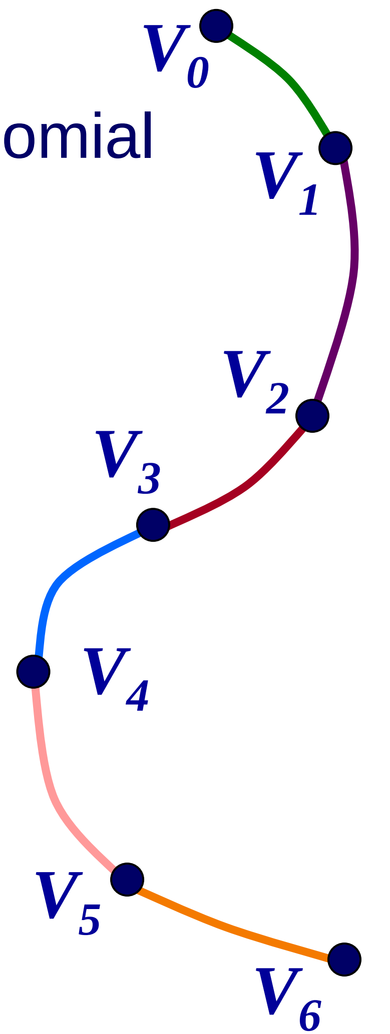


- What degree polynomial?
  - Easy to compute
  - Easy to control
  - Expressive



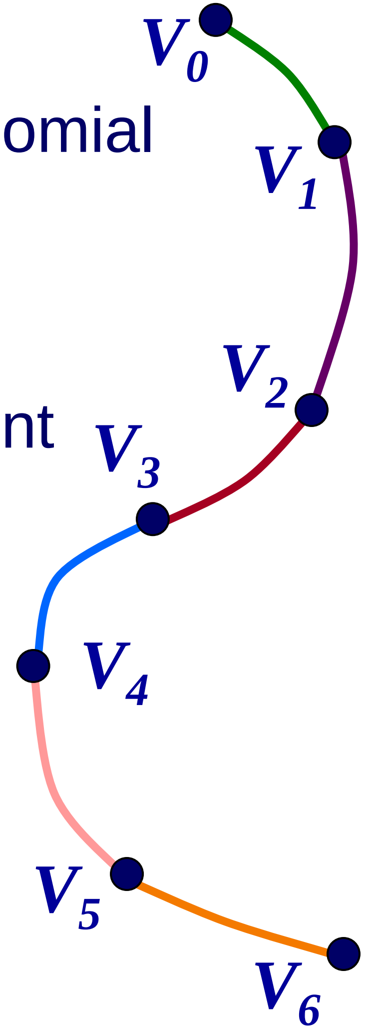
Piecewise Parametric Polynomial Curves

- Splines:
  - Split curve into segments
  - Each segment defined by low-order polynomial blending subset of control vertices



Piecewise Parametric Polynomial Curves

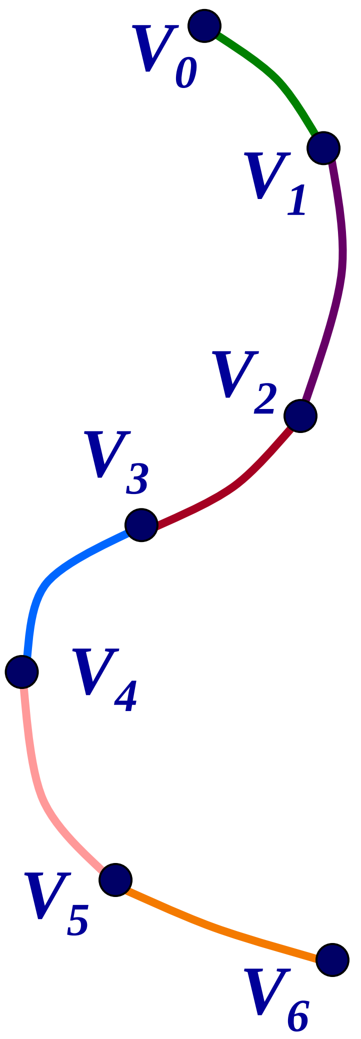
- Splines:
  - Split curve into segments
  - Each segment defined by low-order polynomial blending subset of control vertices
- Motivation:
  - Same blending functions for every segment
  - Prove properties from blending functions
  - Provides local control & efficiency
- Challenges
  - How choose blending functions?
  - How determine properties?



Cubic Splines

- Some properties we might like to have:
  - Local control
  - Continuity
  - Interpolation?
  - Convex hull?

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$



Blending functions determine properties

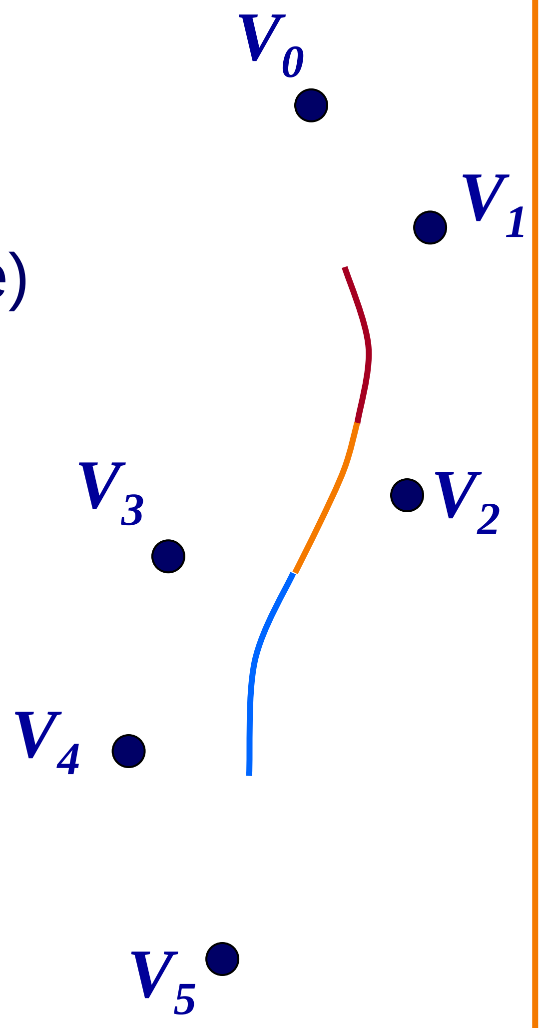
Properties determine blending functions

Outline

- Parametric curves
  - Cubic B-Spline
  - Cubic Bézier
- Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

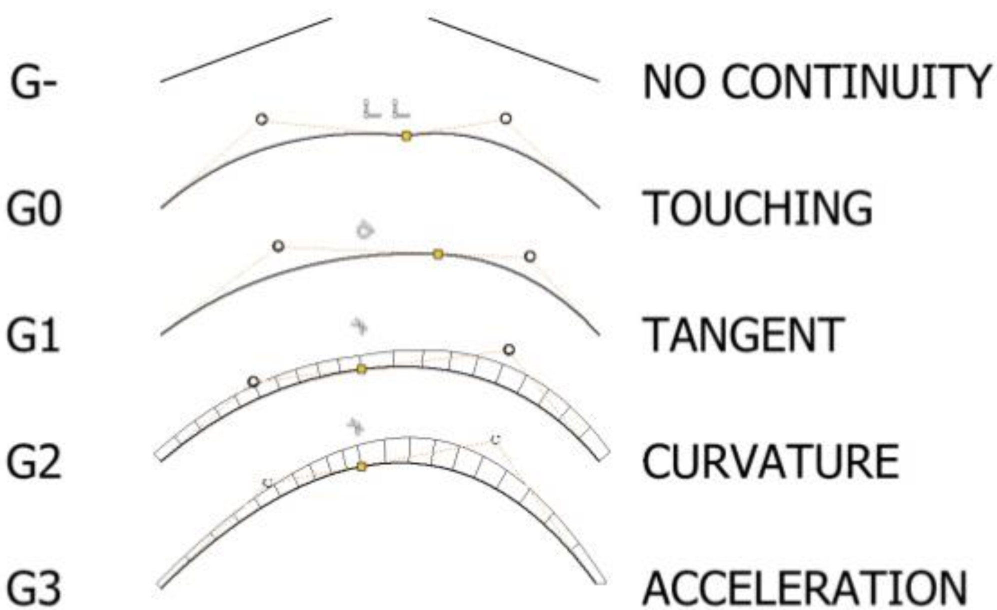
Cubic B-Splines

- Properties:
  - Local control
  - C^2 continuity at joints (infinitely continuous within each piece)
  - Approximating
  - Convex hull

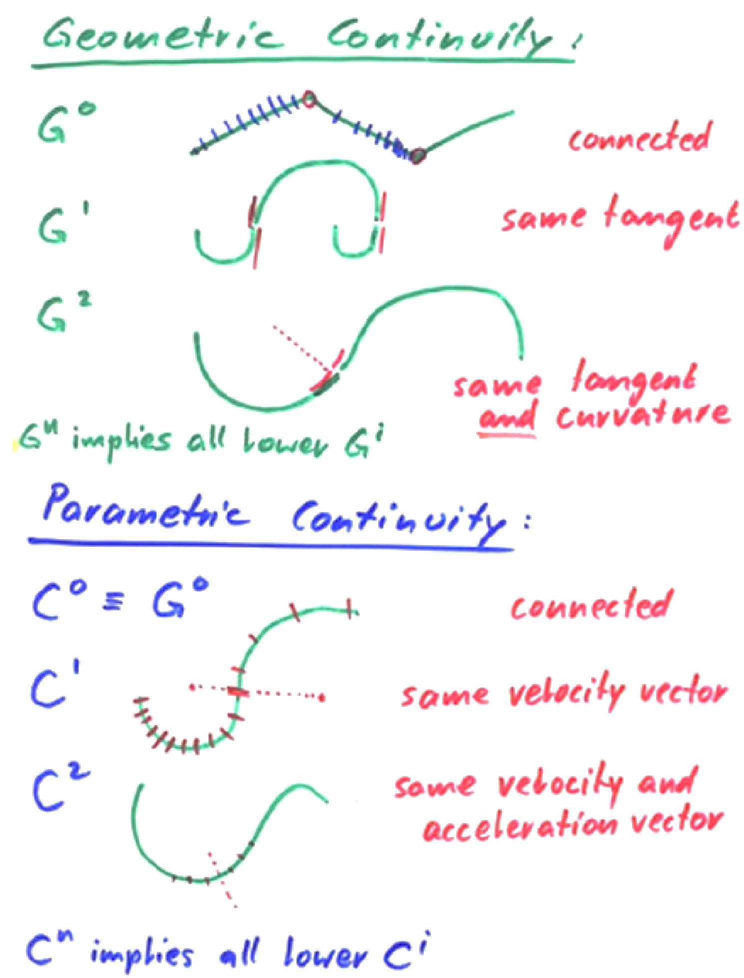


Cubic B-Splines

Notes on continuity



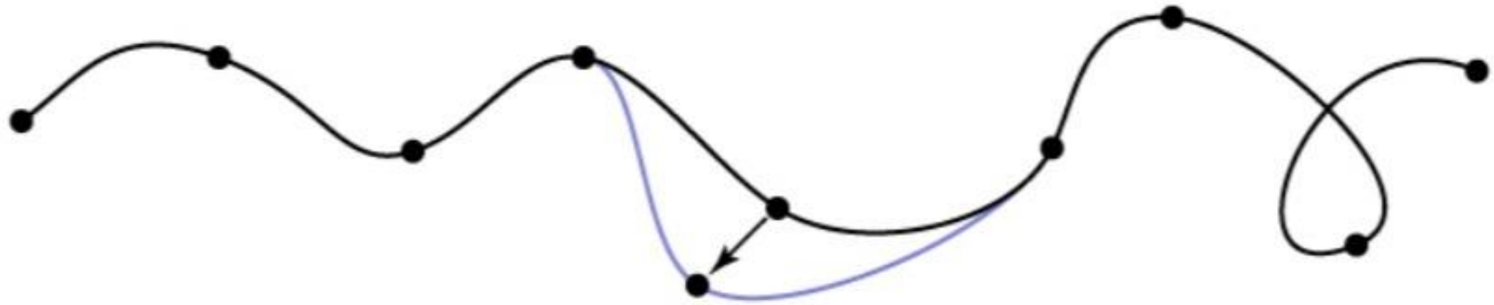
[from Robert Duvall]



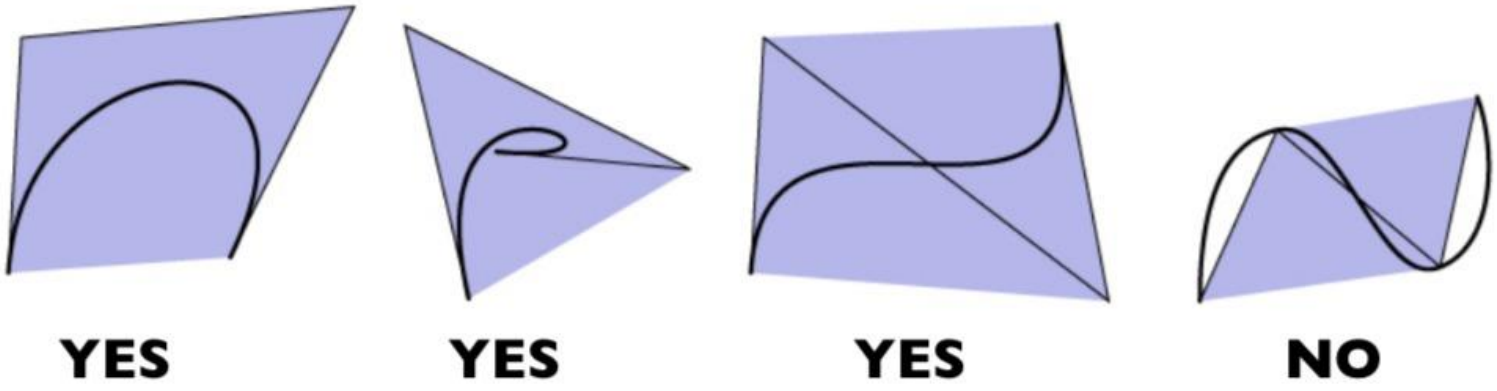
[from Carlo Séquin]

Cubic B-Splines

Notes on local control:



and convex hull property:



[from Steve Marschner]

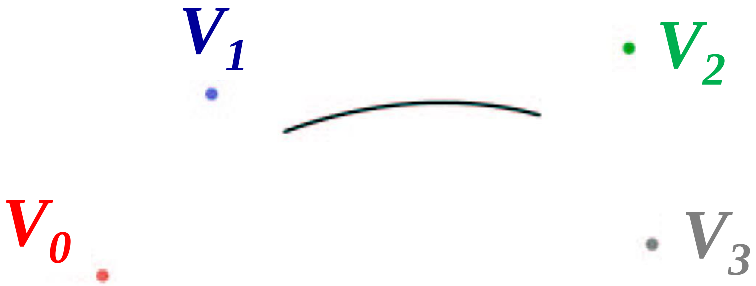
Cubic B-Spline Blending Functions

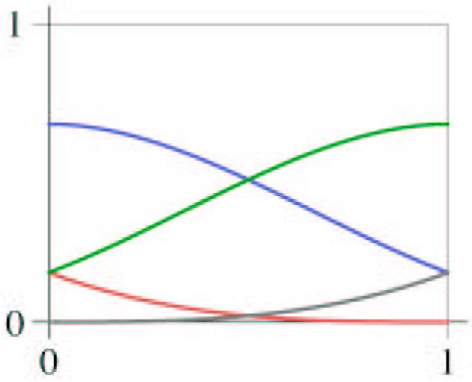
Cubic B-Spline Curve Formation:

$$x(u) = \sum_{i=0}^{n} B_i(u) * Vi_x$$

$$y(u) = \sum_{i=0}^{n} B_i(u) * Vi_y$$

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$

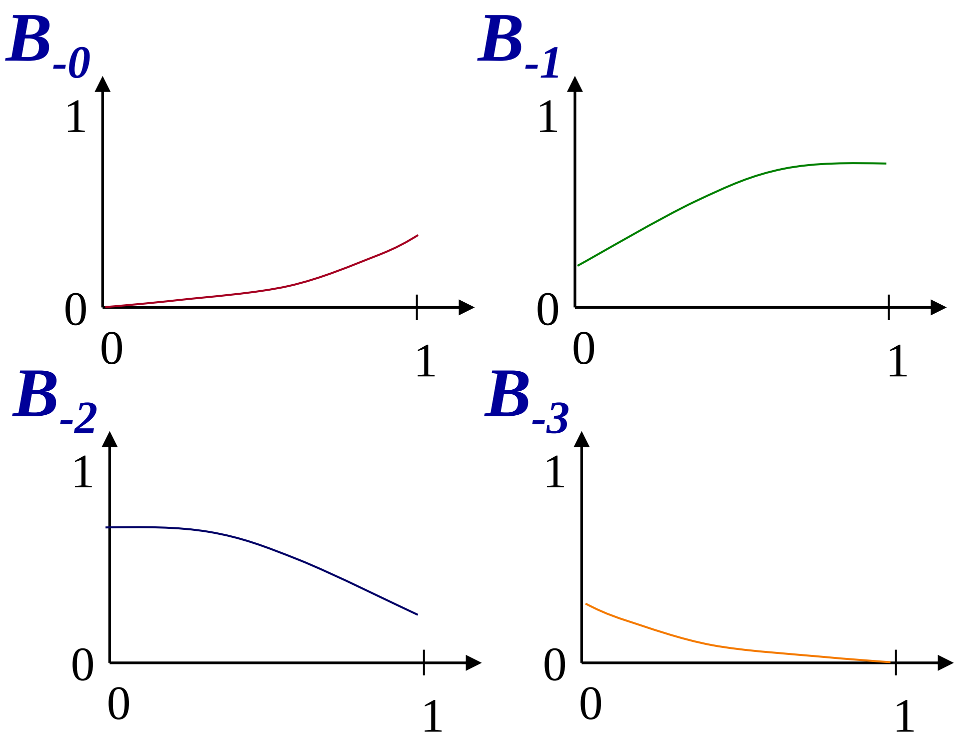


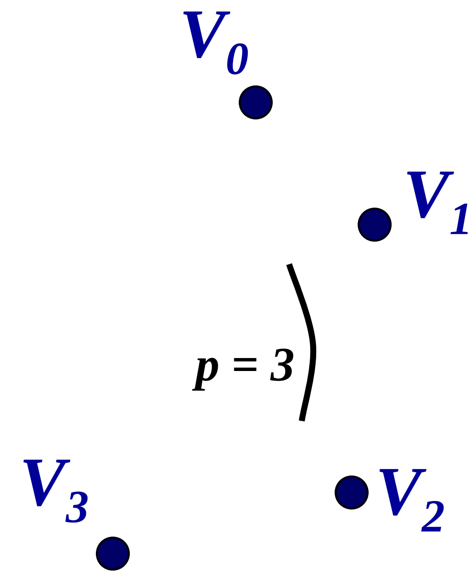


Cubic B-Spline Blending Functions

In plot form backwards (start at 3):

$$B_i(u) = \sum_{j=0}^{m} a_j u^j$$



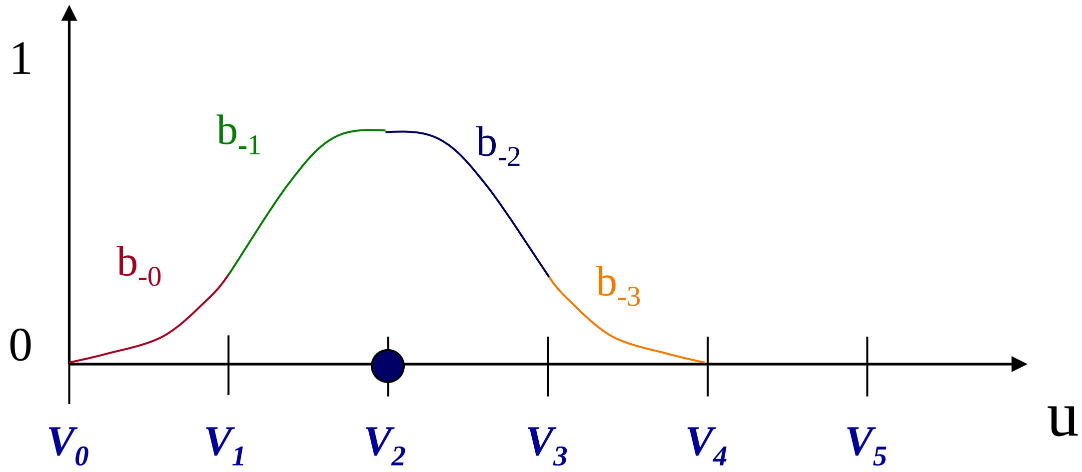


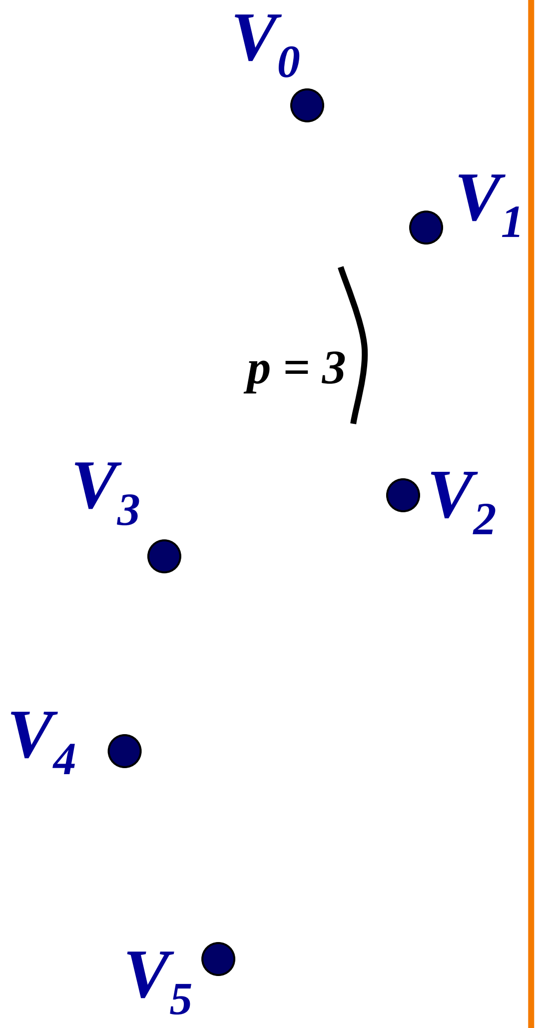
$$y_p(u) = \sum_{i=0}^{n} B_{-i}(u) * V_{p-i_x}$$

$$x_p(u) = \sum_{i=0}^{n} B_{-i}(u) * V_{p-i_y}$$

Cubic B-Spline Blending Functions

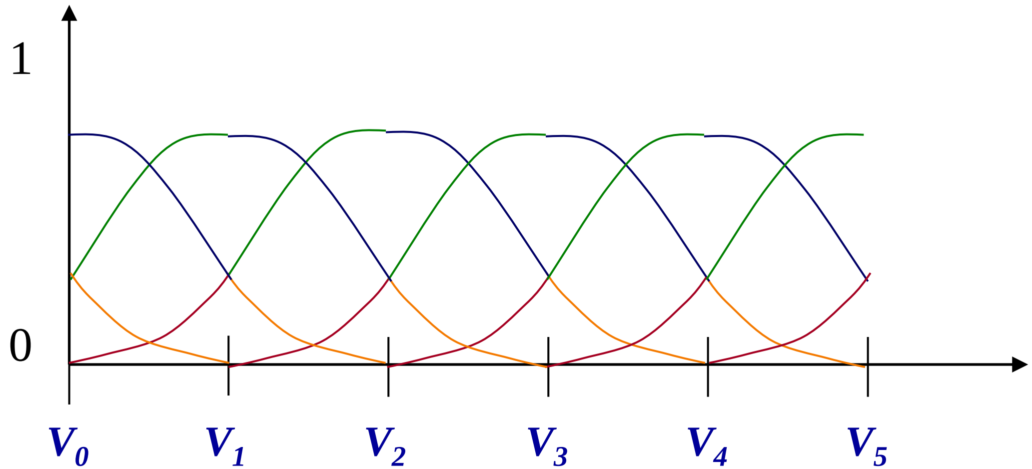
- A single blending function
- Local support
- **Approximating**

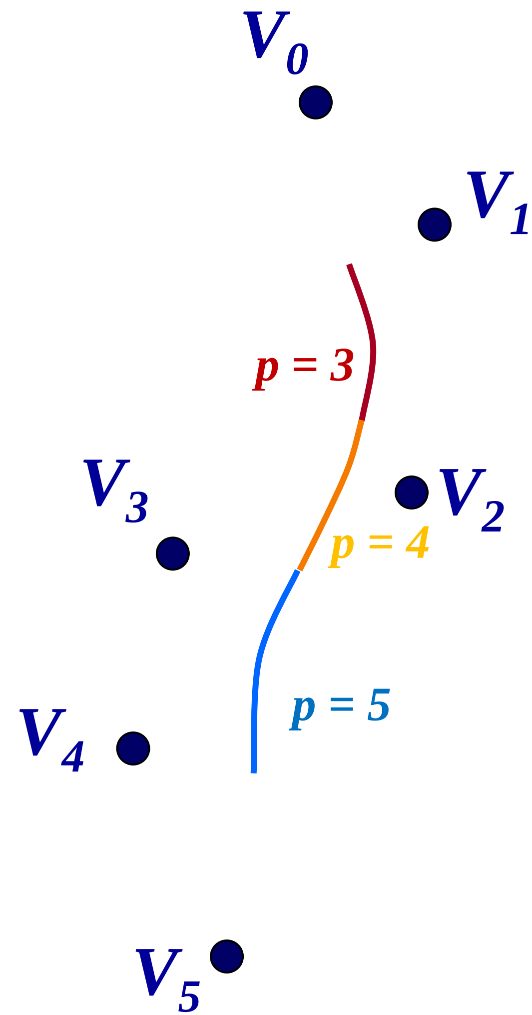




Cubic B-Spline Blending Functions

- Replicate blending functions
- Still local support

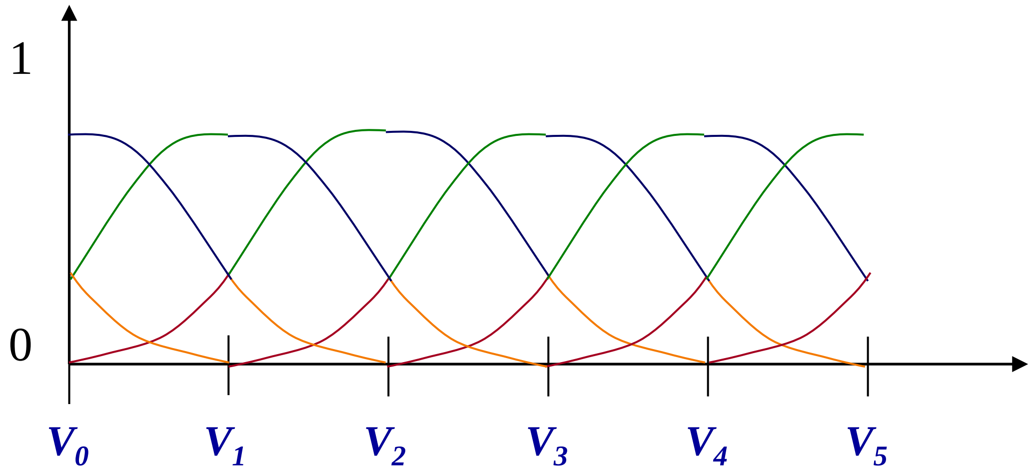


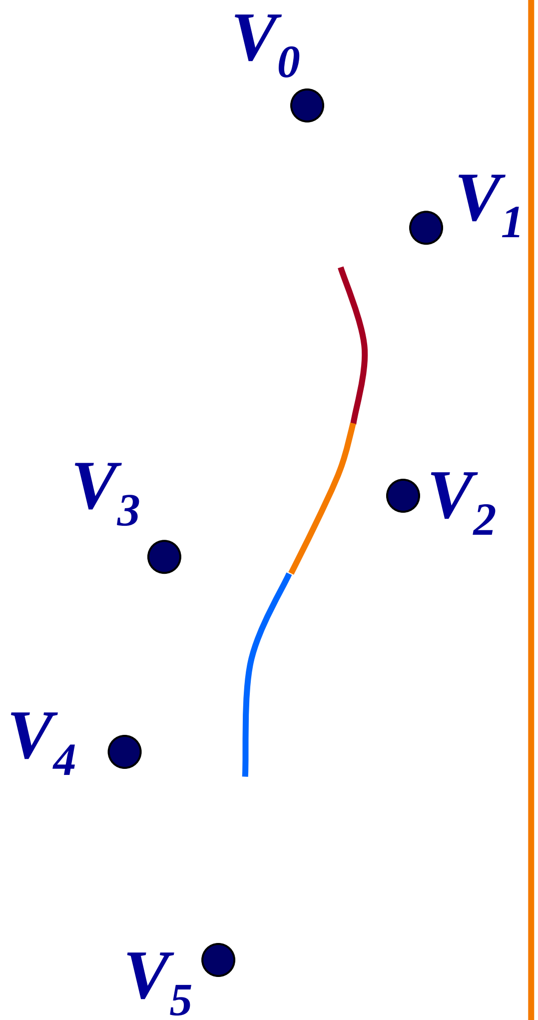


Try online at http://bl.ocks.org/mbostock/4342190

Cubic B-Spline Blending Functions

- How derive blending functions?
  - Cubic polynomials
  - Local control
  - C^2 continuity
  - Convex hull





Cubic B-Spline Blending Functions

- How derive blending functions?
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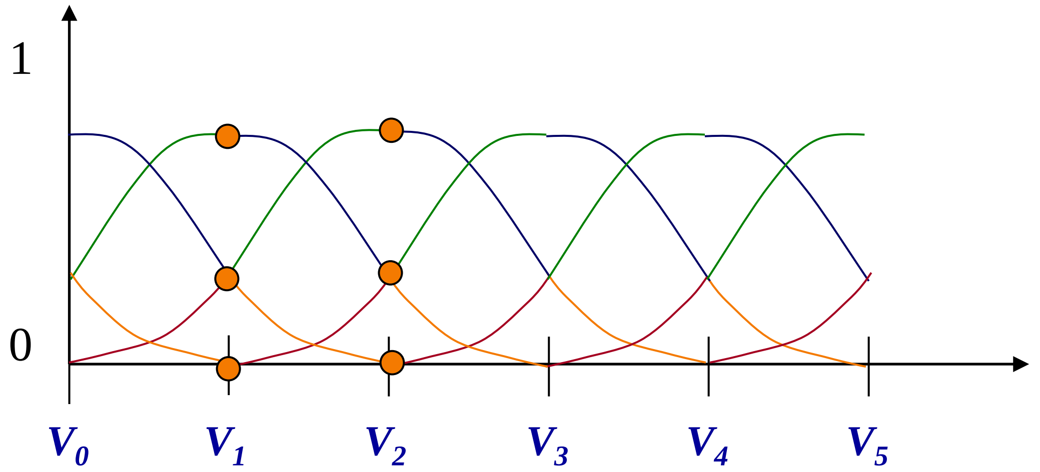
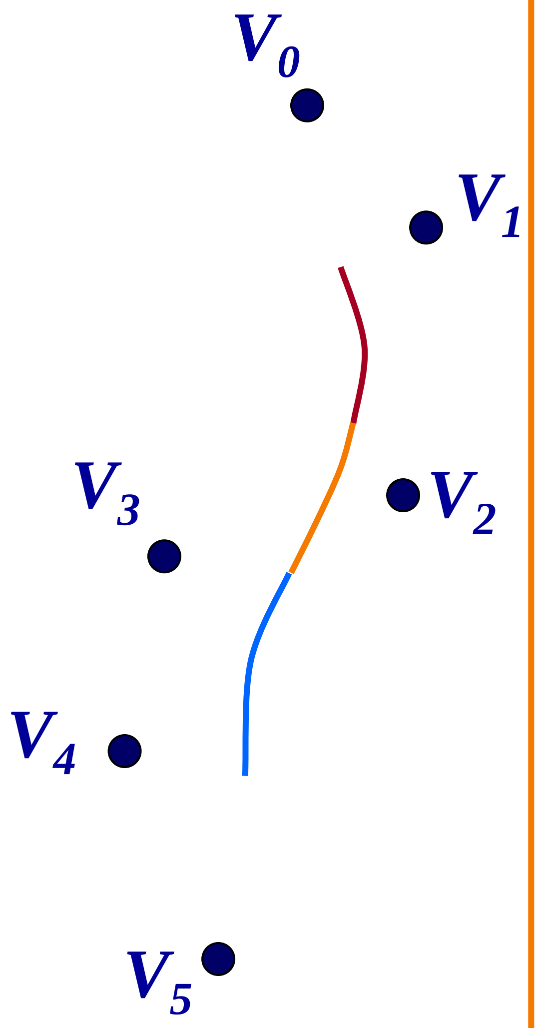
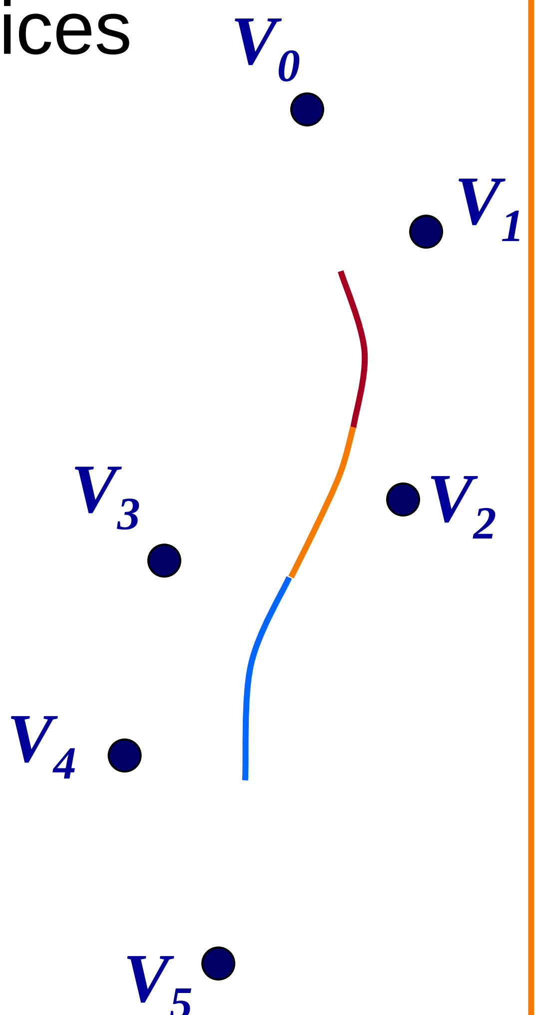




Cubic B-Spline Blending Functions

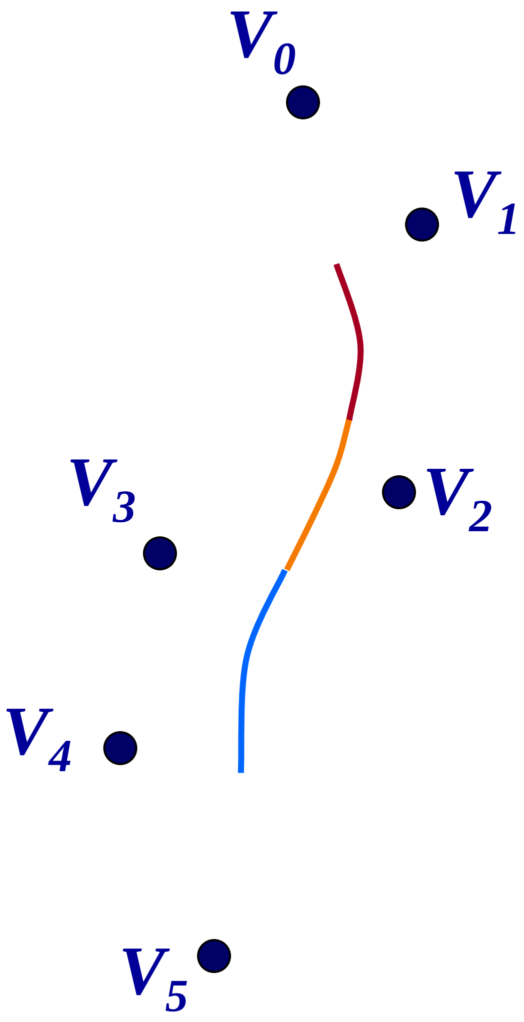
- Four cubic polynomials for four vertices
  - 16 variables (degrees of freedom)
  - Variables are a_i, b_i, c_i, d_i for four blending functions

$$b_{-0}(u) = a_0 u^3 + b_0 u^2 + c_0 u^1 + d_0$$
$$b_{-1}(u) = a_1 u^3 + b_1 u^2 + c_1 u^1 + d_1$$
$$b_{-2}(u) = a_2 u^3 + b_2 u^2 + c_2 u^1 + d_2$$
$$b_{-3}(u) = a_3 u^3 + b_3 u^2 + c_3 u^1 + d_3$$



Cubic B-Spline Blending Functions

- C^2 continuity implies 15 constraints
  - Position of two curves same
  - Derivative of two curves same
  - Second derivatives same



Cubic B-Spline Blending Functions

Fifteen continuity constraints:

$$0 = b_{-0}(0)$$
$$b_{-0}(1) = b_{-1}(0)$$
$$b_{-1}(1) = b_{-2}(0)$$
$$b_{-2}(1) = b_{-3}(0)$$
$$b_{-3}(1) = 0$$

$$0 = b_{-0}'(0)$$
$$b_{-0}'(1) = b_{-1}'(0)$$
$$b_{-1}'(1) = b_{-2}'(0)$$
$$b_{-2}'(1) = b_{-3}'(0)$$
$$b_{-3}'(1) = 0$$

$$0 = b_{-0}''(0)$$
$$b_{-0}''(1) = b_{-1}''(0)$$
$$b_{-1}''(1) = b_{-2}''(0)$$
$$b_{-2}''(1) = b_{-3}''(0)$$
$$b_{-3}''(1) = 0$$

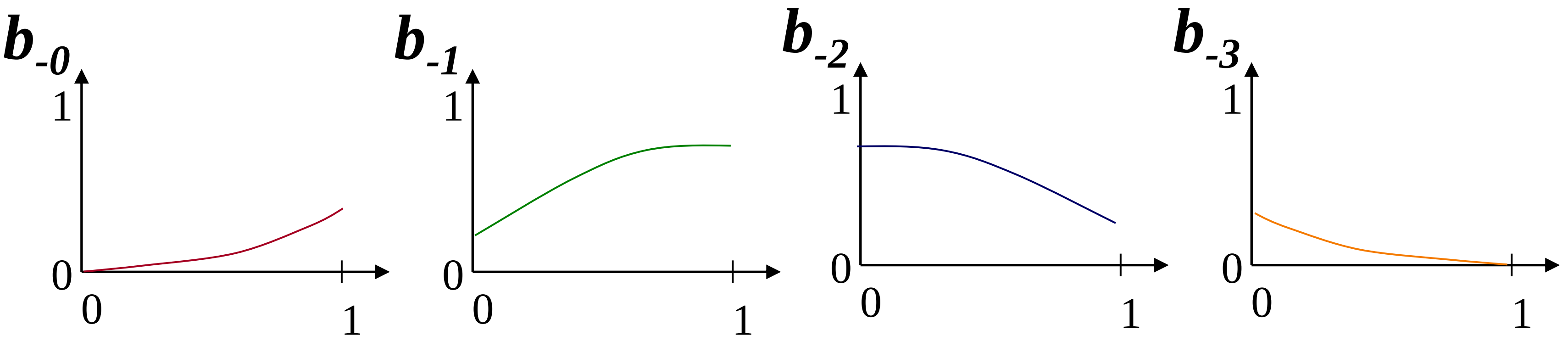
One more convenient constraint:

$$b_{-0}(0) + b_{-1}(0) + b_{-2}(0) + b_{-3}(0) = 1$$

Cubic B-Spline Blending Functions

- Solving the system of equations yields:

$$
\begin{aligned}
b_{-3}(u) &= -\tfrac{1}{6}u^3 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u + \tfrac{1}{6} \\
b_{-2}(u) &= \tfrac{1}{2}u^3 - u^2 + \tfrac{2}{3} \\
b_{-1}(u) &= -\tfrac{1}{2}u^3 + \tfrac{1}{2}u^2 + \tfrac{1}{2}u + \tfrac{1}{6} \\
b_{-0}(u) &= \tfrac{1}{6}u^3
\end{aligned}
$$



Cubic B-Spline Blending Functions

- In matrix form:

$$\underbrace{\mathrm{Q}(u) = \sum_{i=0}^{n} B_i(u) * V_i \qquad B_i(u) = \sum_{j=0}^{m} a_j u^j}$$

Cubic B-Spline Blending Functions

- In matrix form:

$$Q(u) = \sum_{i=0}^{n} B_i(u) * V_i \qquad B_i(u) = \sum_{j=0}^{m} a_j u^j$$

$$Q(u) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

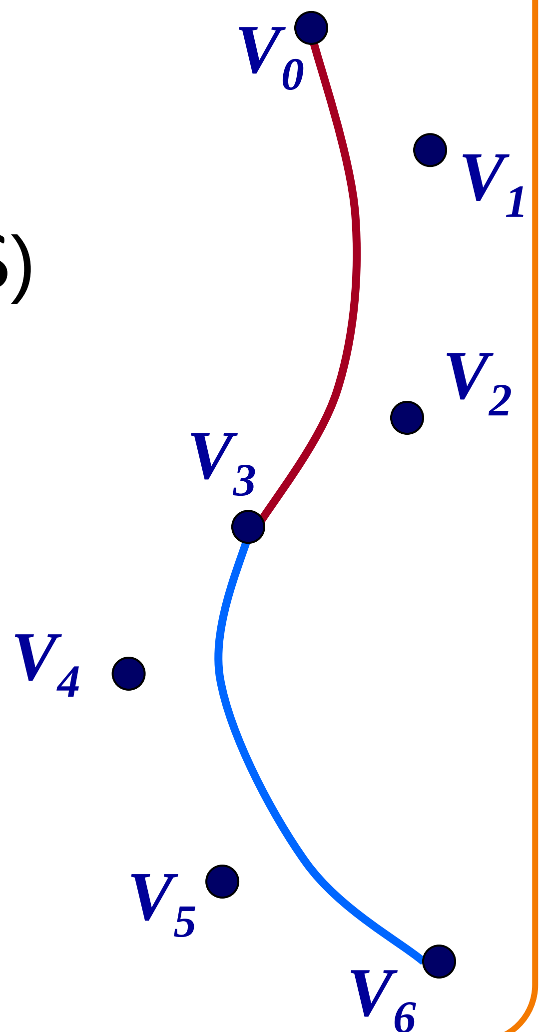
Outline

- Parametric curves
  - Cubic B-Spline
  - Cubic Bézier
- Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

Bézier Curves

- Developed around 1960 by both
  - Pierre Bézier (Renault)
  - Paul de Casteljau (Citroen)
- Today: graphic design (e.g. FONTS)
- Properties:
  - Local control
  - Continuity **depends** on control points
  - **Interpolating** (every third for cubic)

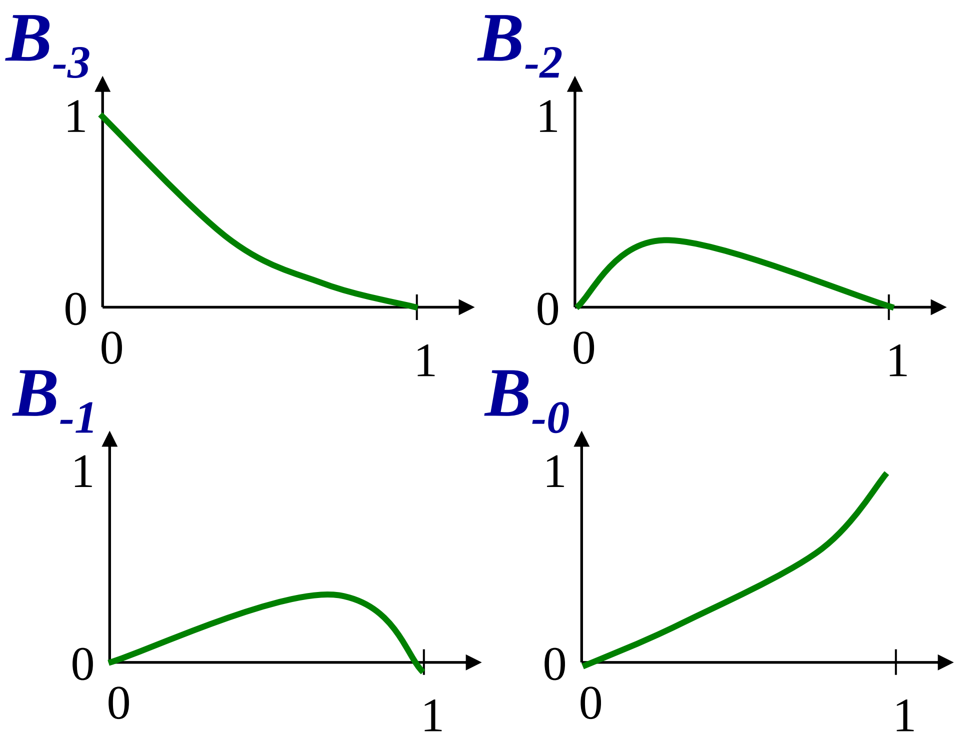
Blending functions determine properties

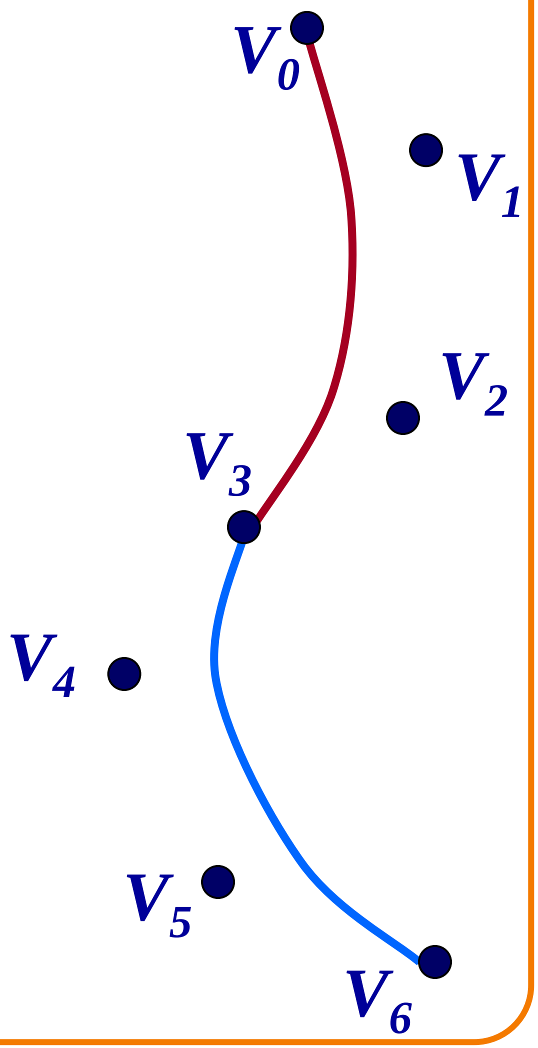


Cubic Bézier Curves

Blending functions:

$$Q(u) = \sum_{i=0}^{n} B_i(u) * V_i \qquad B_i(u) = \sum_{j=0}^{m} a_j u^j$$





Basic properties of Bézier Curves

- Endpoint interpolation:

$$Q(0) = V_0$$

$$Q(1) = V_n$$

- Convex hull:
  - Curve is contained within convex hull of control polygon

- Symmetry

$$Q(u) \text{ defined by } \{V_0, \ldots, V_n\} \;\equiv\; Q(1-u) \text{ defined by } \{V_n, \ldots, V_0\}$$

Cubic Bézier Curves

Bézier curves in matrix form:

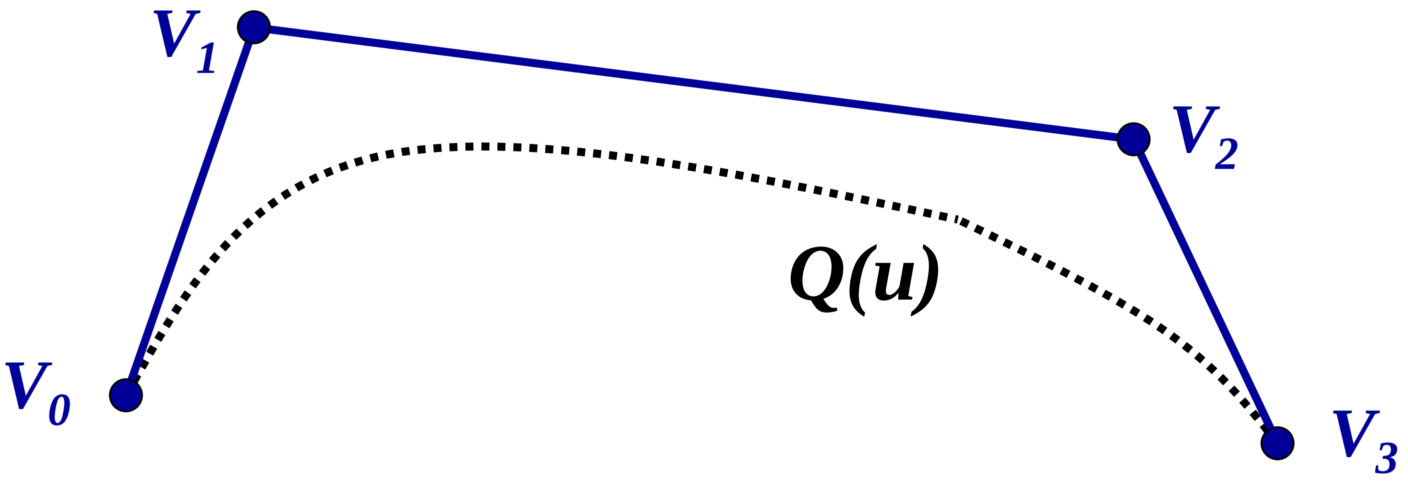
$$Q(u) = \sum_{i=0}^{n} V_i \binom{n}{i} u^i (1-u)^{n-i}$$

$$= (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u) V_2 + u^3 V_3$$

$$= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \underbrace{\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{M}_{\textbf{Bézier}}} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Bézier Curves

- Curve $Q(u)$ can also be defined by nested interpolation:

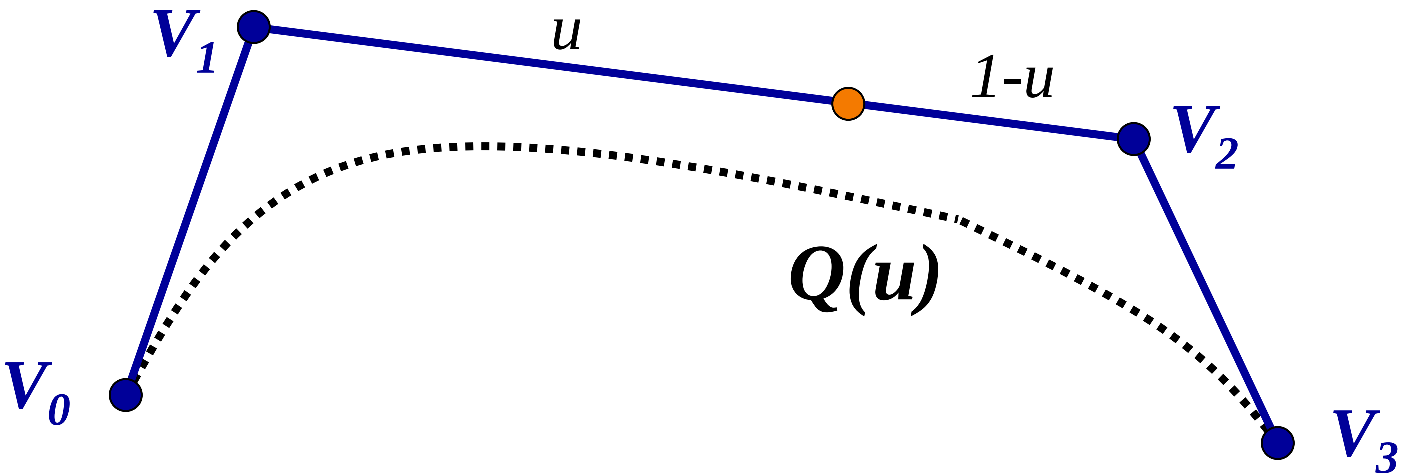


V_i are control points

$\{V_0, V_1, \ldots, V_n\}$ is control polygon

Bézier Curves

- Curve $\boldsymbol{Q(u)}$ can also be defined by nested interpolation:

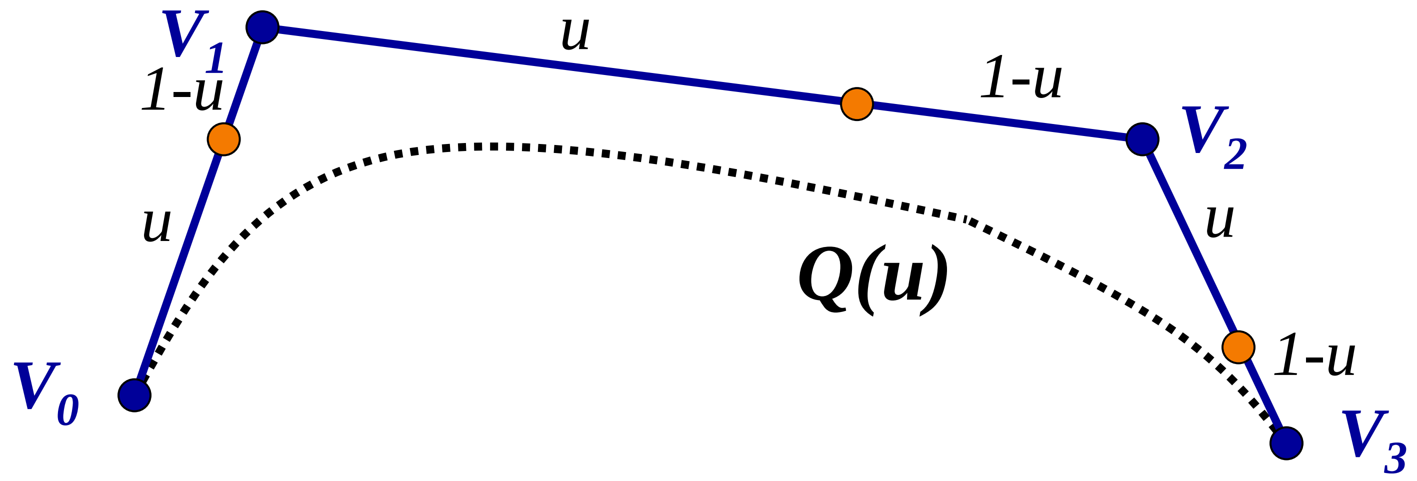


V_i are *control points*

$\{V_0, V_1, \ldots, V_n\}$ is *control polygon*

Bézier Curves

- Curve $Q(u)$ can also be defined by nested interpolation:



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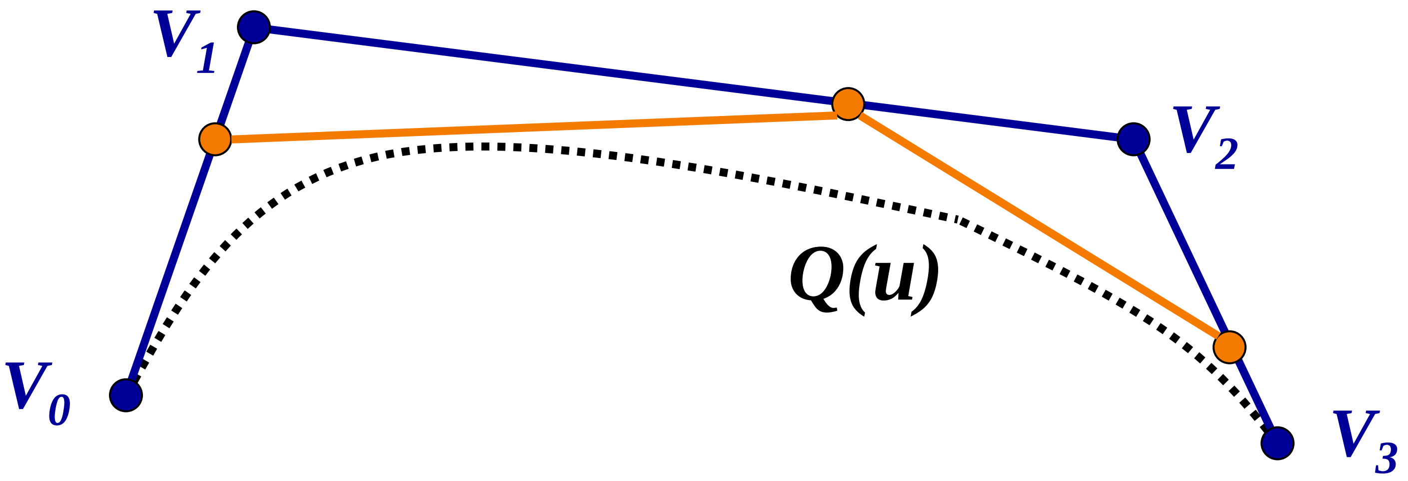
Bézier Curves

- Curve $Q(u)$ can also be defined by nested interpolation:



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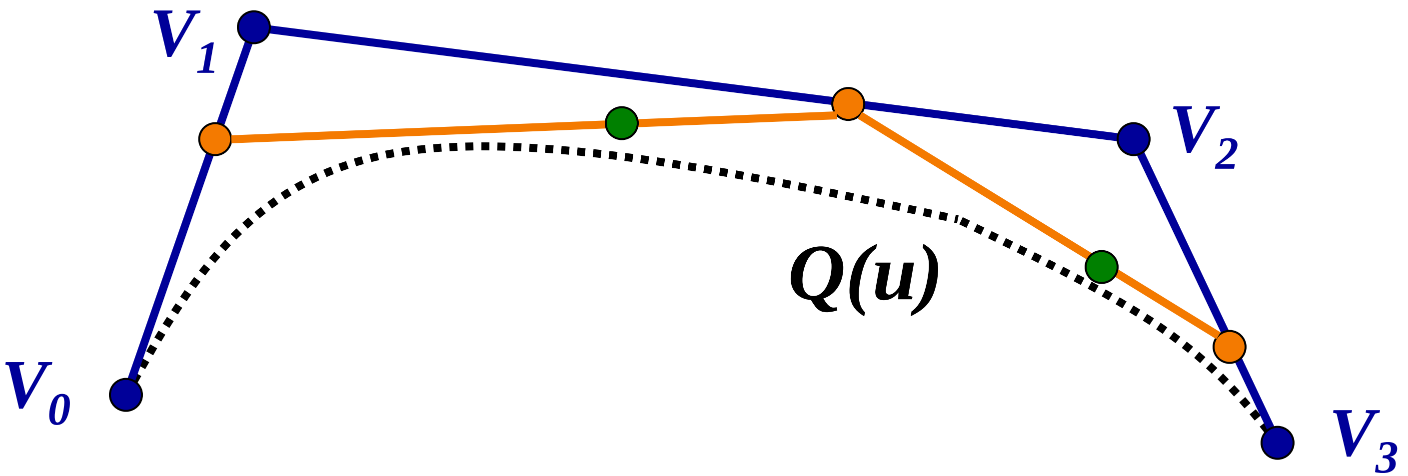
Bézier Curves

- Curve $\boldsymbol{Q(u)}$ can also be defined by nested interpolation:

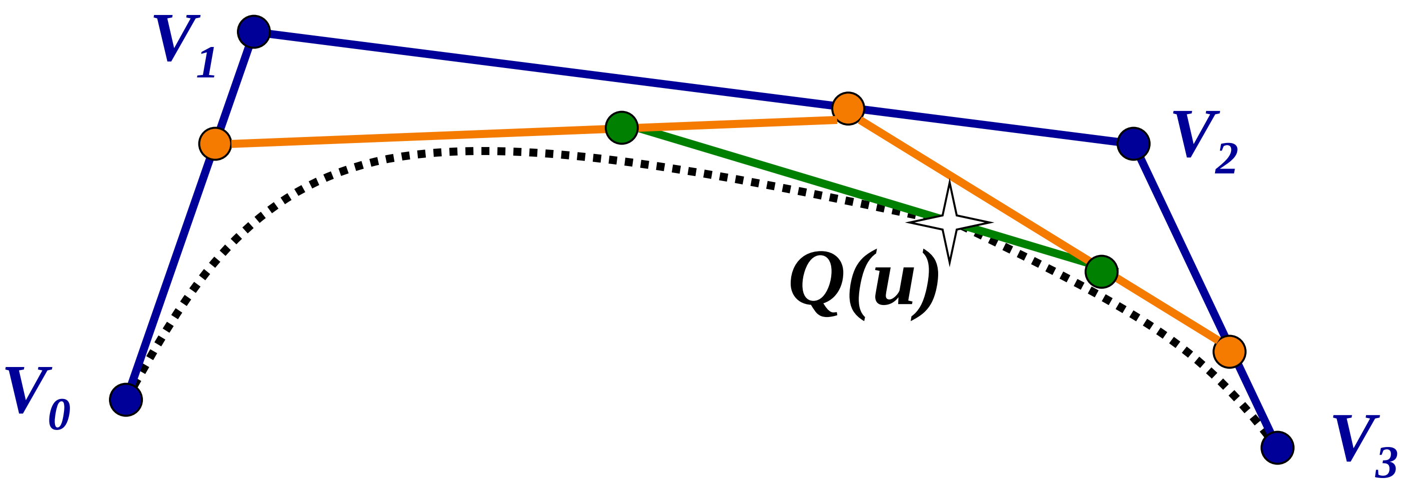


$\boldsymbol{V_i}$ are *control points*

$\{\boldsymbol{V_0}, \boldsymbol{V_1}, \ldots, \boldsymbol{V_n}\}$ is *control polygon*

Bézier Curves

- Curve $Q(u)$ can also be defined by nested interpolation:

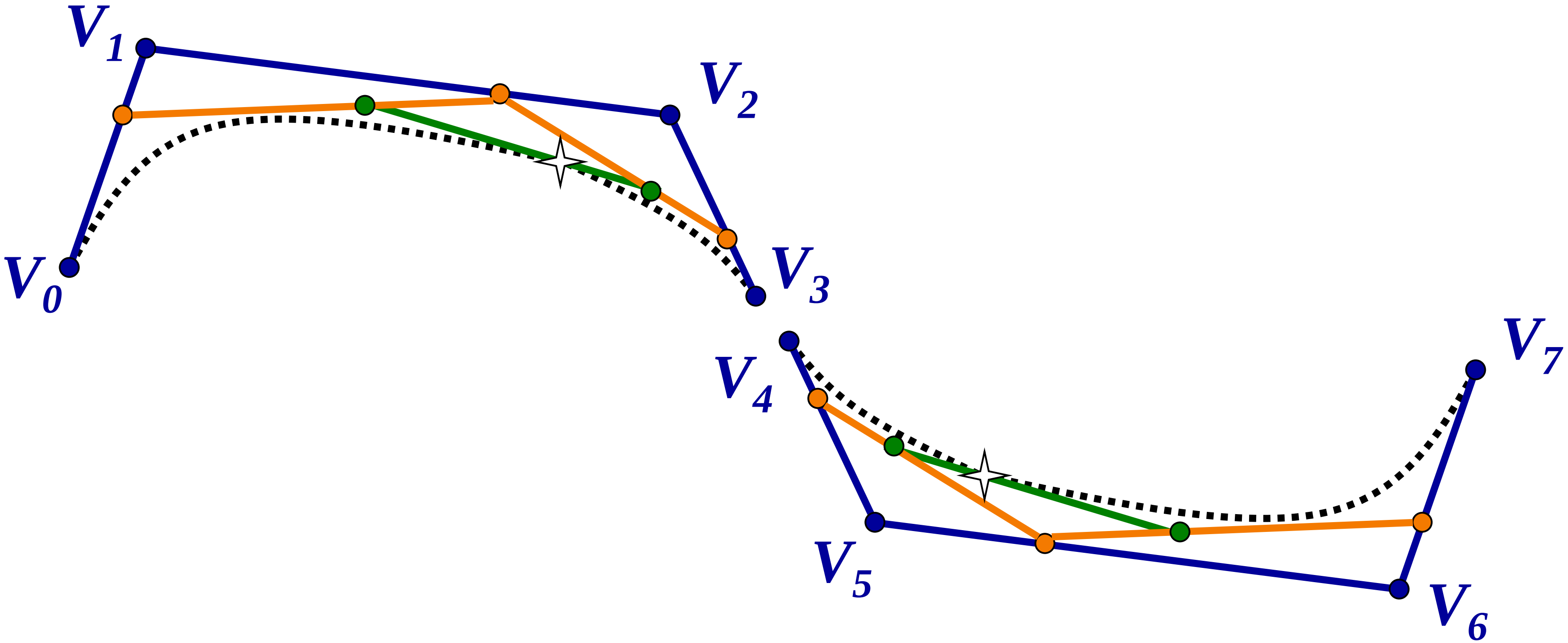


V_i are *control points*

$\{V_0, V_1, \ldots, V_n\}$ is *control polygon*

Enforcing Bézier Curve Continuity

- C^0: $V_3 = V_4$
- C^1: $V_5 - V_4 = V_3 - V_2$
- C^2: $V_6 - 2V_5 + V_4 = V_3 - 2V_2 + V_1$



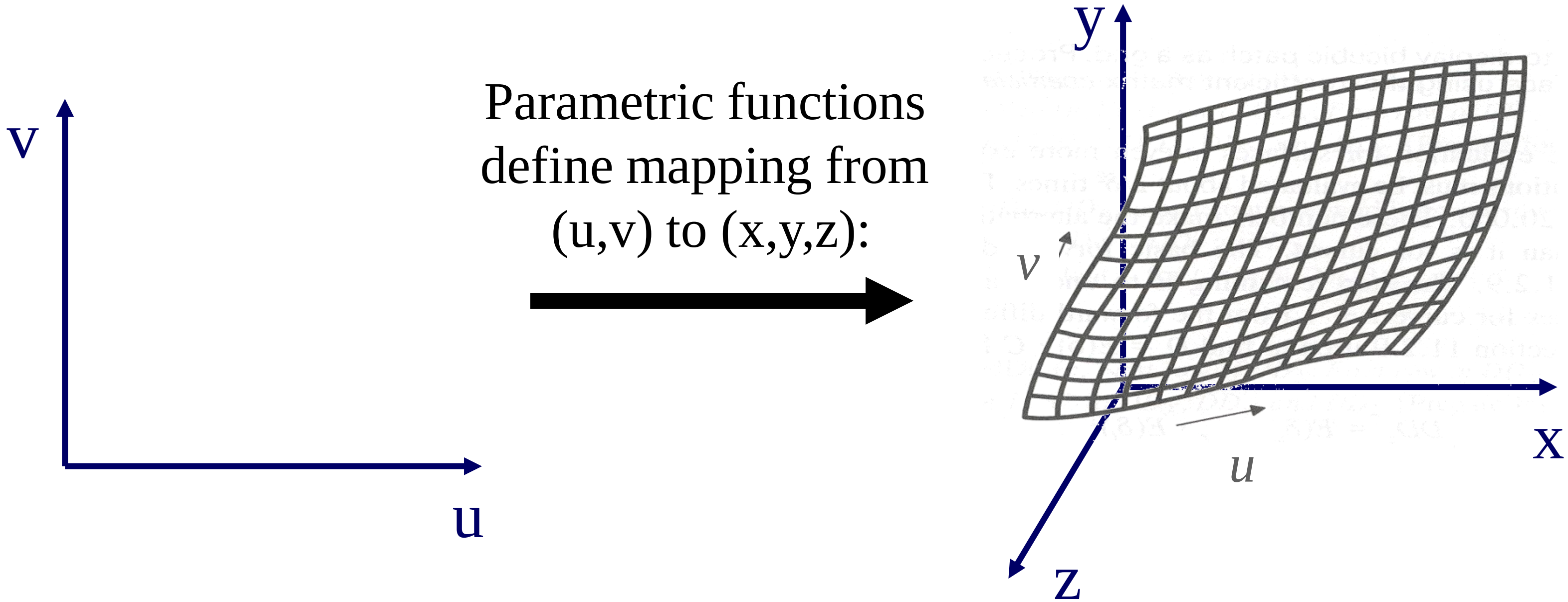
Outline

- Parametric curves
  - Cubic B-Spline
  - Cubic Bézier

➢ Parametric surfaces
  - Bi-cubic B-Spline
  - Bi-cubic Bézier

Parametric Surfaces

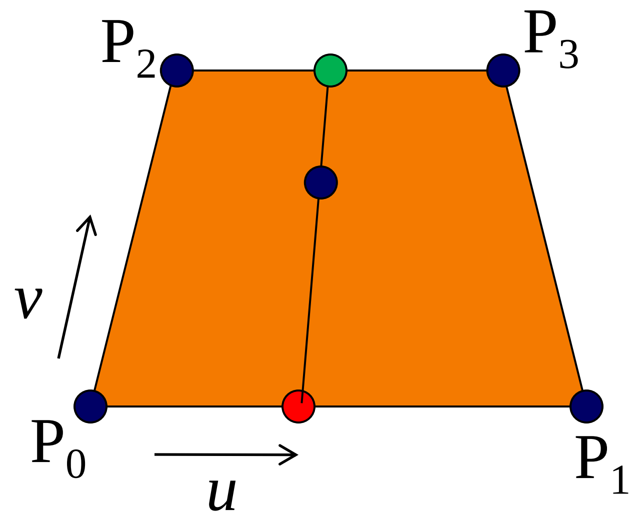
- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$



FvDFH Figure 11.42

Parametric Surfaces

- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$

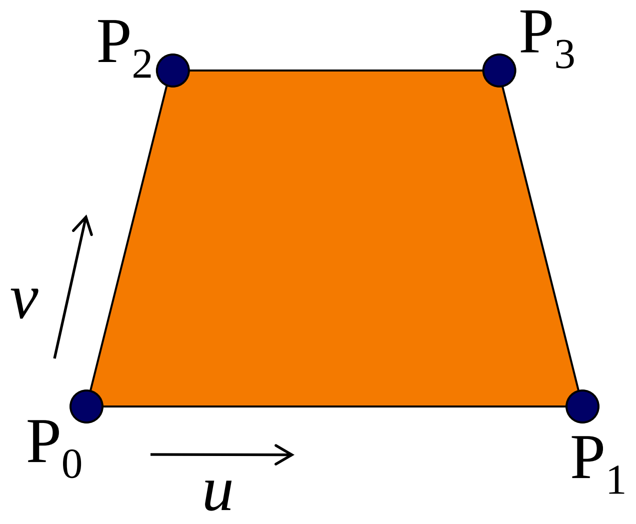


- Example: quadrilateral

$$f_x(u,v) = (1-v)\big((1-u)x_0 + ux_1\big) + v\big((1-u)x_2 + ux_3\big)$$
$$f_y(u,v) = (1-v)\big((1-u)y_0 + uy_1\big) + v\big((1-u)y_2 + uy_3\big)$$
$$f_z(u,v) = (1-v)\big((1-u)z_0 + uz_1\big) + v\big((1-u)z_2 + uz_3\big)$$

Parametric Surfaces

- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$



- Example: quadrilateral

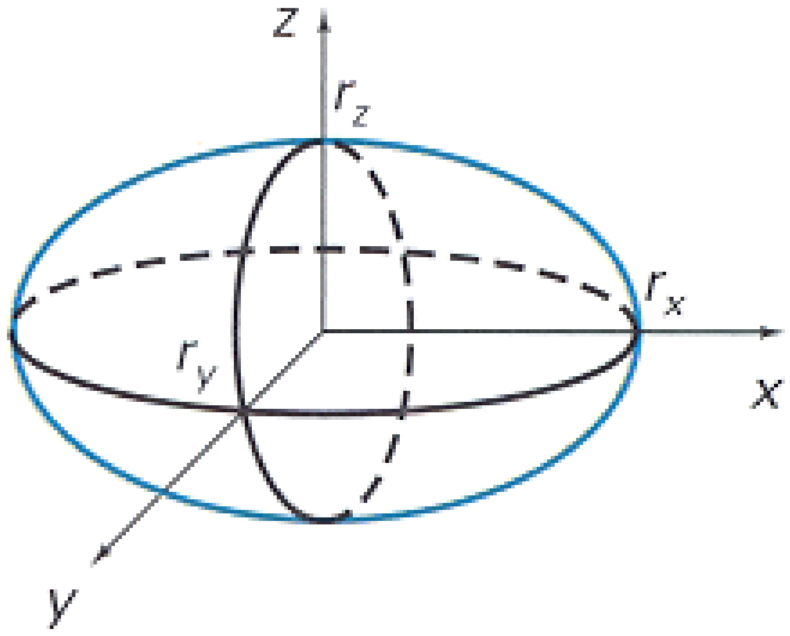
$$f_x(u,v) = (1-v)\left((1-u)x_0 + ux_1\right) + v\left((1-u)x_2 + ux_3\right)$$
$$f_y(u,v) = (1-v)\left((1-u)y_0 + uy_1\right) + v\left((1-u)y_2 + uy_3\right)$$
$$f_z(u,v) = (1-v)\left((1-u)z_0 + uz_1\right) + v\left((1-u)z_2 + uz_3\right)$$

Parametric Surfaces

- Defined by parametric functions:
  - $x = f_x(u,v)$
  - $y = f_y(u,v)$
  - $z = f_z(u,v)$

- Example: ellipsoid

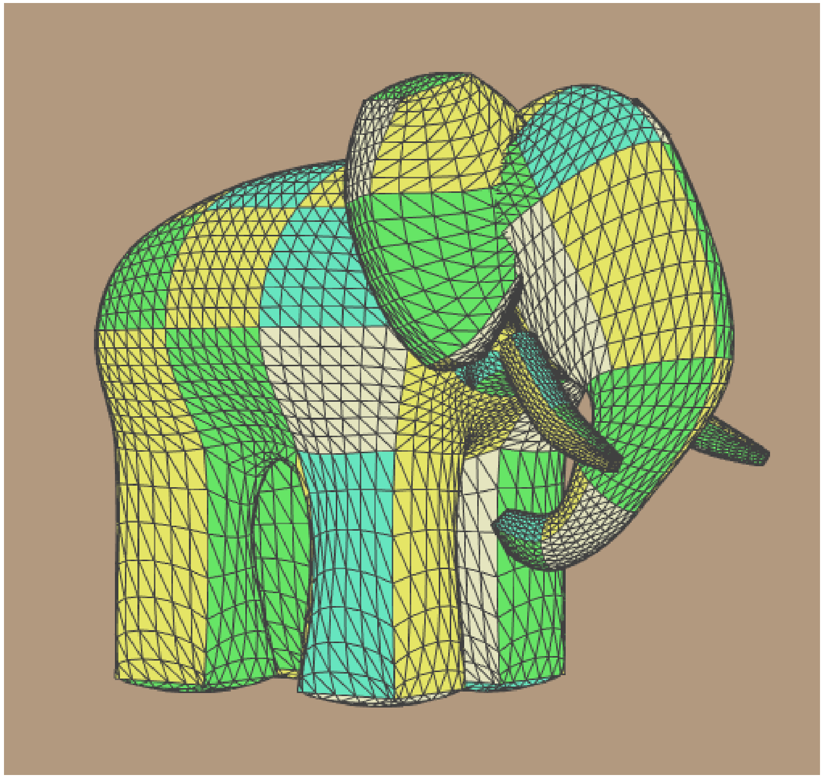
$$f_x(u,v) = r_x \cos v \cos u$$
$$f_y(u,v) = r_y \cos v \sin u$$
$$f_z(u,v) = r_z \sin v$$



H&B Figure 10.10

Parametric Surfaces

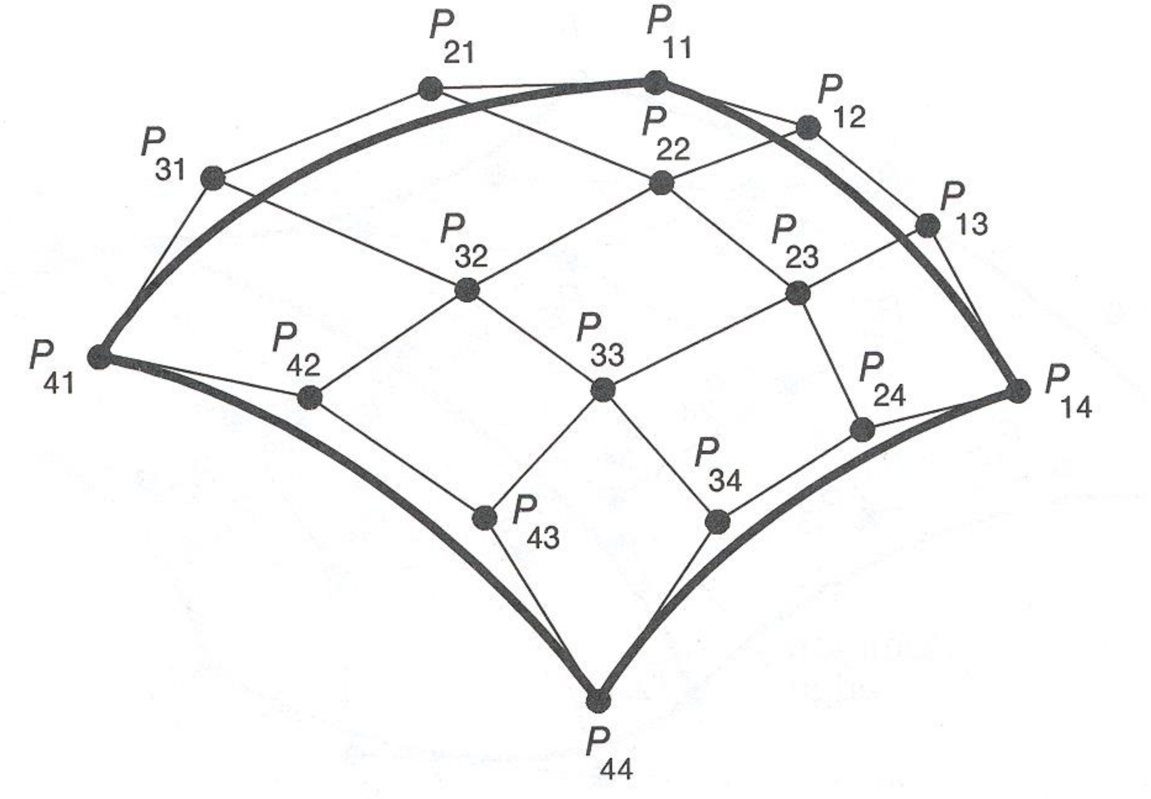
To model arbitrary shapes, surface is partitioned into parametric patches



Catnull

Parametric Patches

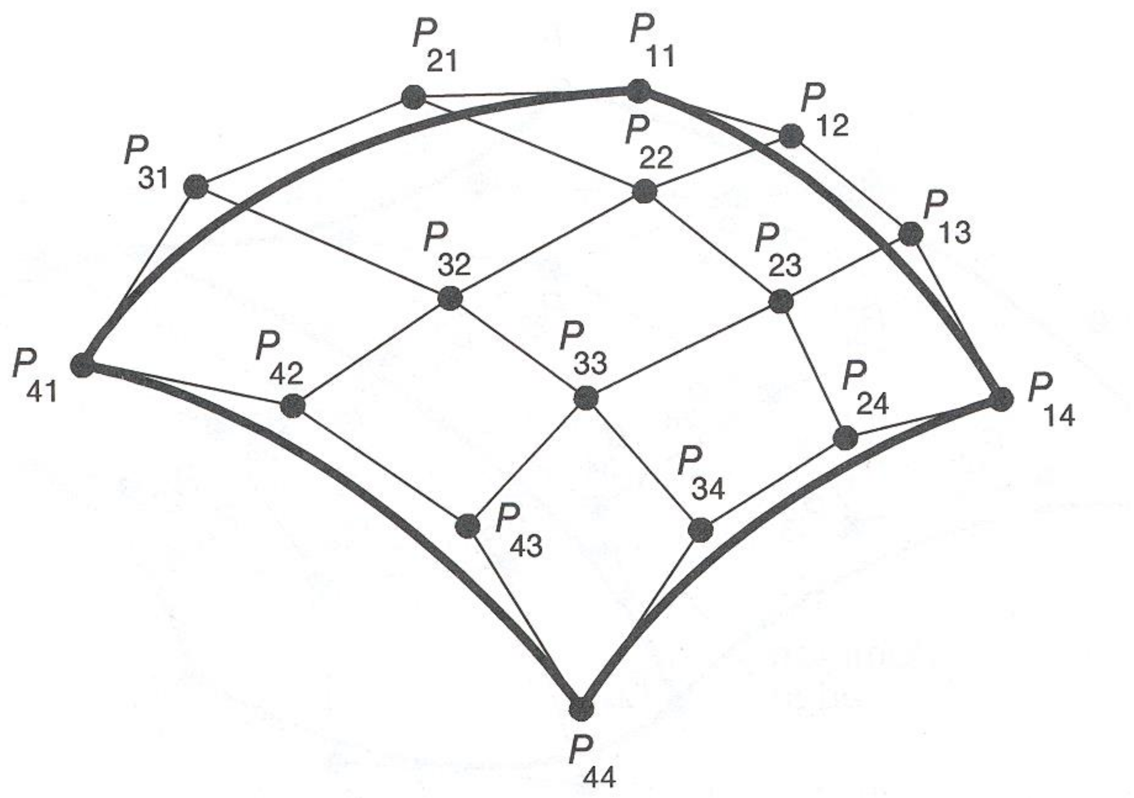
- Each patch is defined by blending control points



FvDFH Figure 11.44

Parametric Patches

- Each patch is defined by blending control points

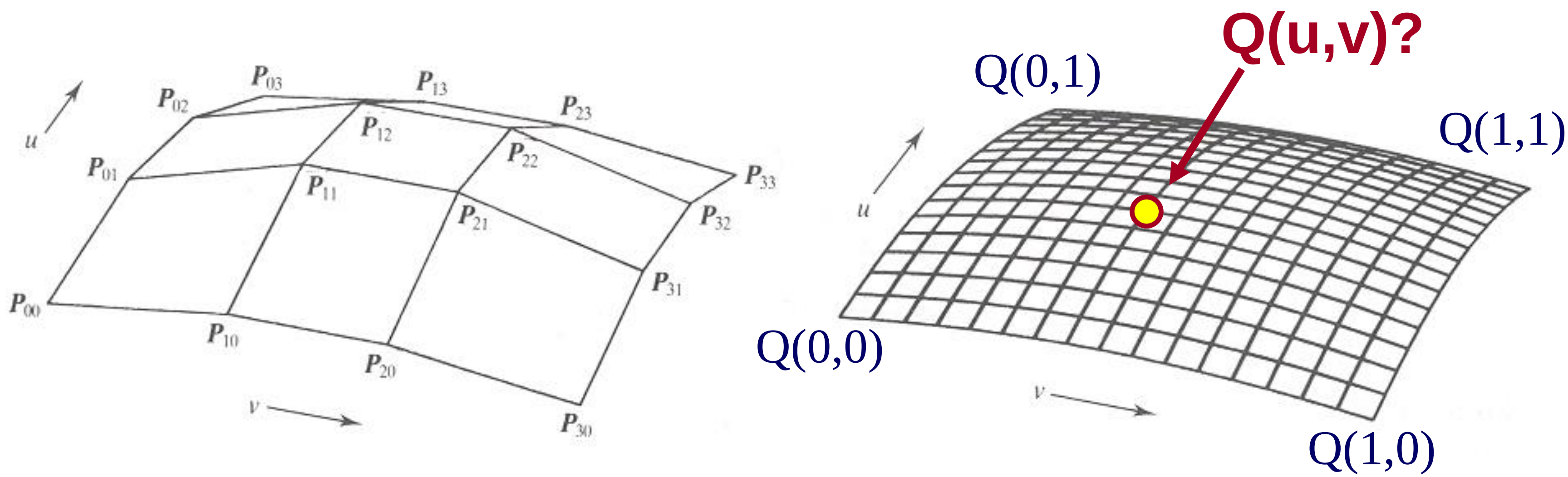


Same ideas as parametric curves!

FvDFH Figure 11.44

Parametric Patches

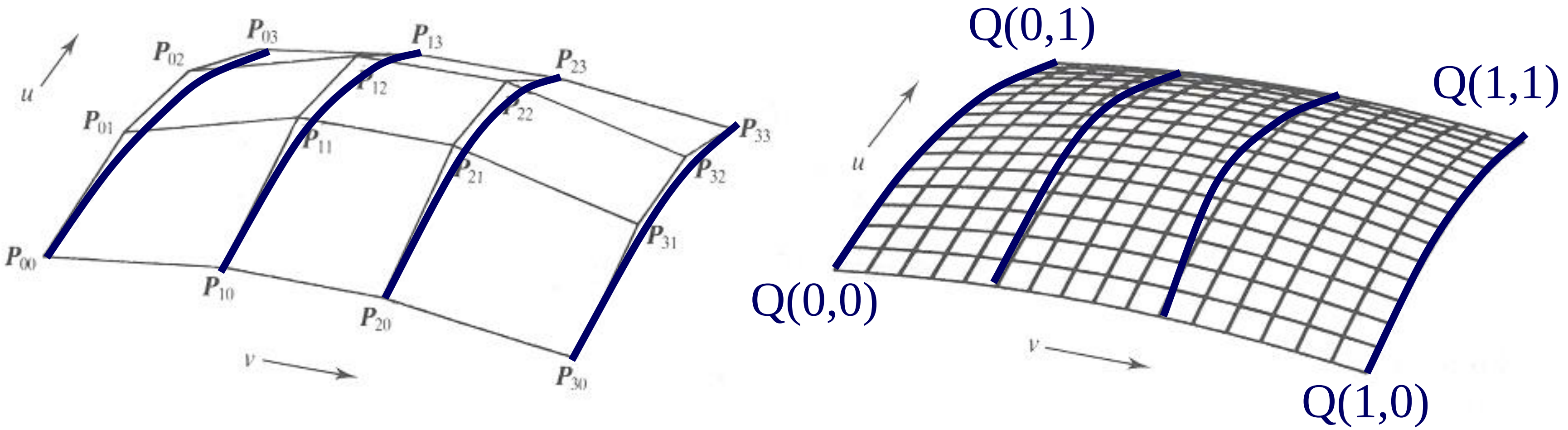
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

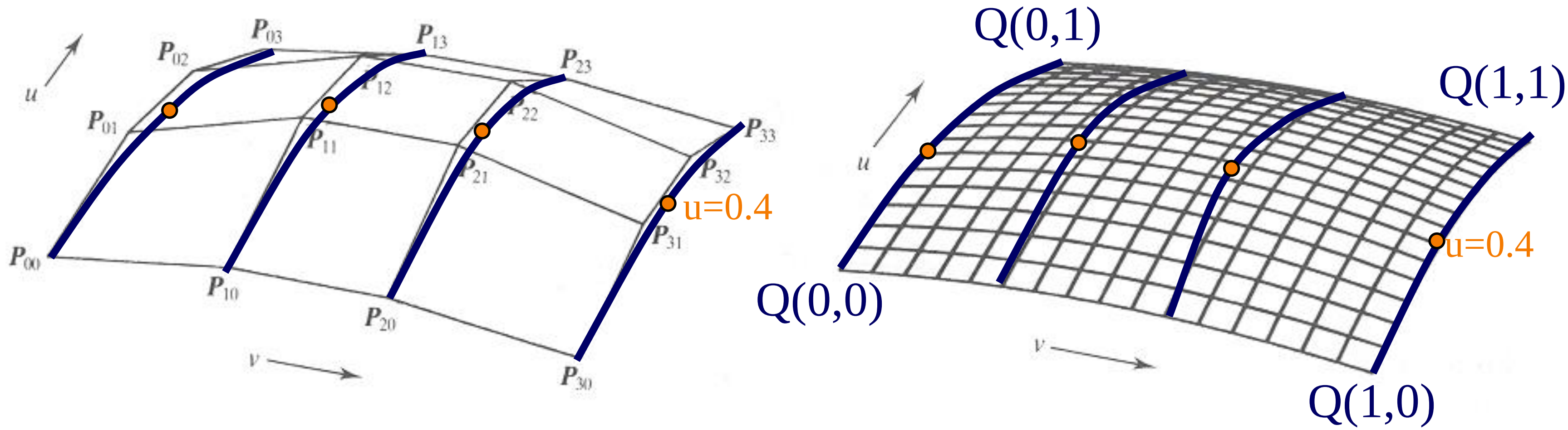
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

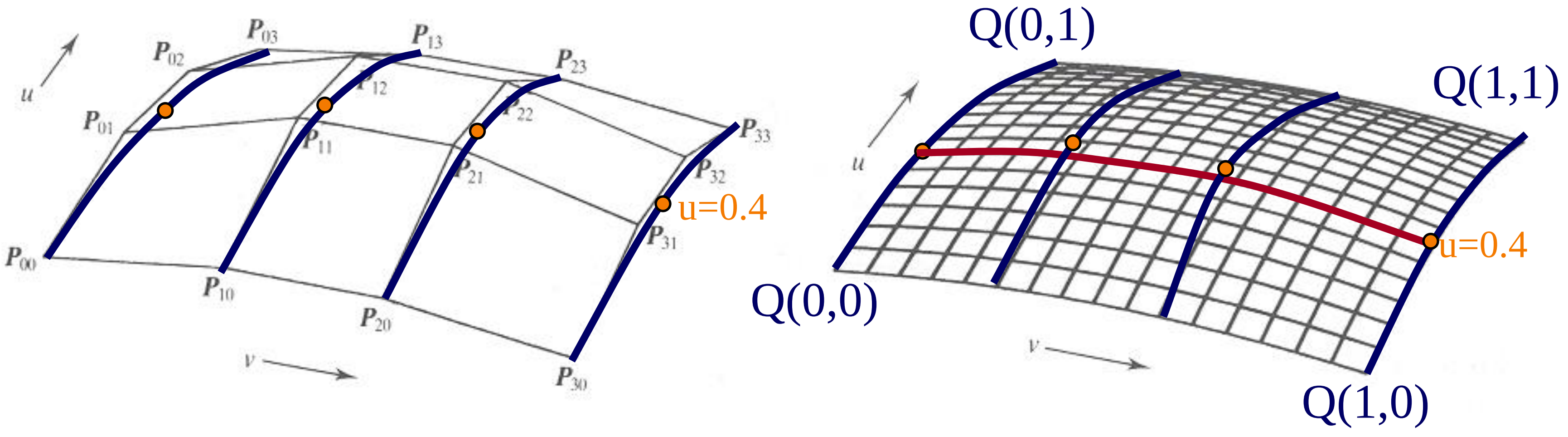
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

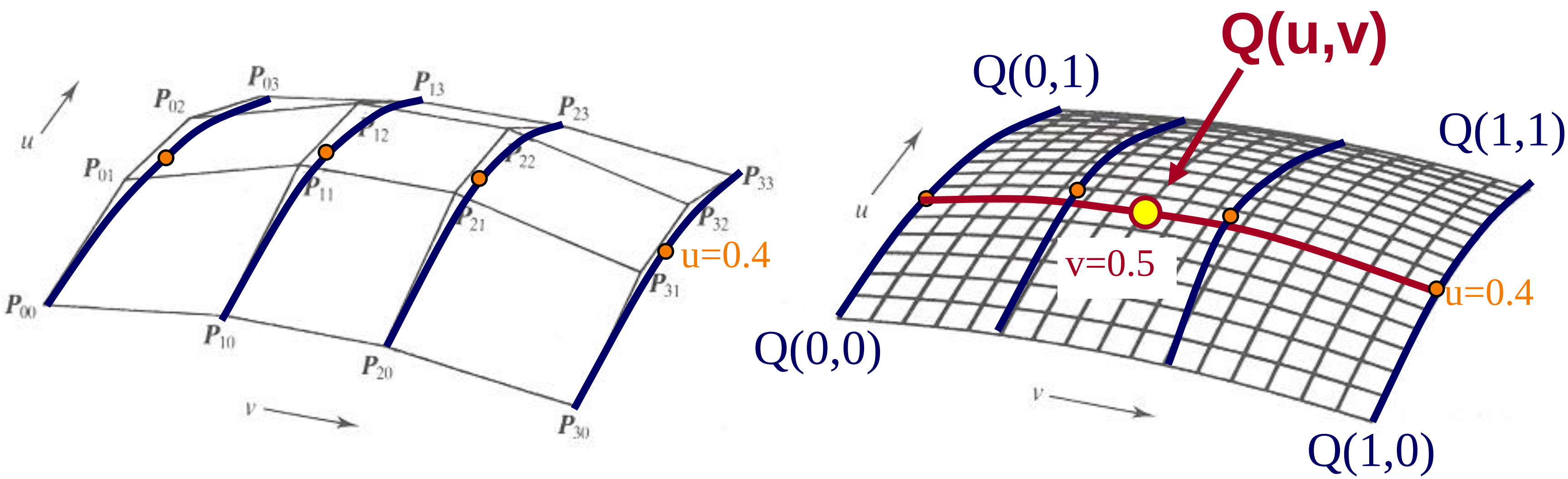
- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Patches

- Point Q(u,v) on the patch is the tensor product of parametric curves defined by the control points



Watt Figure 6.21

Parametric Bicubic Patches

Point Q(u,v) on any patch is defined by combining control points with polynomial blending functions:

$$Q(u,v) = \mathbf{UM}\begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix}\mathbf{M}^{\mathbf{T}}\mathbf{V}^{\mathbf{T}}$$

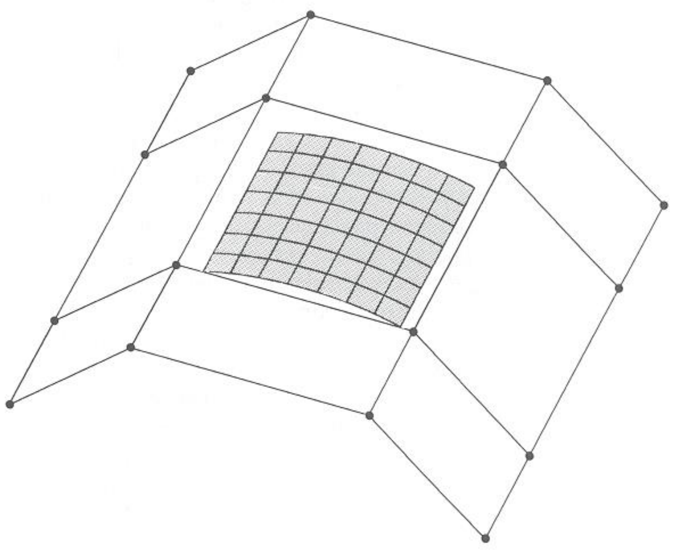
$$\mathbf{U} = [u^3 \quad u^2 \quad u \quad 1] \qquad \mathbf{V} = [v^3 \quad v^2 \quad v \quad 1]$$

Where M is a matrix describing the blending functions for a parametric cubic curve (e.g., Bézier, B-spline, etc.)

B-Spline Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\mathbf{B-Spline}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}_{\mathbf{B-Spline}}^{\mathbf{T}} \mathbf{V}$$

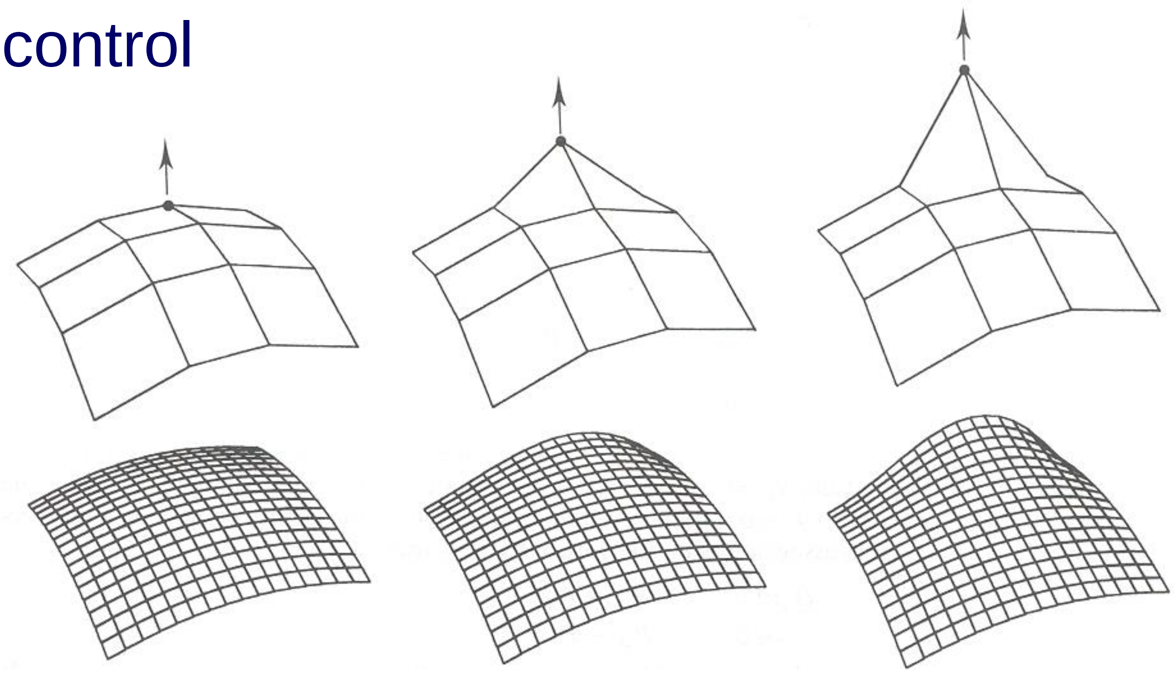
$$\mathbf{M}_{\mathbf{B-Spline}} = \begin{bmatrix} -1/6 & 1/2 & -1/2 & 1/6 \\ 1/2 & -1 & 1/2 & 0 \\ -1/2 & 0 & 1/2 & 0 \\ 1/6 & 2/3 & 1/6 & 0 \end{bmatrix}$$



Watt Figure 6.28

Bézier Patches

- Properties:
  - Interpolates four corner points
  - Convex hull
  - Local control

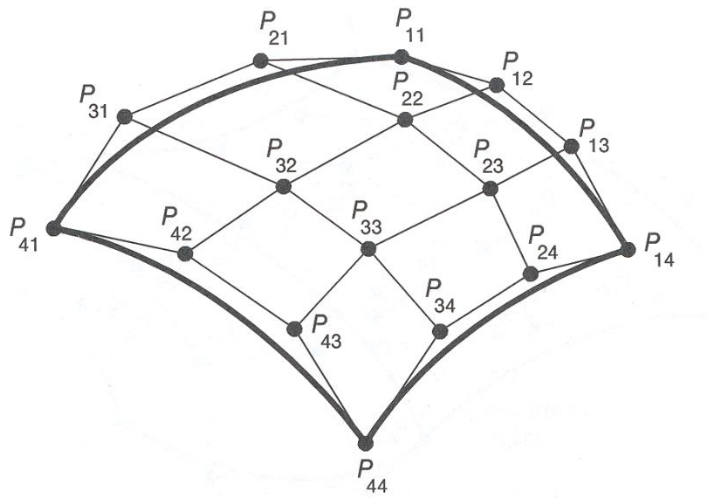


Watt Figure 6.22

Bézier Patches

$$Q(u,v) = \mathbf{U}\mathbf{M}_{\mathbf{Bezier}} \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} & P_{1,4} \\ P_{2,1} & P_{2,2} & P_{2,3} & P_{2,4} \\ P_{3,1} & P_{3,2} & P_{3,3} & P_{3,4} \\ P_{4,1} & P_{4,2} & P_{4,3} & P_{4,4} \end{bmatrix} \mathbf{M}_{\mathbf{Bezier}}^{\mathbf{T}} \mathbf{V}$$

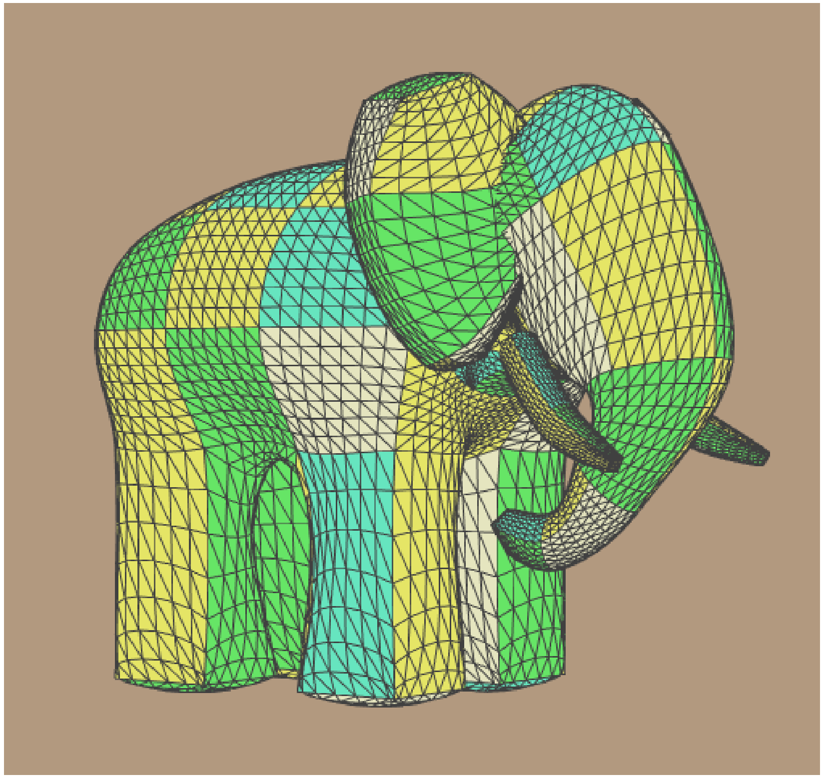
$$\mathbf{M}_{\mathbf{Bezier}} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



FvDFH Figure 11.42

Piecewise Polynomial Parametric Surfaces

Surface is composition of many parametric patches



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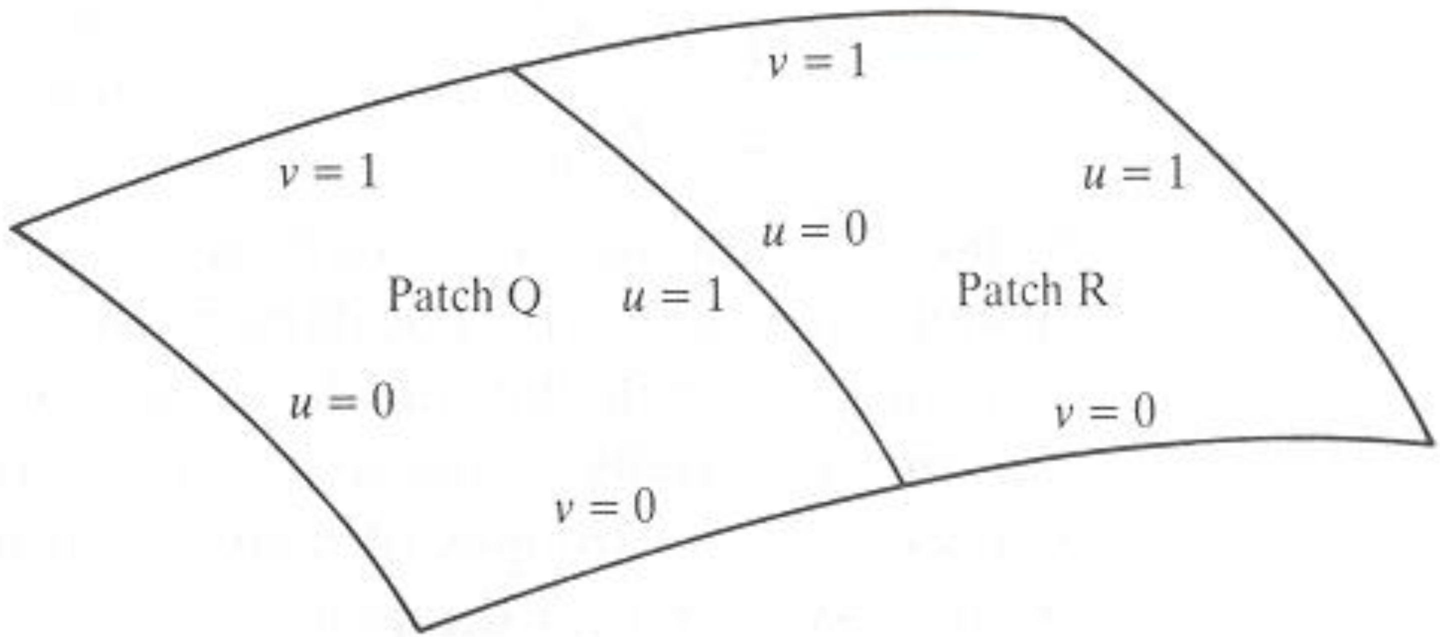
Piecewise Polynomial Parametric Surfaces

Must maintain continuity across seams

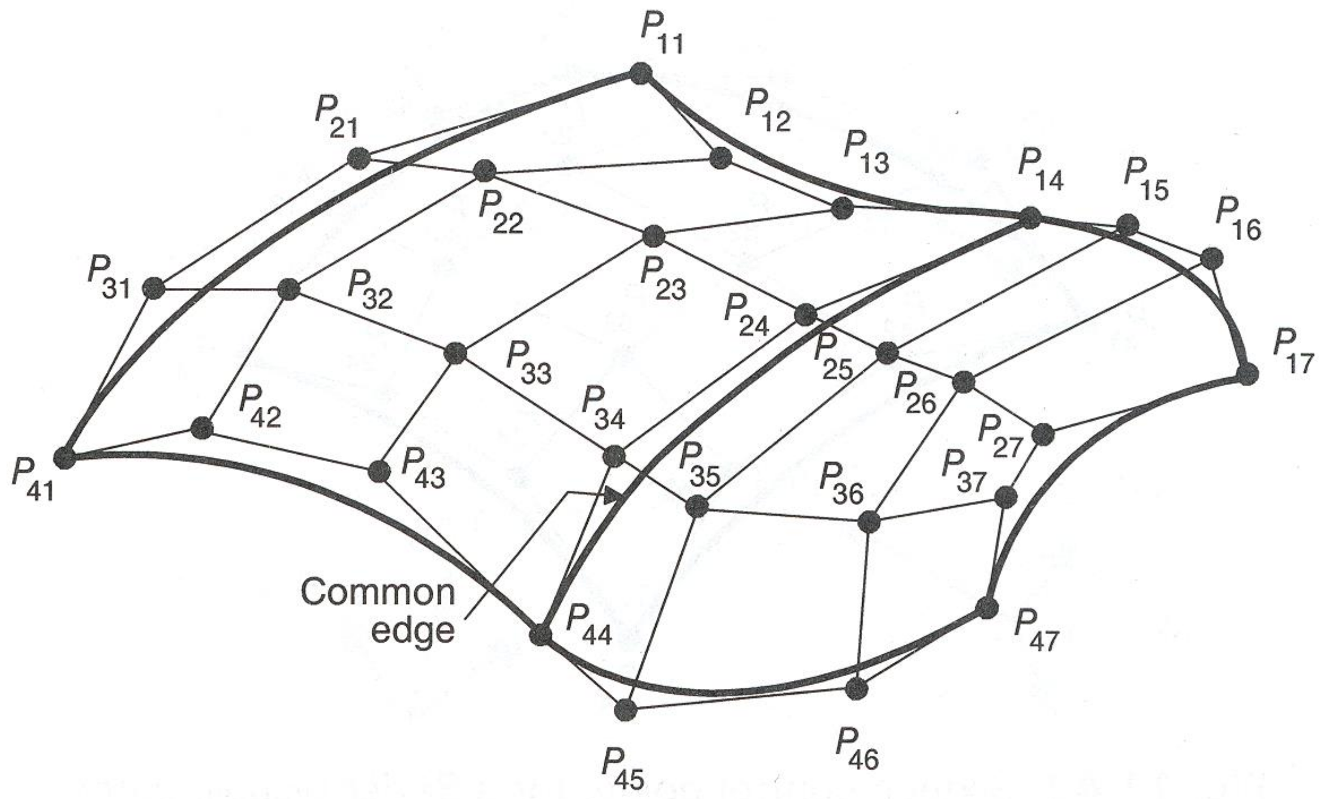


Same ideas as parametric splines!

Watt Figure 6.25

Bézier Surfaces

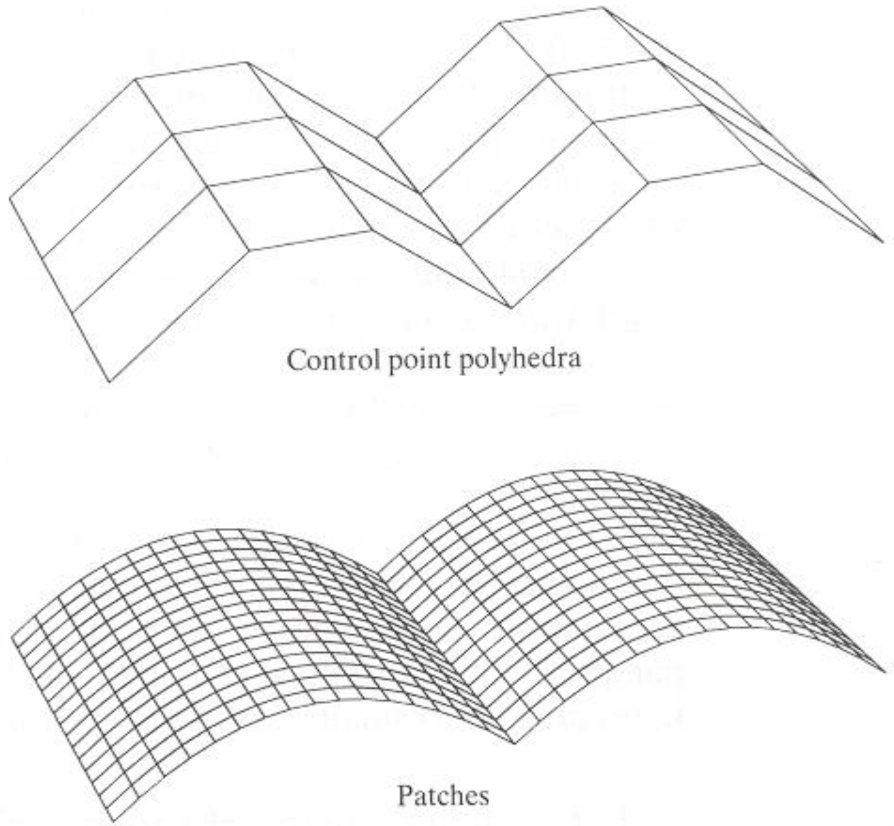
- Continuity constraints are similar to the ones for Bézier splines



FvDFH Figure 11.43

Bézier Surfaces

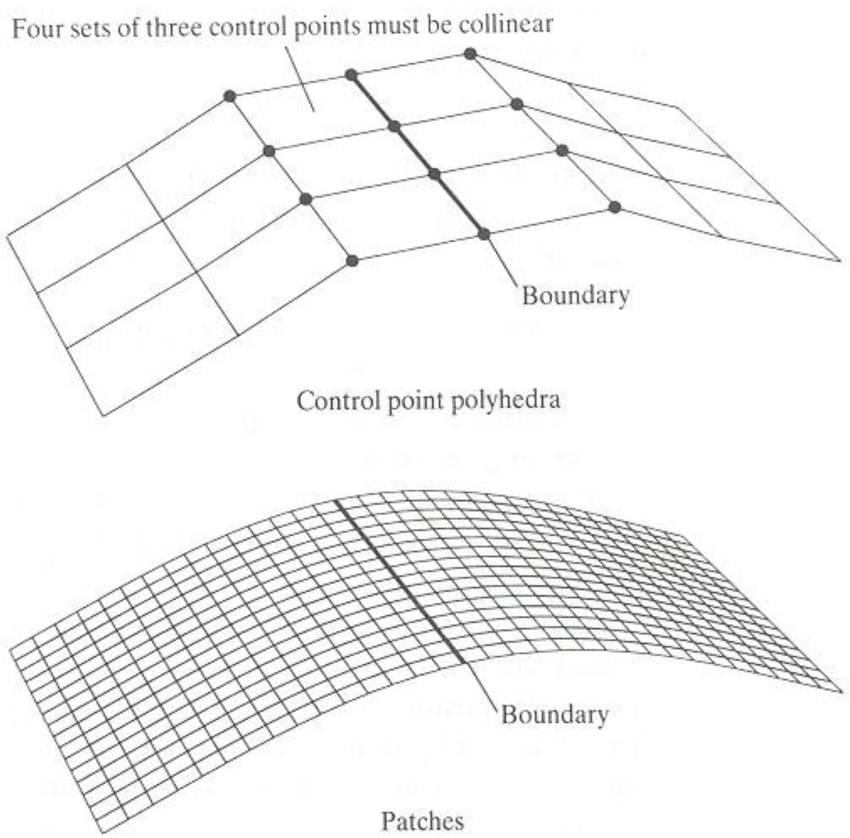
- C^0 continuity requires aligning boundary curves



Watt Figure 6.26a

Bézier Surfaces

- C^1 continuity requires aligning boundary curves and derivatives



Watt Figure 6.26b

Parametric Surfaces

- Properties
  - ? Natural parameterization
  - ? Guaranteed smoothness
  - ? Intuitive editing
  - ? Concise
  - ? Accurate
  - ? Efficient display
  - ? Easy acquisition
  - ? Efficient intersections
  - ? Guaranteed validity
  - ? Arbitrary topology

Parametric Surfaces

- Properties
  - ☺ Natural parameterization
  - ☺ Guaranteed smoothness
  - ☺ Intuitive editing
  - ☺ Concise
  - ☺ Accurate
  - ○ Efficient display
  - ☹ Easy acquisition
  - ☹ Efficient intersections
  - ☹ Guaranteed validity
  - ☹ Arbitrary topology