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2.2 Mergesort

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *divide-and-conquer*

Two classic sorting algorithms: mergesort and quicksort

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

Mergesort. [this lecture]

…

Quicksort. [next lecture]

…

Algorithms

Robert Sedgewick | Kevin Wayne

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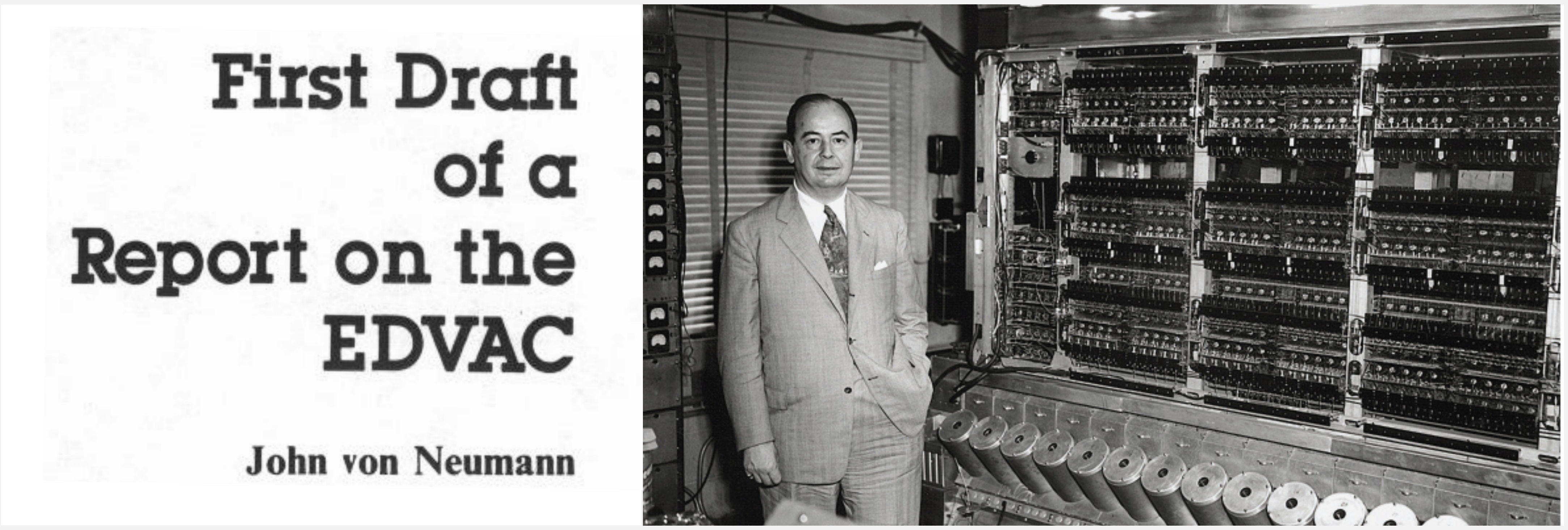
2.2 Mergesort

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *divide-and-conquer*

Mergesort

Basic plan.

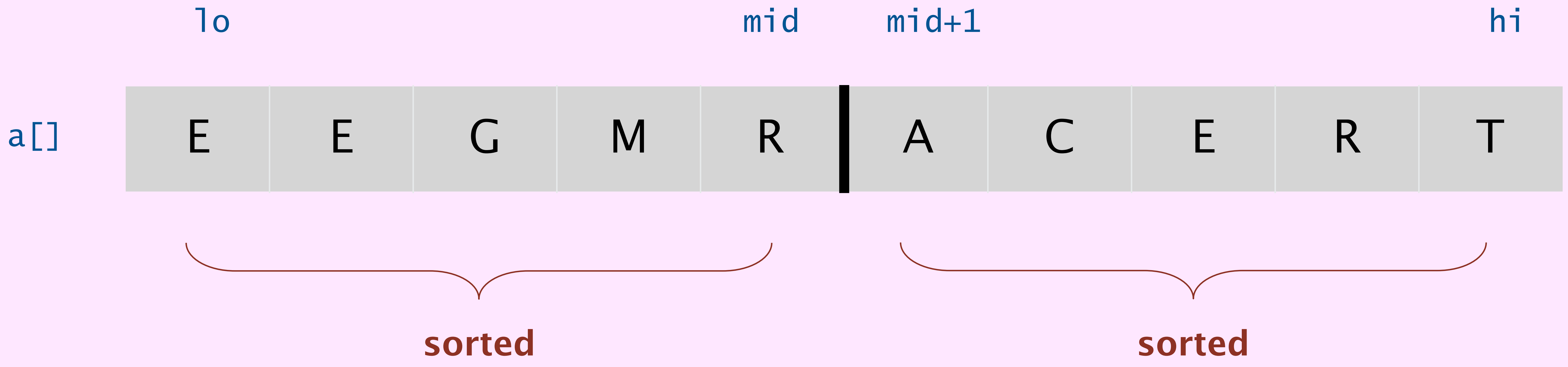
- Divide array into two halves.
- Recursively left half.
- Recursively sort right half.
- Merge two sorted halves.



input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Abstract in-place merge demo

Goal. Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace with sorted subarray `a[lo]` to `a[hi]`.



Merging demo (Transylvanian-Saxon folk dance)

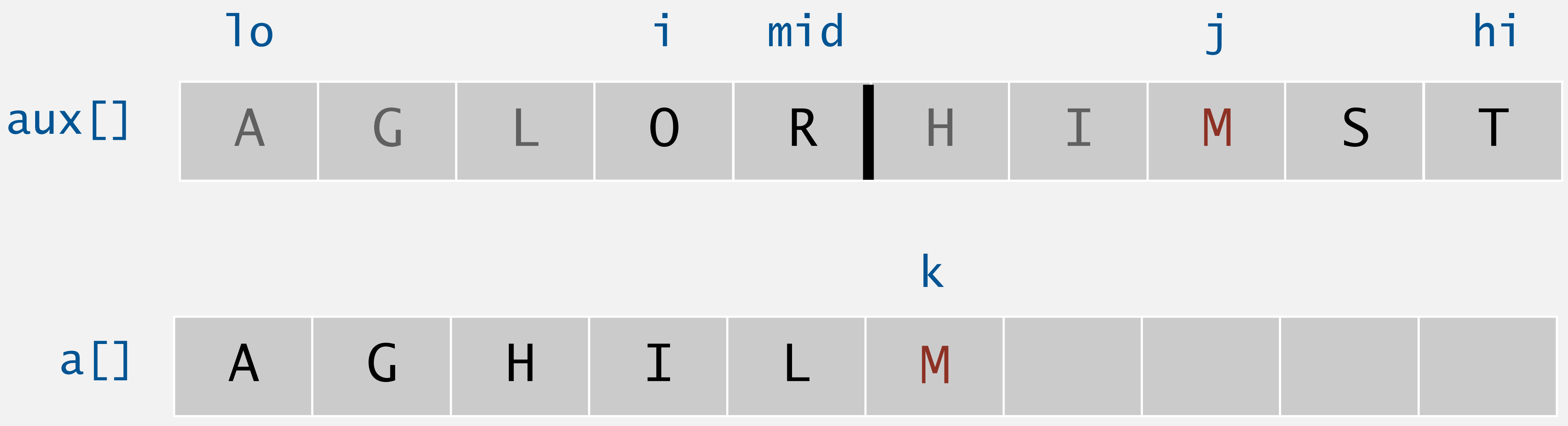
- Given two sorted halves, replace with sorted whole.

https://www.youtube.com/watch?v=XaqR3G_NVoo

Merging: Java implementation

```java
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   for (int k = lo; k <= hi; k++)                          // copy
      aux[k] = a[k];

   int i = lo, j = mid+1;                                  // merge
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```



Mergesort quiz 1

How many calls does `merge()` make to `less()` in order to merge two sorted subarrays, each of length $n / 2$, into a sorted array of length n?

A. $\sim \frac{1}{4} n$ to $\sim \frac{1}{2} n$

B. $\sim \frac{1}{2} n$

C. $\sim \frac{1}{2} n$ to $\sim n$

D. $\sim n$

merging two sorted arrays, each of length n/2

a_0	a_1	a_2	a_3		b_0	b_1	b_2	b_3

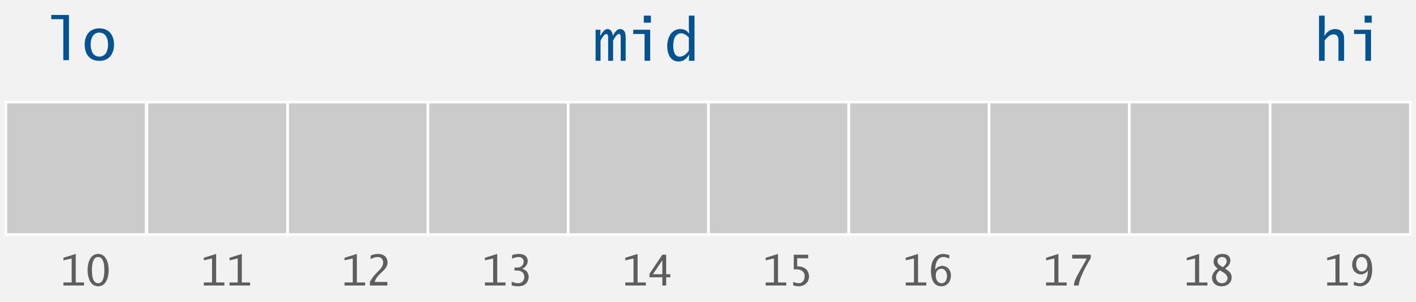
Mergesort: Java implementation

```
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      Comparable[] aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```

avoid array allocation in inner loop



Mergesort: trace

	a[]															
lo → … hi →	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
merge(a, aux, 8, 9, 11)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, aux, 12, 12, 13)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, aux, 14, 14, 15)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge(a, aux, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

← result after recursive call

Mergesort quiz 2

Which subarray lengths will arise when mergesorting an array of length 12?

A. $\{\ 1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12\ \}$

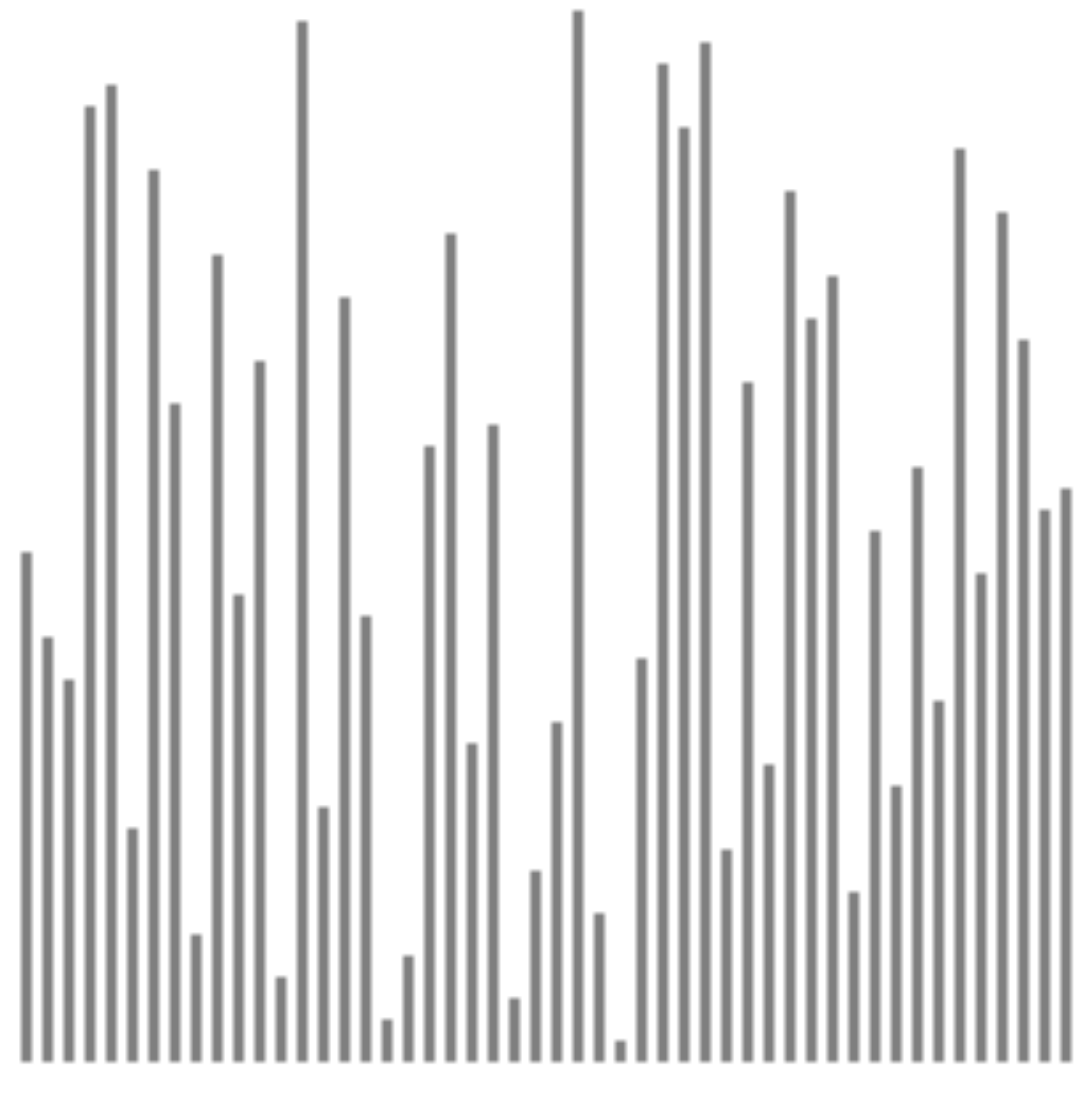
B. $\{\ 1,\ 2,\ 3,\ 6,\ 12\ \}$

C. $\{\ 1,\ 2,\ 4,\ 8,\ 12\ \}$

D. $\{\ 1,\ 3,\ 6,\ 9,\ 12\ \}$

Mergesort: animation

50 random items

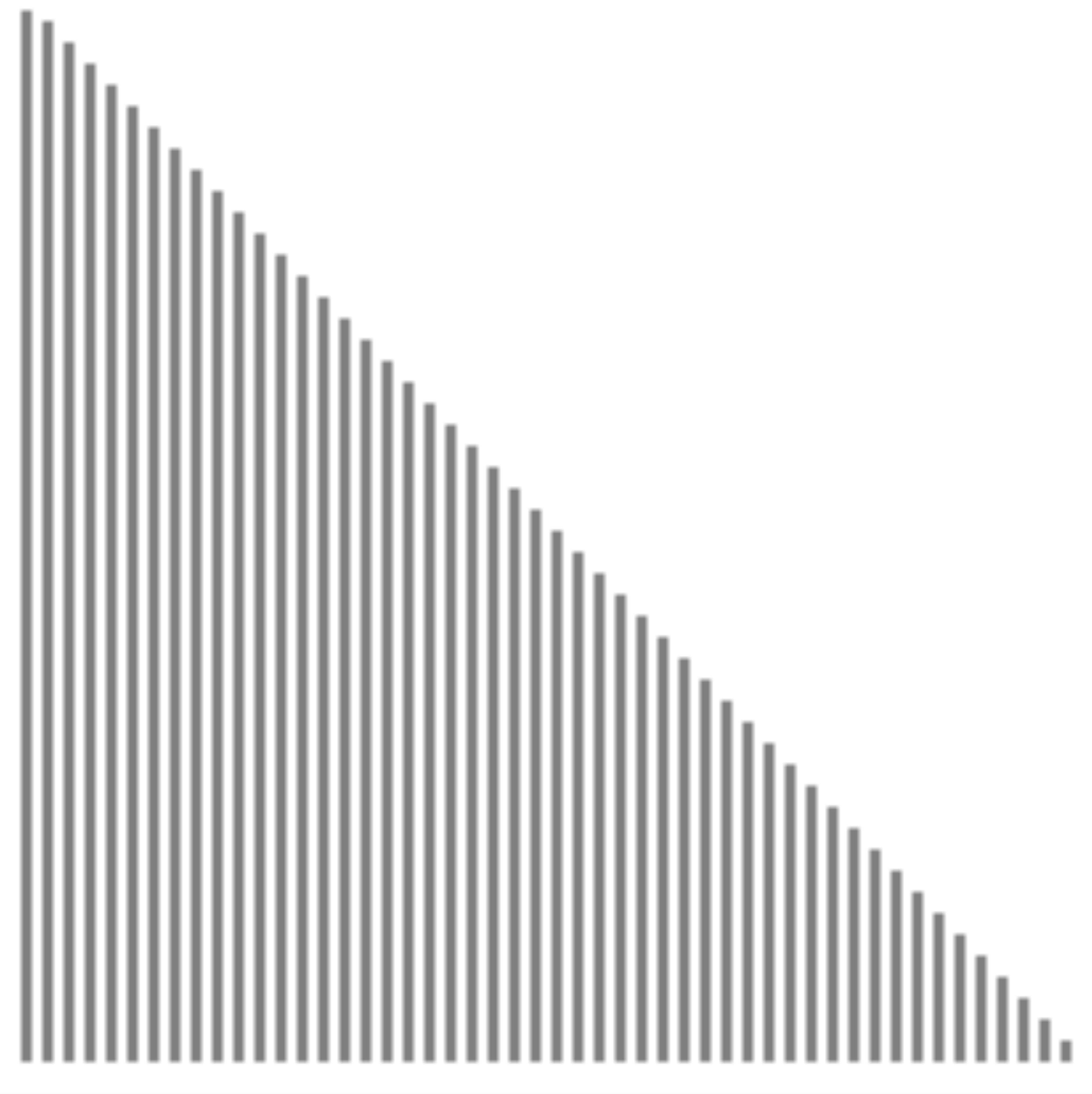


http://www.sorting-algorithms.com/merge-sort

▲ algorithm position
in order
current subarray
not in order

Mergesort: animation

50 reverse-sorted items



http://www.sorting-algorithms.com/merge-sort

algorithm position

in order

current subarray

not in order

Mergesort: empirical analysis

Running time estimates:

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (n^2)			mergesort (n log n)		
computer	**thousand**	**million**	**billion**	**thousand**	**million**	**billion**
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

Bottom line. Good algorithms are better than supercomputers.

Mergesort analysis: number of compares

Proposition. Mergesort uses $\le n \log_2 n$ compares to sort any array of length n.

Pf sketch. The number of compares $C(n)$ to mergesort any array of length n satisfies the recurrence:

$$C(n) \le C(\lceil n/2 \rceil) + C(\lfloor n/2 \rfloor) + n - 1 \quad \text{for } n > 1, \text{ with } C(1) = 0.$$

- $C(\lceil n/2 \rceil)$: sort left half
- $C(\lfloor n/2 \rfloor)$: sort right half
- $n - 1$: merge

For simplicity: Assume n is a power of 2 and solve this recurrence:

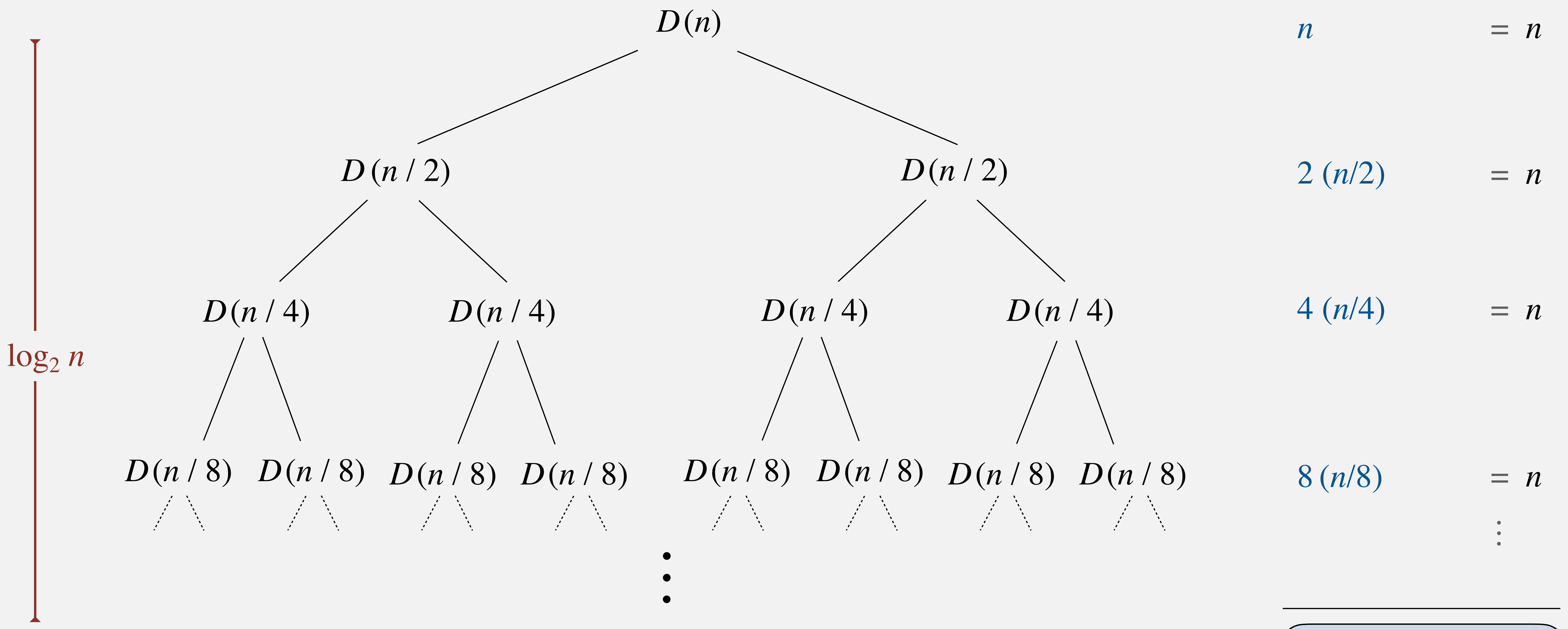
proposition holds even when n is not a power of 2 (but analysis cleaner in this case)

$$D(n) = 2\, D(n/2) + n, \text{ for } n > 1, \text{ with } D(1) = 0.$$

Divide-and-conquer recurrence

Proposition. If $D(n)$ satisfies $D(n) = 2\,D(n/2) + n$ for $n > 1$, with $D(1) = 0$, then $D(n) = n \log_2 n$.

Pf by picture. [assuming n is a power of 2]



$D(n) = n \log_2 n$

Mergesort analysis: number of array accesses

Proposition. Mergesort makes $\Theta(n \log n)$ array accesses.

Pf sketch. The number of array accesses $A(n)$ satisfies the recurrence:

$$A(n) = A(\lceil n/2 \rceil) + A(\lfloor n/2 \rfloor) + \Theta(n)$$ for $n > 1$, with $A(1) = 0$.

Key point. Any algorithm with the following structure takes $\Theta(n \log n)$ time:

```
public static void f(int n)
{
    if (n == 0) return;
    f(n/2);          ← solve two problems
    f(n/2);          ← of half the size
    linear(n);       ← do Θ(n) work
}
```

Famous examples. FFT and convolution, hidden-line removal, Kendall-tau distance, ...

Mergesort analysis: memory

Proposition. Mergesort uses $\Theta(n)$ extra space.

Pf. The array aux[] needs to be of length n for the last merge.

two sorted subarrays

A	C	D	G	H	I	M	N	U	V

B	E	F	J	O	P	Q	R	S	T

merged result

A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

essentially negligible

Def. A sorting algorithm is in-place if it uses $\le c \log n$ extra space.

Ex. Insertion sort and selection sort.

Challenge 1 (not hard). Use aux[] array of length $\sim \frac{1}{2} n$ instead of n.

Challenge 2 (very hard). In-place merge. [Kronrod 1969]

Mergesort quiz 3

Consider the following modified version of mergesort.

How much total memory is allocated over all recursive calls?

A. $\Theta(n)$

B. $\Theta(n \log n)$

C. $\Theta(n^2)$

D. $\Theta(2^n)$

```
private static void sort(Comparable[] a, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   int n = hi - lo + 1;
   Comparable[] aux = new Comparable[n];
   sort(a, lo, mid);
   sort(a, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Mergesort: practical improvement

Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.

```
private static void sort(...)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }

   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

makes mergesort about 20% faster

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2.2 Mergesort

- mergesort
- *bottom-up mergesort*
- sorting complexity
- divide-and-conquer

Bottom-up mergesort

Basic plan.

- Pass through array, merging subarrays of length 1.
- Repeat for subarrays of length 2, 4, 8, ...

a[i]

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
merge(a, aux, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
merge(a, aux, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Bottom-up mergesort: Java implementation

```java
public class MergeBU
{
   private static void merge(...)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int n = a.length;
      Comparable[] aux = new Comparable[n];
      for (int sz = 1; sz < n; sz = sz+sz)
         for (int lo = 0; lo < n-sz; lo += sz+sz)
            merge(a, aux, lo, lo+sz-1, Math.min(lo+sz+sz-1, n-1));
   }
}
```

Proposition. At most $n \log_2 n$ compares; $\Theta(n)$ extra space.

Bottom line. Simple and non-recursive version of mergesort.

Mergesort quiz 4

Which is faster in practice for $n = 2^{20}$, top–down mergesort or bottom–up mergesort?

A. Top-down (recursive) mergesort.

B. Bottom-up (non-recursive) mergesort.

C. No difference.

D. *I don't know.*

Natural mergesort

Idea. Exploit pre-existing order by identifying naturally occurring runs.

input

1	5	10	16	3	4	23	9	13	2	7	8	12	14

first run

1	5	10	16	3	4	23	9	13	2	7	8	12	14

second run

1	5	10	16	3	4	23	9	13	2	7	8	12	14

merge two runs

1	3	4	5	10	16	23	9	13	2	7	8	12	14

Tradeoff. Fewer passes vs. extra compares per pass to identify runs.

Timsort (2002)

- Natural mergesort.
- Use binary insertion sort to make initial runs (if needed).
- A few more clever optimizations.

```
Intro
-----
This describes an adaptive, stable, natural mergesort, modestly called
timsort (hey, I earned it <wink>).  It has supernatural performance on many
kinds of partially ordered arrays (less than lg(n!) comparisons needed, and
as few as n-1), yet as fast as Python's previous highly tuned samplesort
hybrid on random arrays.

In a nutshell, the main routine marches over the array once, left to right,
alternately identifying the next run, then merging it into the previous
runs "intelligently".  Everything else is complication for speed, and some
hard-won measure of memory efficiency.
...
```

Tim Peters

Consequence. Linear time on many arrays with pre-existing order.

Now widely used. Python, Java 7–11, GNU Octave, Android, ….

Timsort bug (February 2015)

Envisage: Engineering Virtualized Services

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Proving that Android's, Java's and Python's sorting algorithm is broken (and showing how to fix it)

February 24, 2015 Envisage Written by Stijn de Gouw. $s

Tim Peters developed the Timsort hybrid sorting algorithm in 2002. It is a clever combination of ideas from merge sort and insertion sort, and designed to perform well on real world data. TimSort was first developed for Python, but later ported to Java (where it appears as java.util.Collections.sort and java.util.Arrays.sort) by Joshua Bloch (the designer of Java Collections who also pointed out that most binary search algorithms were broken). TimSort is today used as the default sorting algorithm for Android SDK, Sun's JDK and OpenJDK. Given the popularity of these platforms this means that the number of computers, cloud services and mobile phones that use TimSort for sorting is well into the billions.

http://envisage-project.eu/proving-android-java-and-python-sorting-algorithm-is-broken-and-how-to-fix-it

Timsort bug (May 2018)

JDK / JDK-8203864

Execution error in Java's Timsort

Details

Type:	Bug
Status:	RESOLVED
Priority:	3 P3
Resolution:	Fixed
Affects Version/s:	None
Fix Version/s:	11
Component/s:	core-libs
Labels:	None
Subcomponent:	java.util:collections
Introduced In Version:	6
Resolved In Build:	b20

Description

Carine Pivoteau wrote:
While working on a proper complexity analysis of the algorithm, we realised that there was an error in the last paper reporting such a bug (http://envisage-project.eu/wp-content/uploads/2015/02/sorting.pdf).
This implies that the correction implemented in the Java source code (changing Timsort stack size) is wrong and that it is still possible to make it break.
This is explained in full details in our analysis: https://arxiv.org/pdf/1805.08612.pdf.
We understand that coming upon data that actually causes this error is very unlikely, but we thought you'd still like to know and do something about it.
As the authors of the previous article advocated for, we strongly believe that you should consider modifying the algorithm as explained in their article (and as was done in Python) rather than trying to fix the stack size.

https://bugs.openjdk.java.net/browse/JDK-8203864

Sorting summary

	in-place?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2}\, n^2$	$\frac{1}{2}\, n^2$	$\frac{1}{2}\, n^2$	n exchanges
insertion	✔	✔	n	$\frac{1}{4}\, n^2$	$\frac{1}{2}\, n^2$	use for small n or partially sorted
merge		✔	$\frac{1}{2}\, n \log_2 n$	$n \log_2 n$	$n \log_2 n$	$\Theta(n \log n)$ guarantee; stable
timsort		✔	n	$n \log_2 n$	$n \log_2 n$	improves mergesort when pre-existing order
?	✔	✔	n	$n \log_2 n$	$n \log_2 n$	holy sorting grail

number of compares to sort an array of n elements

$n \log_2 \varrho$, where ϱ = # runs (proved in August 2018)

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2.2 Mergesort

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- *divide-and-conquer*

Computational complexity

A framework to study efficiency of algorithms for solving a particular problem X.

Model of computation. Allowable operations.

Cost model. Operation counts.

Upper bound. Cost guarantee provided by some algorithm for X.

Lower bound. Proven limit on cost guarantee of all algorithms for X.

Optimal algorithm. Algorithm with best possible cost guarantee for X.

lower bound ~ upper bound

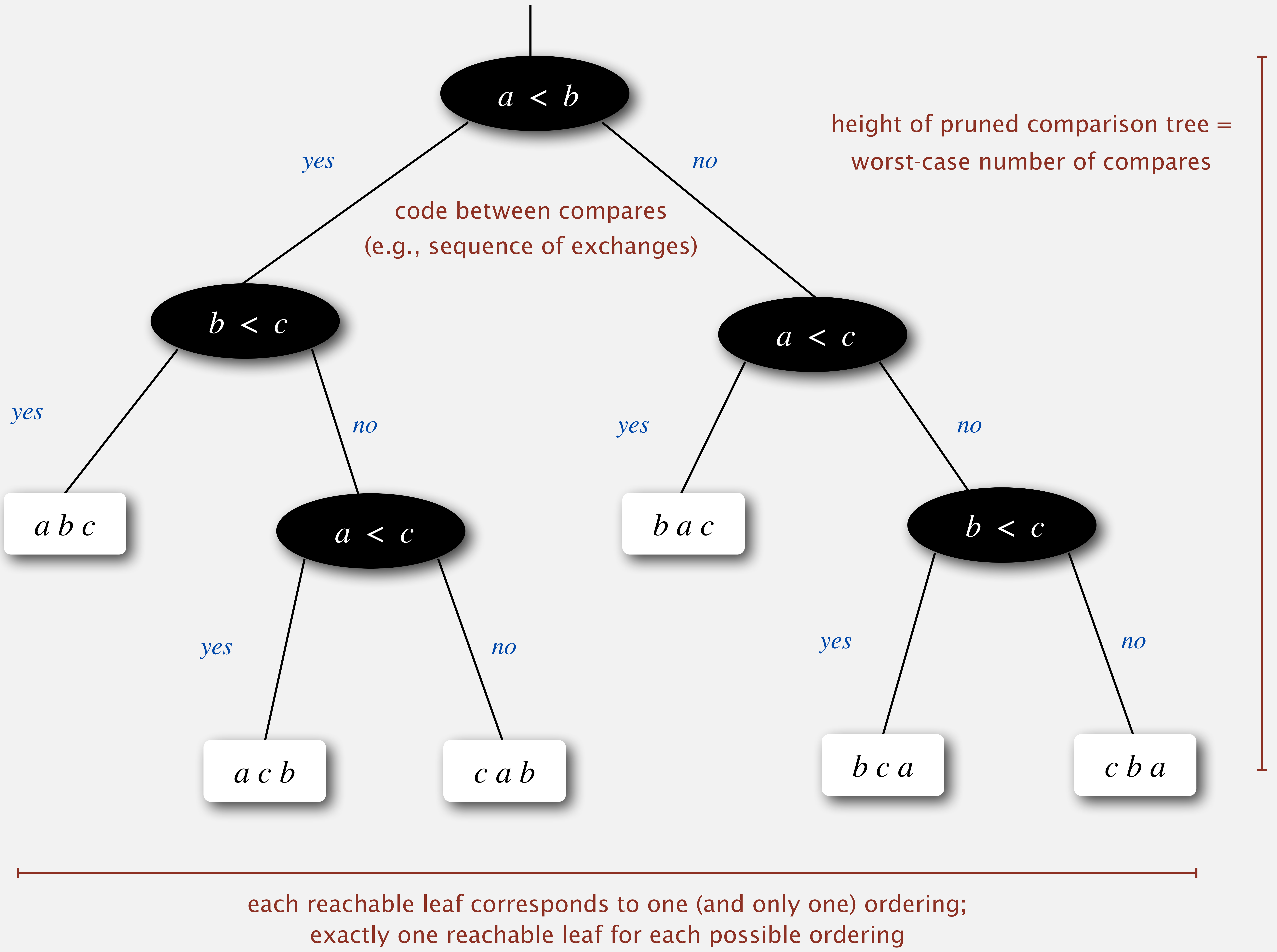
model of computation	*comparison tree*
cost model	*# compares*
upper bound	$\sim n \log_2 n$
lower bound	?
optimal algorithm	?

computational complexity of sorting

comparison tree: can access information only through compares (e.g., Java `Comparable` framework)

upper bound: from mergesort

Comparison tree (for 3 distinct keys a, b, and c)

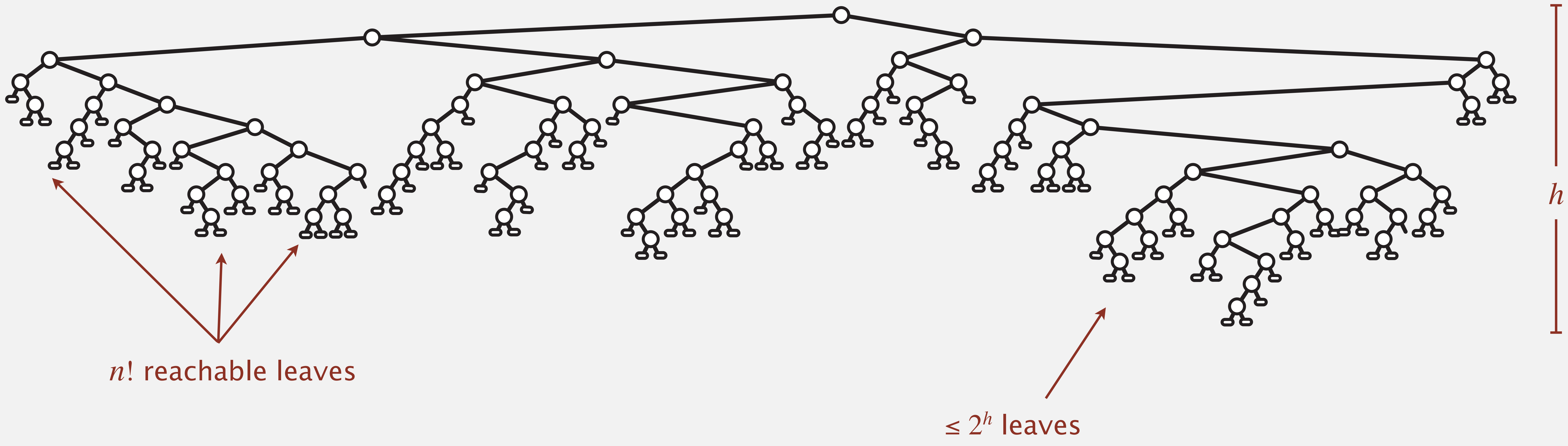


Compare-based lower bound for sorting

Proposition. In the worst case, any compare-based sorting algorithm must make at least $\log_2(n!) \sim n \log_2 n$ compares.

Pf.

- Assume array consists of n distinct values a_1 through a_n.
- Worst-case number of compares = height h of pruned comparison tree.
- Binary tree of height h has $\leq 2^h$ leaves.
- $n!$ different orderings $\Rightarrow$ $n!$ reachable leaves.



Compare-based lower bound for sorting

Proposition. In the worst case, any compare-based sorting algorithm must make at least $\log_2(n!) \sim n \log_2 n$ compares.

Pf.

- Assume array consists of n distinct values a_1 through a_n.
- Worst-case number of compares = height h of pruned comparison tree.
- Binary tree of height h has $\leq 2^h$ leaves.
- $n!$ different orderings $\Rightarrow$ $n!$ reachable leaves.

$$2^h \geq \#\text{ reachable leaves } = n!$$

$$\Rightarrow \; h \geq \log_2(n!)$$

$$\sim n \log_2 n$$

↑ Stirling's formula

Complexity of sorting

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by some algorithm for X.

Lower bound. Proven limit on cost guarantee of all algorithms for X.

Optimal algorithm. Algorithm with best possible cost guarantee for X.

model of computation	*comparison tree*	
cost model	*# compares*	
upper bound	$\sim n \log_2 n$	← from mergesort
lower bound	$\sim n \log_2 n$	← comparison tree
optimal algorithm	*mergesort*	

complexity of sorting

First goal of algorithm design: optimal algorithms.

Complexity results in context

Compares? Mergesort is optimal with respect to number compares.

Space? Mergesort is not optimal with respect to space usage.

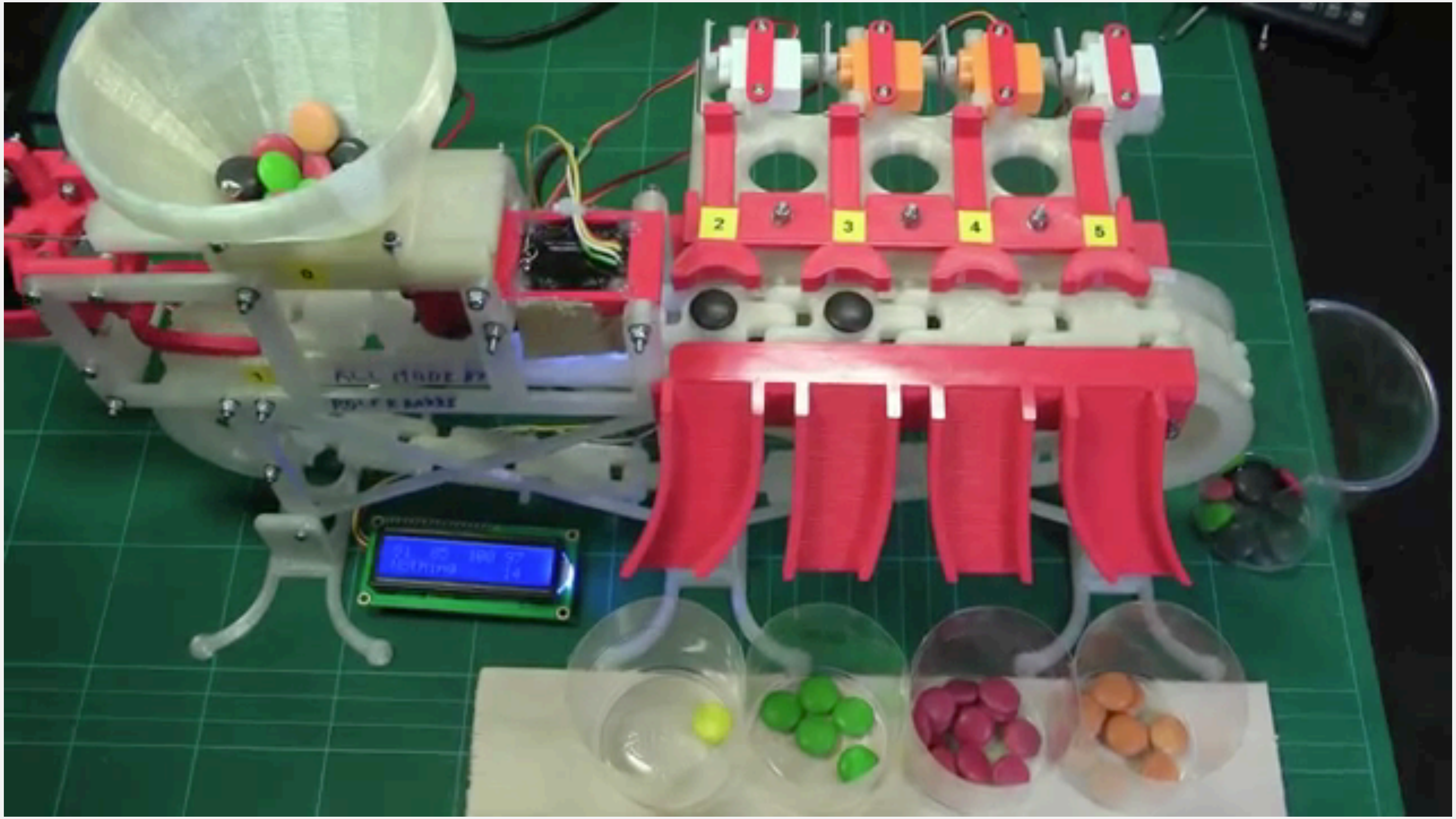
Lessons. Use theory as a guide.

Ex. Design sorting algorithm that makes $\sim \frac{1}{2}\, n \log_2 n$ compares in worst case?

Ex. Design sorting algorithm that makes $\Theta(n \log n)$ compares and uses $\Theta(1)$ extra space.

Commercial break (sponsored by Skittles)

Q. Why doesn't this Skittles sorter violate the sorting lower bound?



https://www.youtube.com/watch?v=tSEHDBSynVo

Complexity results in context (continued)

Lower bound may not hold if the algorithm can take advantage of:

- The initial order of the input array.

  Ex: insertion sort makes only $\Theta(n)$ compares on partially sorted arrays.

- The distribution of key values.

  Ex: 3-way quicksort makes only $\Theta(n)$ compares on arrays
  with a constant number of distinct keys. [stay tuned]

- The representation of the keys.

  Ex: radix sorts do not make any key compares;
  they access the data via character/digit compares. [stay tuned]

Asymptotic notations

Warning: many programmers abuse big O to mean Θ

notation	provides	example	shorthand for	
tilde	*leading term*	$\sim \frac{1}{2} n^2$	$\frac{1}{2} n^2$ $\frac{1}{2} n^2 + 3n + 22$	← ignore lower-order terms
big Theta	*order of growth*	$\Theta(n^2)$	$\frac{1}{2} n^2$ $7 n^2 + n^{1/2}$ $5 n^2 - 3n$	← also ignore leading coefficient
big O	*upper bound*	$O(n^2)$	$10 n^2$ $22 n$ $\log_2 n$	← $\Theta(n^2)$ or smaller
big Omega	*lower bound*	$\Omega(n^2)$	$\frac{1}{2} n^2$ $n^3 + 3n$ 2^n	← $\Theta(n^2)$ or larger

SORTING LOWER BOUND

Interviewer. Give a formal description of the sorting lower bound for sorting an array of n elements.

Algorithms

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2.2 Mergesort

- *mergesort*
- *bottom-up mergesort*
- *sorting complexity*
- *divide-and-conquer*

Sorting a linked list

Problem. Given a singly linked list, rearrange its nodes in sorter order.

Application. Sort list of inodes to garbage collect in Linux kernel.

Version 0. $\Theta(n \log n)$ time, $\Theta(n)$ extra space.

Version 1. $\Theta(n \log n)$ time, $\Theta(\log n)$ extra space.

Version 2. $\Theta(n \log n)$ time, $\Theta(1)$ extra space.