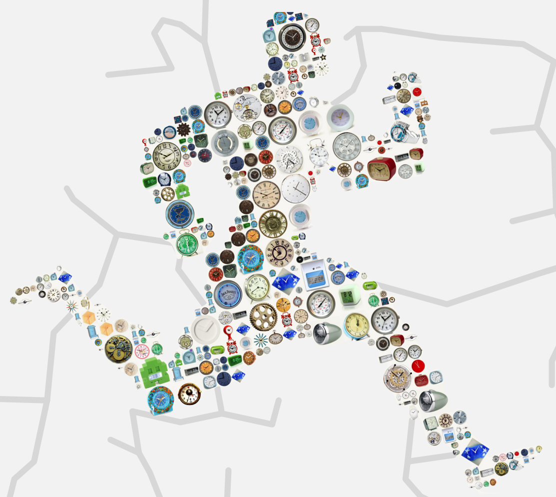

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1.4 Analysis of Algorithms

- *introduction*
- *running time (experimental analysis)*
- *running time (mathematical models)*
- *memory usage*

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1.4 Analysis of Algorithms

- *introduction*
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- *running time (mathematical models)*
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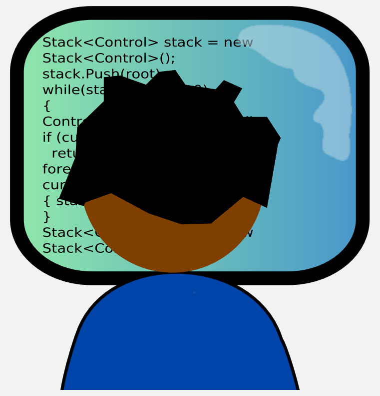
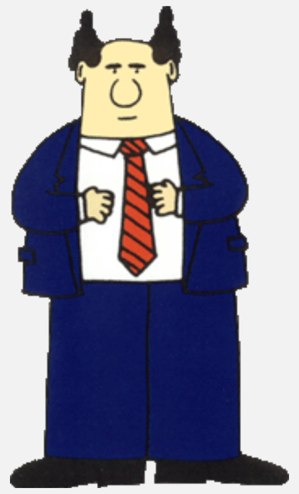
Cast of characters

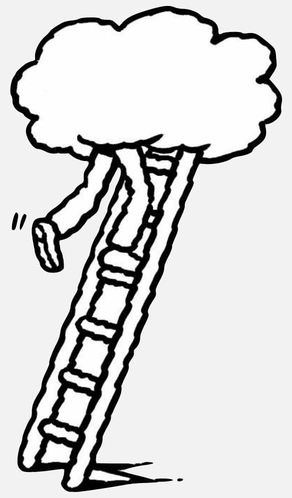


programmer needs to
develop a working solution

student (you)
might play all of
these roles someday



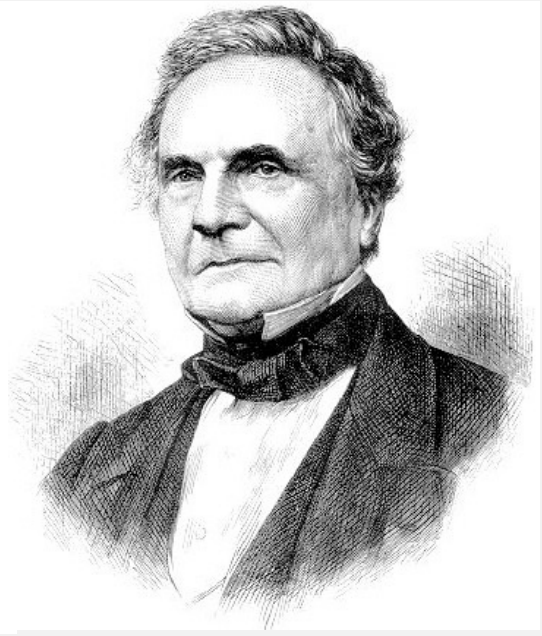
client wants to solve
problem efficiently

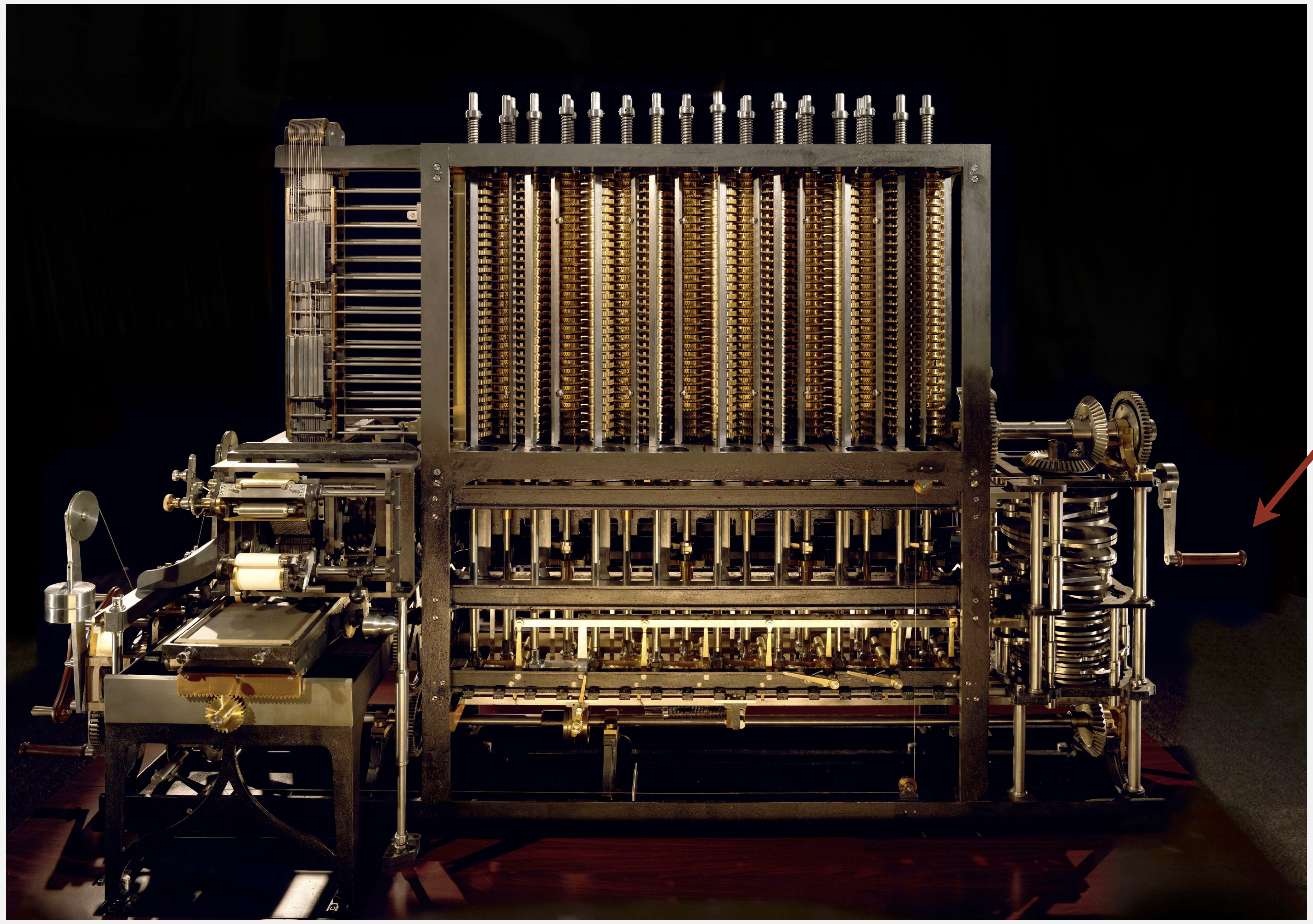


theoretician seeks
to understand

Running time

" *As soon as an Analytical Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will then arise—By what course of calculation can these results be arrived at by the machine in the shortest time?* " — *Charles Babbage (1864)*

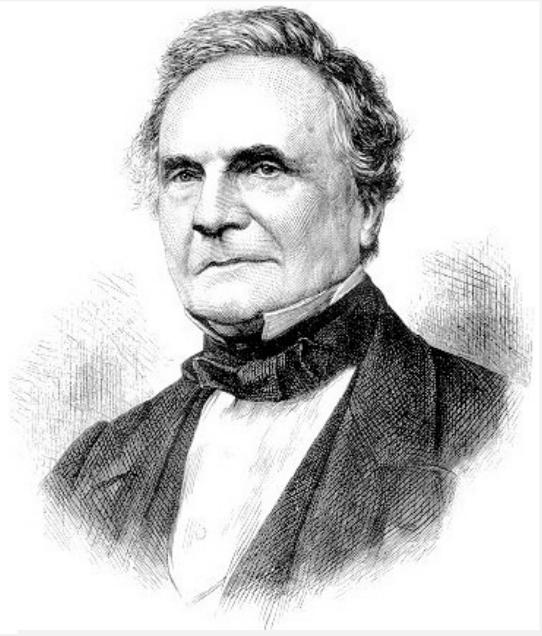




how many times do you have to turn the crank?

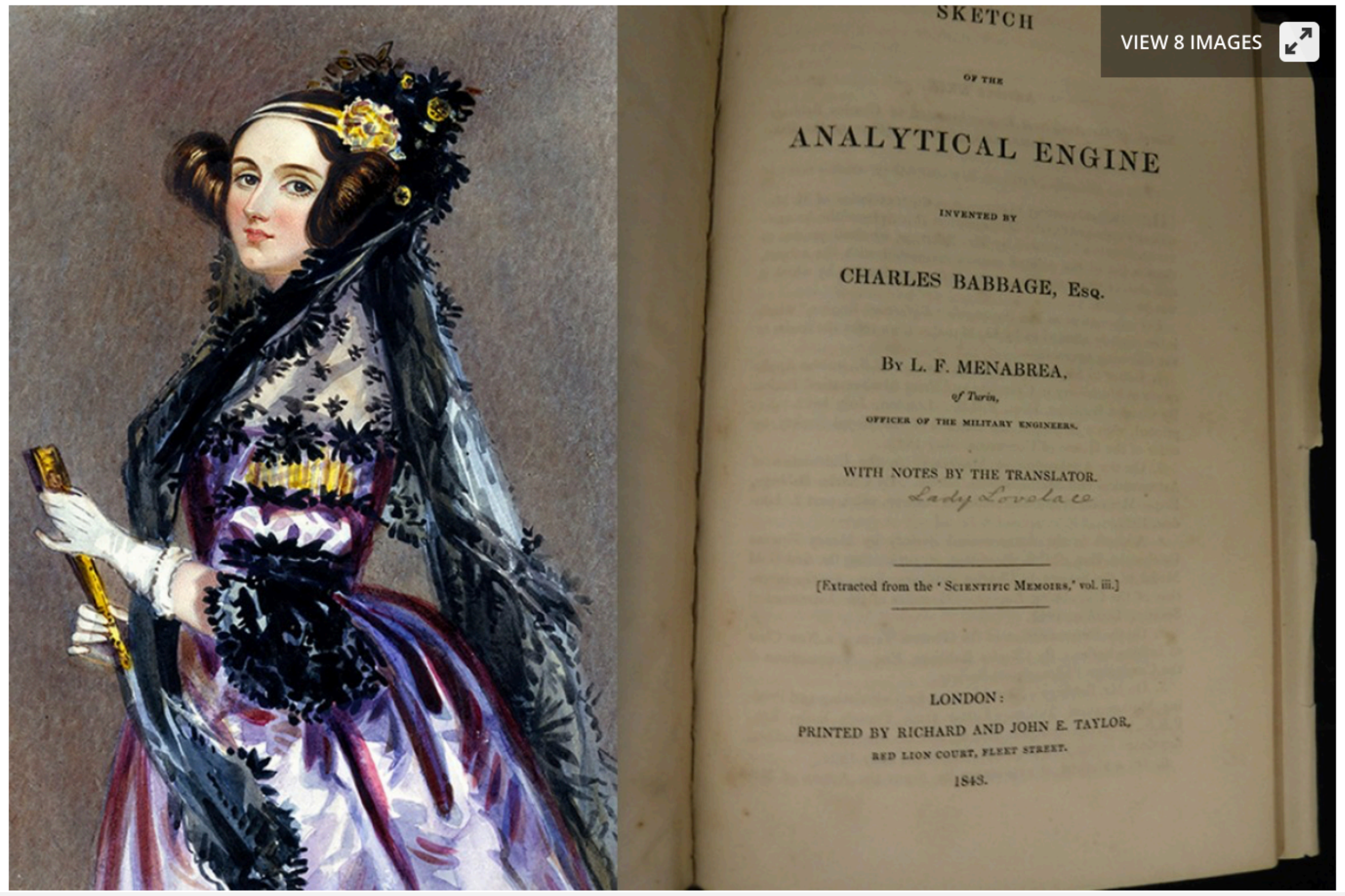
Running time

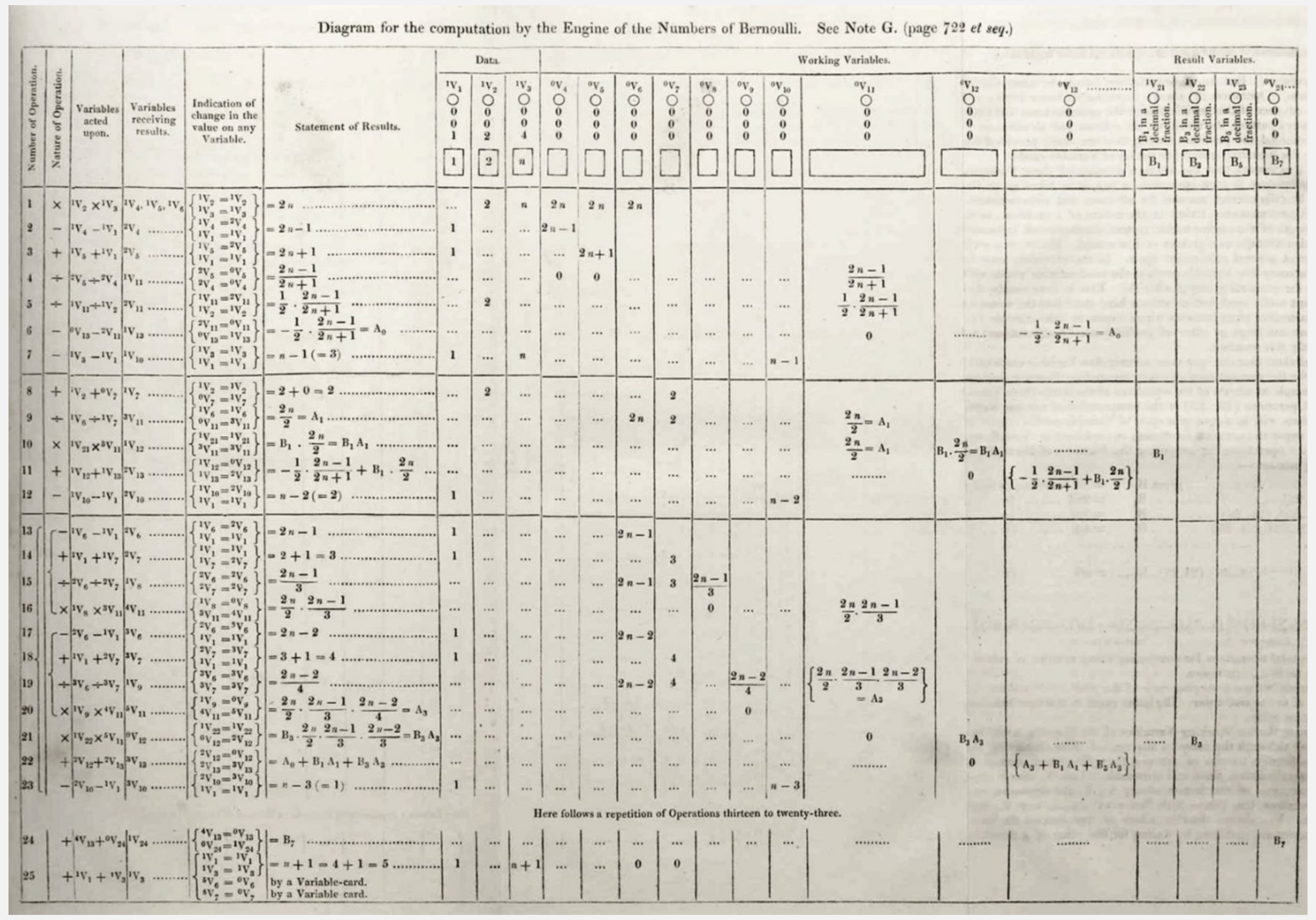
" *As soon as an Analytical Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will then arise—By what course of calculation can these results be arrived at by the machine in the shortest time?* " — *Charles Babbage (1864)*



Rare book containing the world's first computer algorithm earns $125,000 at auction

By Matt Kennedy
July 25, 2018





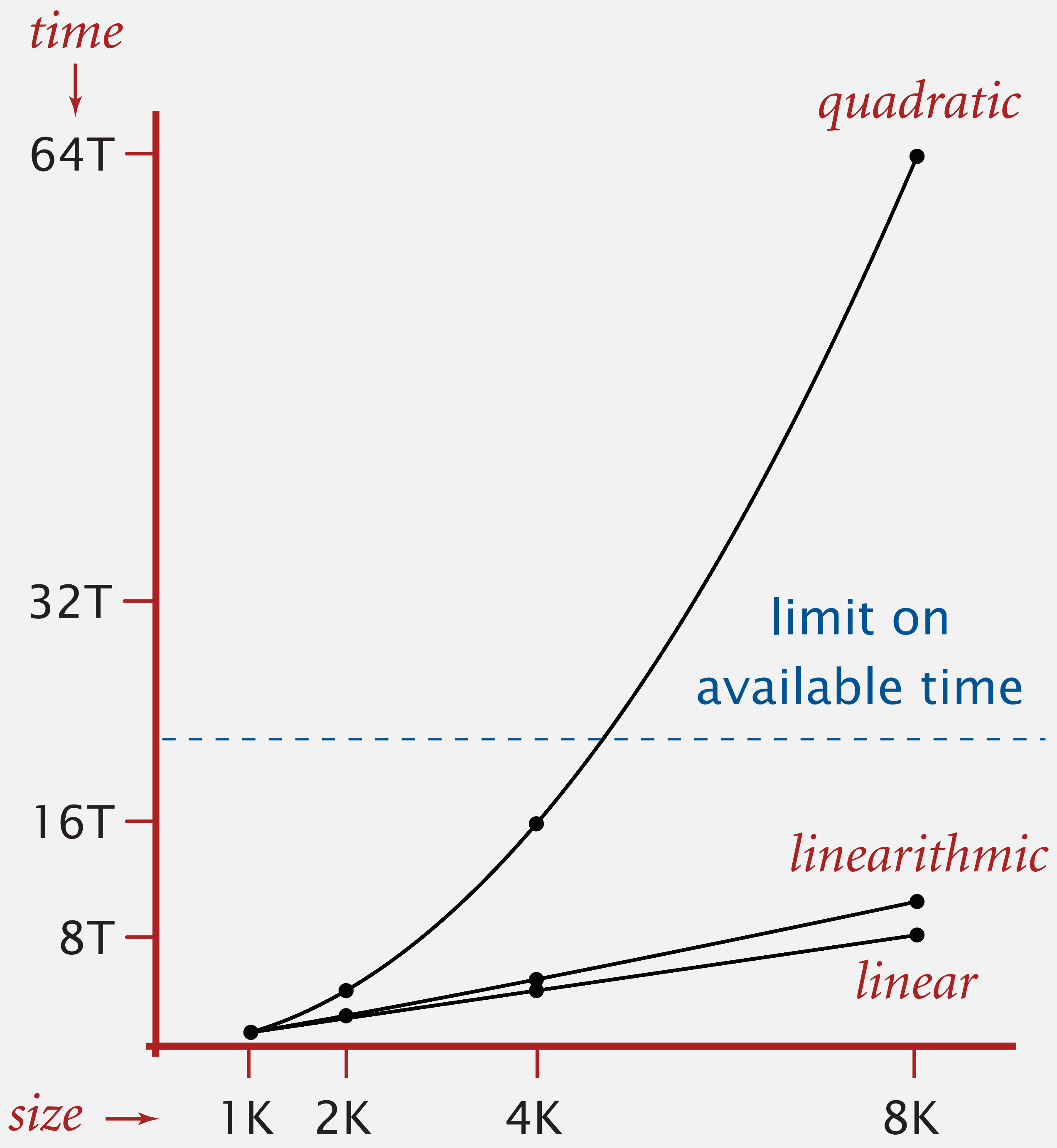
Ada Lovelace's algorithm
to compute Bernoulli numbers
on Analytic Engine (1843)

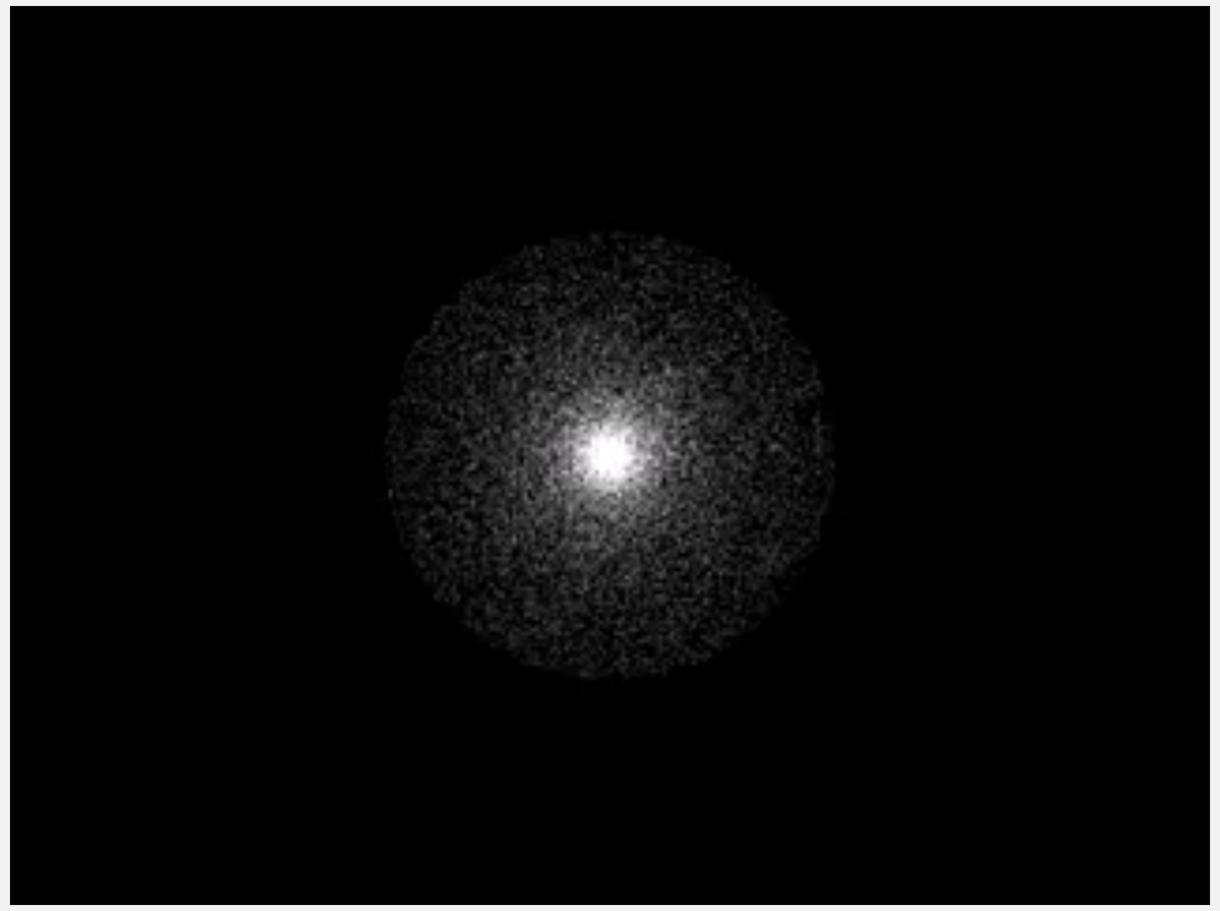
An algorithmic success story

N-body simulation.

- Simulate gravitational interactions among n bodies.
- Applications: cosmology, fluid dynamics, semiconductors, ...
- Brute force: n^2 steps.
- Barnes–Hut algorithm: $n \log n$ steps, enables new research.

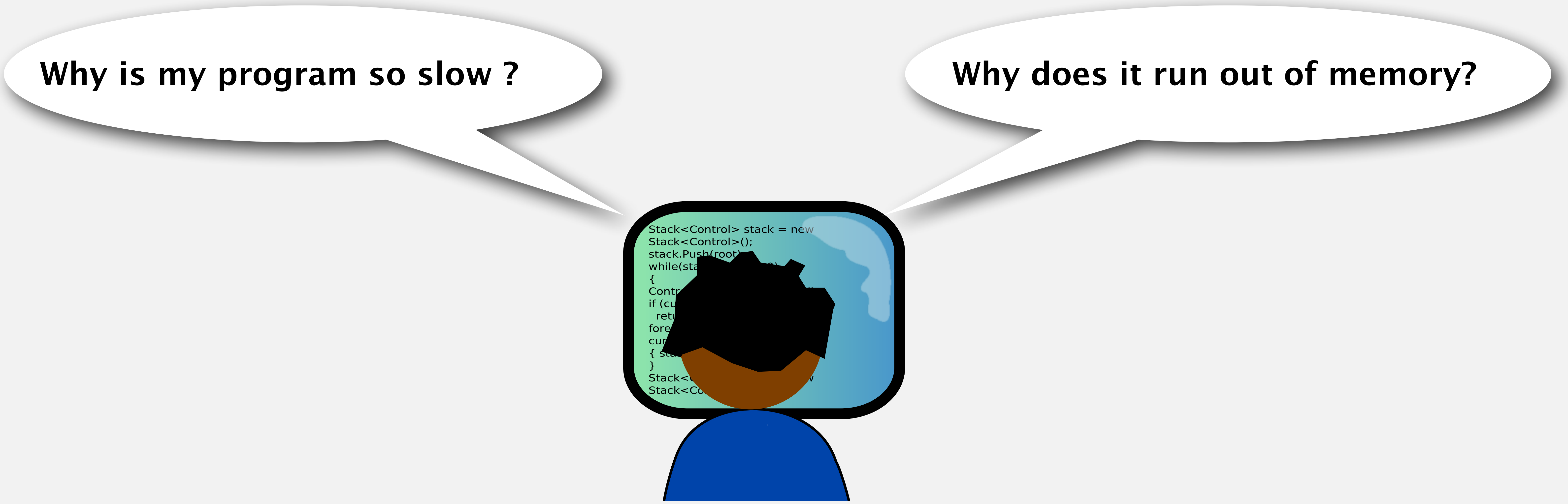
Andrew Appel
PU ’81





The challenge

Q. Will my program be able to solve a large practical input?



Our approach. Combination of experiments and mathematical modeling.

Example: 3-SUM

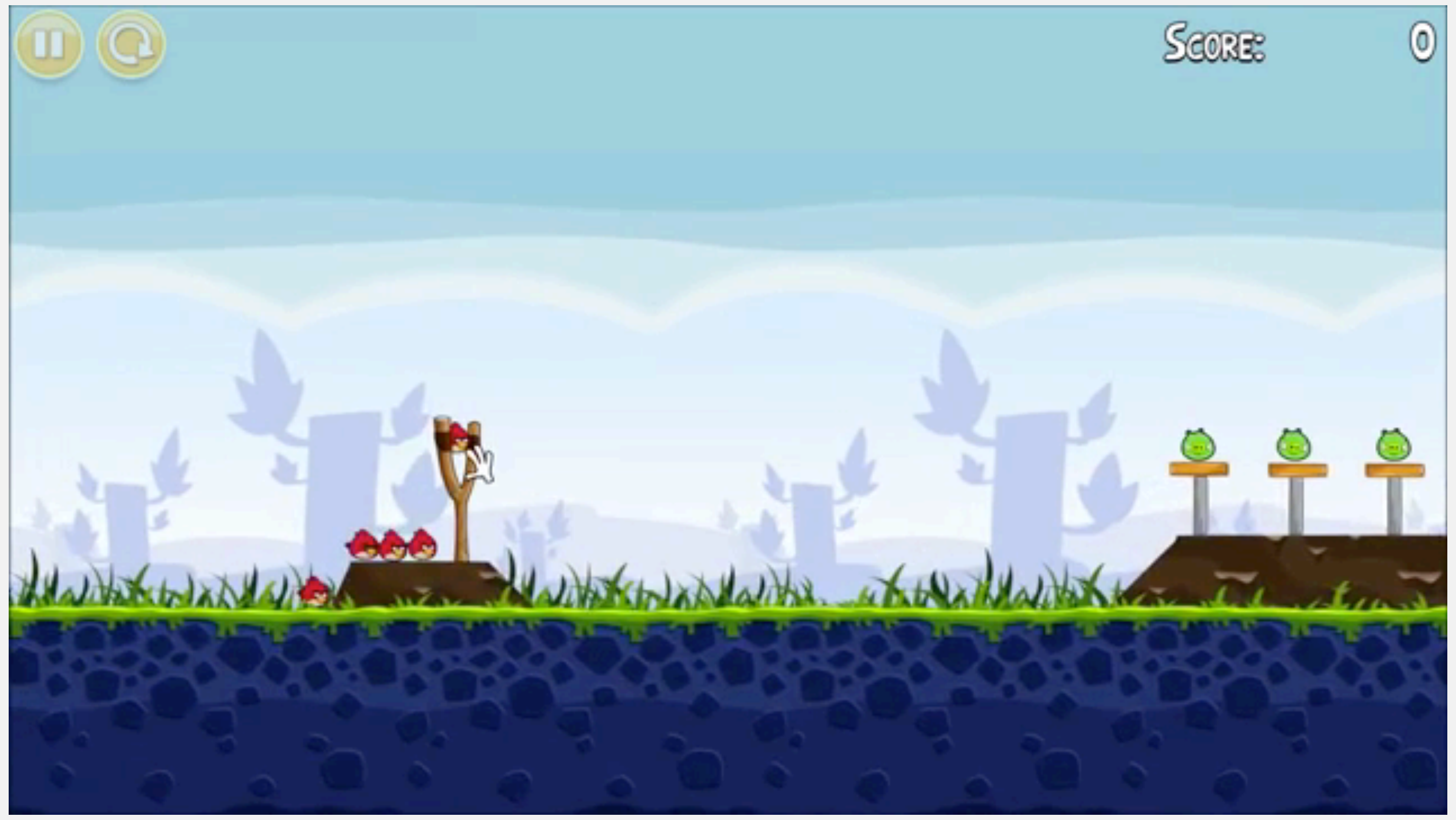
3-SUM. Given n distinct integers, how many triples sum to exactly zero?

```
~/Desktop/3sum> more 8ints.txt
8
30 -40 -20 -10 40 0 10 5

~/Desktop/3sum> java ThreeSum 8ints.txt
4
```

	a[i]	a[j]	a[k]	sum	
1	30	−40	10	0	✔
2	30	−20	−10	0	✔
3	−40	40	0	0	✔
4	−10	0	10	0	✔

Context. Related to problems in computational geometry.



3-SUM: brute-force algorithm

```
public class ThreeSum
{
   public static int count(int[] a)
   {
      int n = a.length;
      int count = 0;
      for (int i = 0; i < n; i++)
         for (int j = i+1; j < n; j++)
            for (int k = j+1; k < n; k++)
               if (a[i] + a[j] + a[k] == 0)
                  count++;
      return count;
   }

   public static void main(String[] args)
   {
      In in = new In(args[0]);
      int[] a = in.readAllInts();
      StdOut.println(count(a));
   }
}
```

check distinct triples

for simplicity,
ignore integer overflow

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1.4 Analysis of Algorithms

- introduction
- running time (experimental analysis)
- running time (mathematical models)
- memory usage

Measuring the running time

Q. How to time a program?

A. Manual.

`% java ThreeSum 1Kints.txt`

tick tick tick

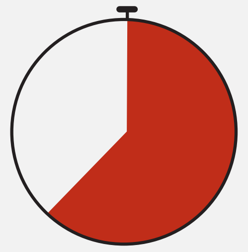
`70`

`% java ThreeSum 2Kints.txt`

tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick
tick tick tick tick tick tick tick tick

`528`

`% java ThreeSum 4Kints.txt`



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`4039`

Measuring the running time

Q. How to time a program?

A. Automatic.

```
import edu.princeton.cs.algs4.StdOut;
import edu.princeton.cs.algs4.Stopwatch;

public static void main(String[] args)
{
   In in = new In(args[0]);
   int[] a = in.readAllInts();
   Stopwatch stopwatch = new Stopwatch();
   StdOut.println(ThreeSum.count(a));
   double time = stopwatch.elapsedTime();
   StdOut.println("elapsed time = " + time);
}
```

Empirical analysis

Run the program for various input sizes and measure running time.

```
%
```

Empirical analysis

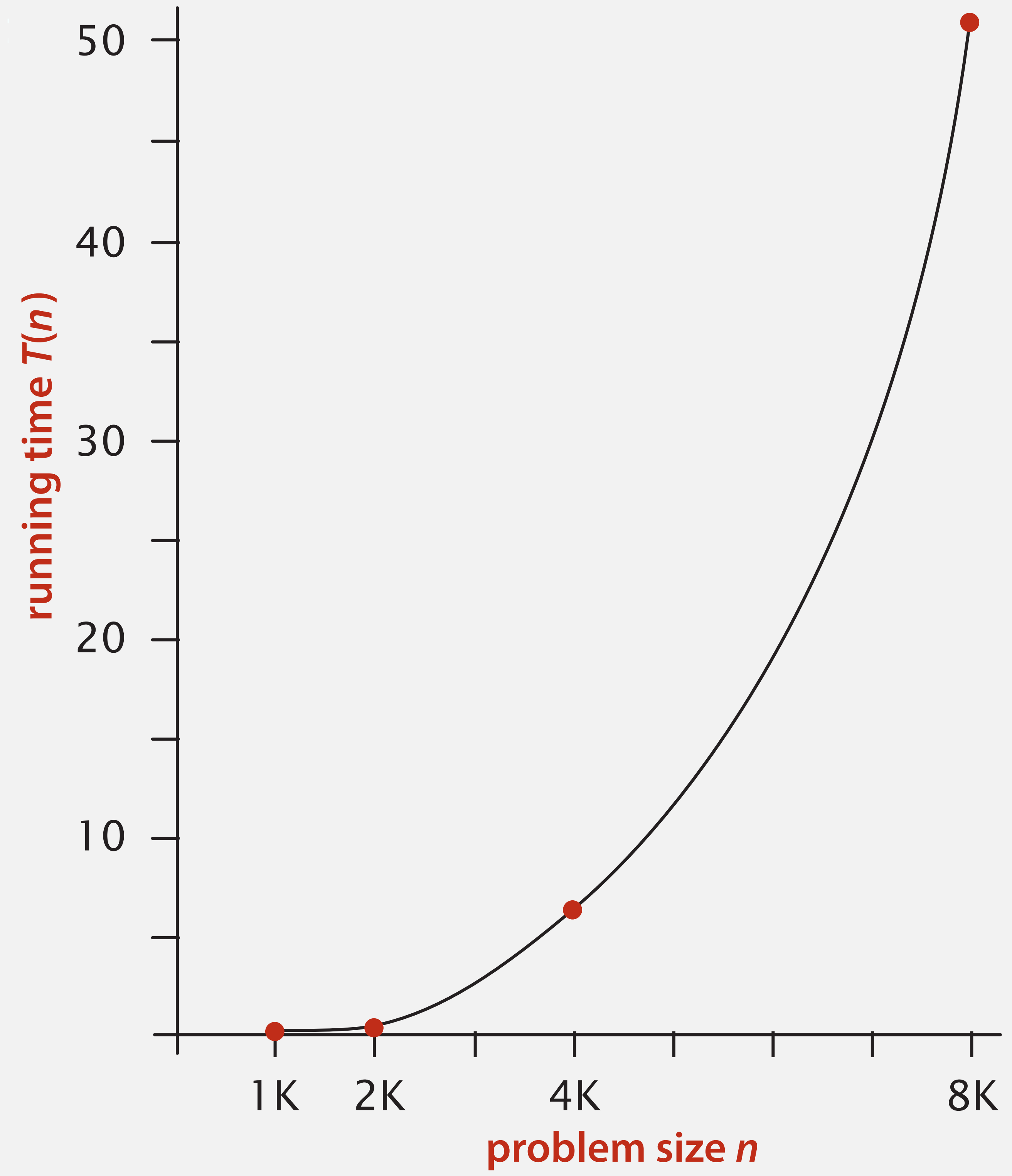
Run the program for various input sizes and measure running time.

n	time (seconds) †
250	0
500	0
1,000	0.1
2,000	0.8
4,000	6.4
8,000	51.1
16,000	?

† on a 2.8GHz Intel PU-226 with 64GB DDR E3 memory and 32MB L3 cache; running Oracle Java 1.7.0_45-b18 on Springdale Linux v. 6.5

Data analysis

Standard plot. Plot running time $T(n)$ vs. input size n.

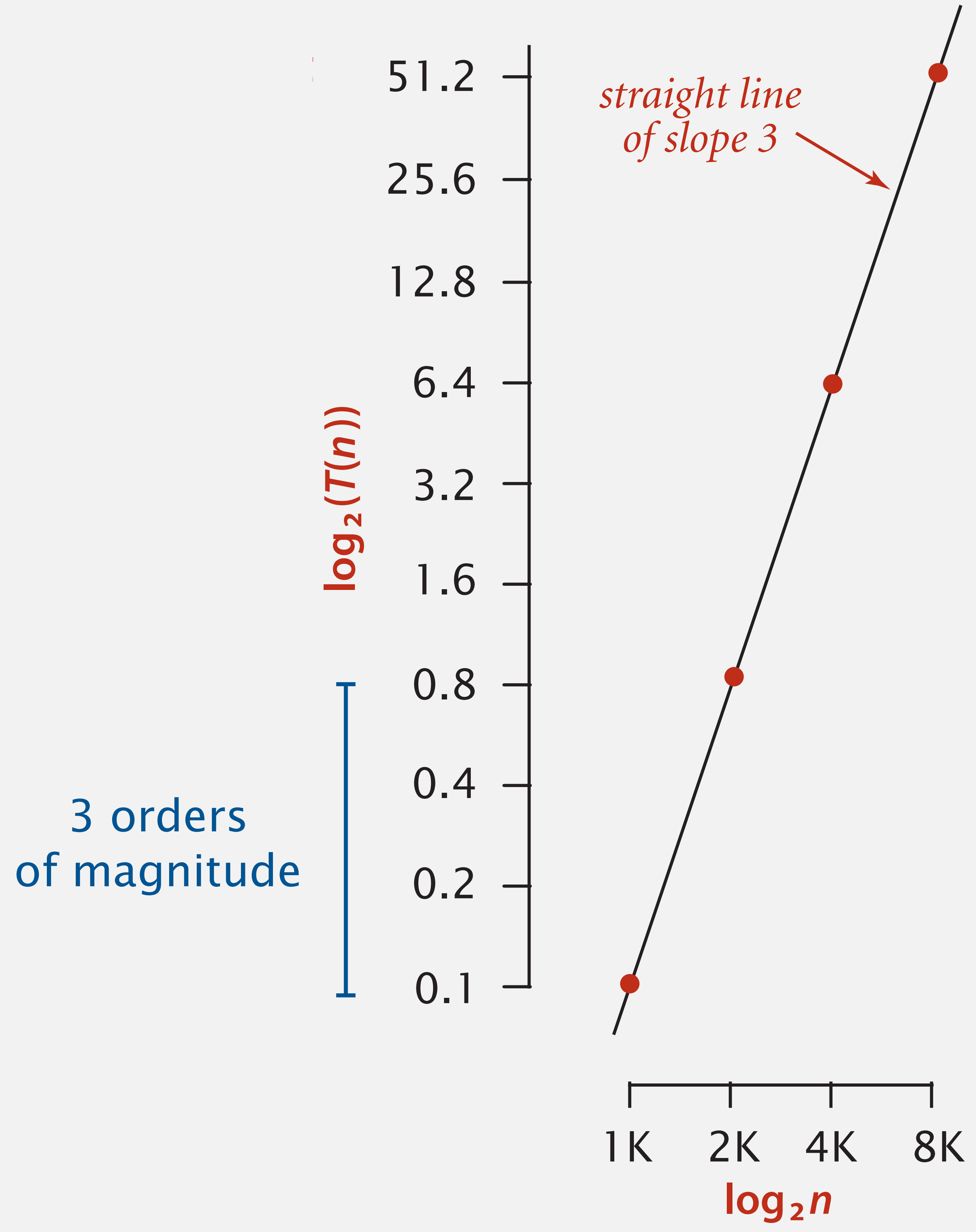


Hypothesis (power law). $T(n) = a\, n^b$.

Questions. How to validate hypothesis? How to estimate a and b ?

Data analysis

Log–log plot. Plot running time $T(n)$ vs. input size n using log–log scale.



slope

$$\log_2(T(n)) = 2.999 \log_2 n + (-33.21)$$

$$\begin{aligned} T(n) &= 2^{-33.21} \times n^{2.999} \\ &= 1.006 \times 10^{-10} \times n^{2.999} \end{aligned}$$

Regression. Fit straight line through data points.

Hypothesis. The running time is about $1.006 \times 10^{-10} \times n^{2.999}$ seconds.

Prediction and validation

Hypothesis. The running time is about $1.006 \times 10^{-10} \times n^{2.999}$ seconds.

"order of growth"
of running time is about n^3
[stay tuned]

Predictions.

- 51.0 seconds for $n = 8{,}000$.
- 408.1 seconds for $n = 16{,}000$.

Observations.

n	time (seconds) †
8,000	51.1
8,000	51.0
8,000	51.1
16,000	410.8

validates hypothesis!

Doubling hypothesis

Doubling hypothesis. Quick way to estimate exponent b in a power-law relationship.

Run program, doubling the size of the input.

n	time (seconds) †	ratio	$\log_2$ ratio
250	0		–
500	0	4.8	2.3
1,000	0.1	6.9	2.8
2,000	0.8	7.7	2.9
4,000	6.4	8	3.0
8,000	51.1	8	3.0

$\log_2 (6.4 / 0.8) = 3.0$

seems to converge to a constant $b \approx 3$

$$\frac{T(n)}{T(n/2)} = \frac{an^b}{a(n/2)^b} = 2^b$$

$$\implies b = \log_2 \frac{T(n)}{T(n/2)}$$

Hypothesis. Running time is about $a\, n^b$ with $b = \log_2$ ratio.

Caveat. Cannot identify logarithmic factors with doubling hypothesis.

Doubling hypothesis

Doubling hypothesis. Quick way to estimate exponent b in a power-law relationship.

Q. How to estimate a (assuming we know b) ?

A. Run the program (for a sufficient large value of n) and solve for a.

n	time (seconds) †
8,000	51.1
8,000	51.0
8,000	51.1

$$51.1 = a \times 8000^3$$

$$\Rightarrow \quad a = 0.998 \times 10^{-10}$$

Hypothesis. Running time is about $0.998 \times 10^{-10} \times n^3$ seconds.

almost identical hypothesis
to one obtained via regression
(but less work)

Analysis of algorithms: quiz 1

Estimate the running time to solve a problem of size $n = 96{,}000$.

A. 39 *seconds*

B. 52 *seconds*

C. 117 *seconds*

D. 350 *seconds*

n	time (seconds)
1,000	0.02
2,000	0.05
4,000	0.20
8,000	0.81
16,000	3.25
32,000	13.01

Experimental algorithmics

System independent effects.

- Algorithm.
- Input data.

determines exponent b in power law $a\,n^b$

determines constant a in power law $a\,n^b$

System dependent effects.

- Hardware: CPU, memory, cache, …
- Software: compiler, interpreter, garbage collector, …
- System: operating system, network, other apps, …

14.04 LTS

Bad news. Sometimes difficult to get accurate measurements.

Context: the scientific method

Experimental algorithmics is an example of the scientific method.

Chemistry
(1 experiment)

Biology
(1 experiment)

Physics
(1 experiment)

Computer Science
(1 million experiments)

Good news. Experiments are easier and cheaper than other sciences.

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1.4 Analysis of Algorithms

- *introduction*
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- *running time (mathematical models)*
- *memory usage*

Mathematical models for running time

Total running time: sum of cost × frequency for all operations.

- Need to analyze program to determine set of operations.
- Cost depends on machine, compiler.
- Frequency depends on algorithm, input data.

Warning. No general-purpose method (e.g., halting problem).

Example: 1-SUM

Q. How many operations as a function of input size n ?

```
int count = 0;
for (int i = 0; i < n; i++)
   if (a[i] == 0)
      count++;
```

exactly n array accesses

operation	cost (ns) †	frequency
variable declaration	2/5	2
assignment statement	1/5	2
less than compare	1/5	$n + 1$
equal to compare	1/10	n
array access	1/10	n
increment	1/10	n to $2n$

in practice, depends on caching, bounds checking, … (see COS 217)

† representative estimates (with some poetic license)

How many array accesses as a function of n?

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      if (a[i] + a[j] == 0)
         count++;
```

A. $\frac{1}{2}\, n\,(n-1)$

B. $n\,(n-1)$

C. $2\,n^2$

D. $2\,n\,(n-1)$

Example: 2-SUM

Q. How many operations as a function of input size n ?

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      if (a[i] + a[j] == 0)
         count++;
```

$$0 + 1 + 2 + \ldots + (n-1) = \frac{1}{2} n(n-1) = \binom{n}{2}$$

operation	cost (ns)	frequency
variable declaration	2/5	$n + 2$
assignment statement	1/5	$n + 2$
less than compare	1/5	$½ (n + 1) (n + 2)$
equal to compare	1/10	$½ n (n - 1)$
array access	1/10	$n (n - 1)$
increment	1/10	$½ n (n + 1)$ to n^2

$1/4\ n^2 + 13/20\ n + 13/10$ ns

to

$3/10\ n^2 + 3/5\ n + 13/10$ ns

(tedious to count exactly)

Simplification 1: cost model

Cost model. Use some elementary operation as a proxy for running time.

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      if (a[i] + a[j] == 0)
         count++;
```

operation	cost (ns)	frequency
variable declaration	2/5	$n + 2$
assignment statement	1/5	$n + 2$
less than compare	1/5	$\frac{1}{2} (n + 1) (n + 2)$
equal to compare	1/10	$\frac{1}{2} n (n - 1)$
array access	1/10	$n (n - 1)$
increment	1/10	$\frac{1}{2} n (n + 1)$ to n^2

cost model = array accesses

(we're assuming compiler/JVM does not optimize any array accesses away!)

Simplification 2: asymptotic notations

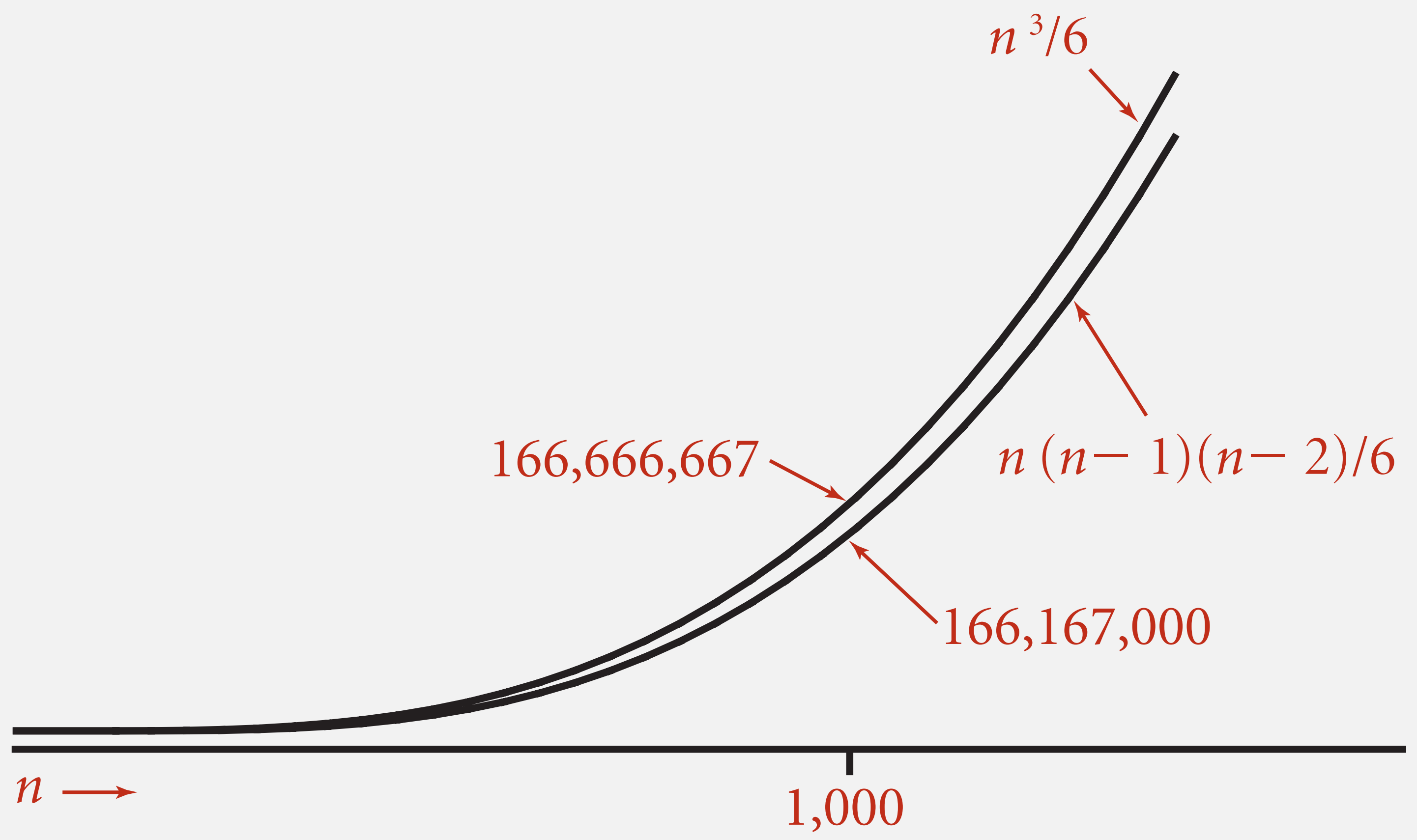
Tilde notation. Discard lower-order terms.

Big Theta notation. Also discard leading coefficient.

formal definitions involve limits

function	tilde	big Theta
$4\,n^5 + 20\,n + 16$	$\sim 4\,n^5$	$\Theta(n^5)$
$7\,n^2 + 100\,n^{4/3} + 56$	$\sim 7\,n^2$	$\Theta(n^2)$
$\frac{1}{6}\,n^3 \underbrace{-\ \frac{1}{2}\,n^2 + \frac{1}{3}\,n}$	$\sim \frac{1}{6}\,n^3$	$\Theta(n^3)$

discard lower-order terms
(e.g., n = 1,000: 166.67 million vs. 166.17 million)



Leading-term approximation

Rationale.

- When n is large, lower-order terms are negligible.
- When n is small, we don't care.

Common order-of-growth classifications

order of growth	name	typical code framework	description	example	$T(2n) / T(n)$
$\Theta(1)$	**constant**	`a = b + c;`	statement	*add two numbers*	1
$\Theta(\log n)$	**logarithmic**	`while (n > 1) { n = n/2; ... }`	divide in half	*binary search*	~ 1
$\Theta(n)$	**linear**	`for (int i = 0; i < n; i++) { ... }`	single loop	*find the maximum*	2
$\Theta(n \log n)$	**linearithmic**	*see mergesort lecture*	divide and conquer	*mergesort*	~ 2
$\Theta(n^2)$	**quadratic**	`for (int i = 0; i < n; i++) for (int j = 0; j < n; j++) { ... }`	double loop	*check all pairs*	4
$\Theta(n^3)$	**cubic**	`for (int i = 0; i < n; i++) for (int j = 0; j < n; j++) for (int k = 0; k < n; k++) { ... }`	triple loop	*check all triples*	8
$\Theta(2^n)$	**exponential**	*see combinatorial search lecture*	exhaustive search	*check all subsets*	2^n

Example: 2-SUM

Q. Approximately how many array accesses as a function of input size n?

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      if (a[i] + a[j] == 0)
         count++;
```

"inner loop"

$$0 + 1 + 2 + \ldots + (n-1) \;=\; \frac{1}{2}\, n(n-1) \;=\; \binom{n}{2}$$

A. $\sim n^2$ array accesses.

Example: 3-SUM

Q. Approximately how many array accesses as a function of input size n?

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      for (int k = j+1; k < n; k++)
         if (a[i] + a[j] + a[k] == 0)
            count++;
```

"inner loop"

A. $\sim \frac{1}{2} n^3$ array accesses.

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!}$$

$$\sim \frac{1}{6} n^3$$

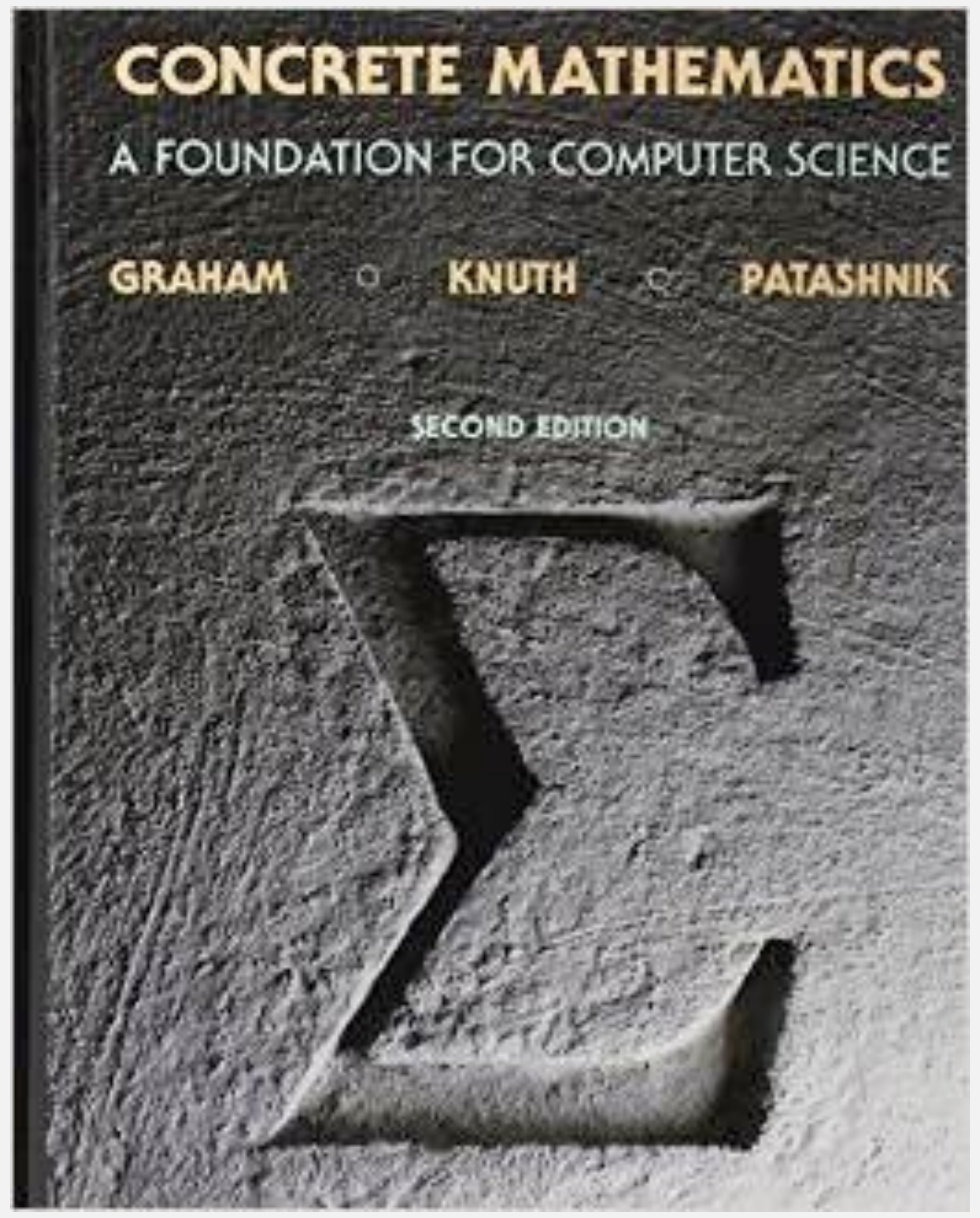
see COS 340

Bottom line. Use cost model and asymptotic notation to simplify analysis.

Estimating a discrete sum

Q. How to estimate a discrete sum?

A1. Take a discrete mathematics course (COS 340).



Estimating a discrete sum

Q. How to estimate a discrete sum?

A2. Replace the sum with an integral; use calculus!

Ex 1. $1 + 2 + \ldots + n$.

(triangular sum)

$$\sum_{i=1}^{n} i \sim \int_{x=1}^{n} x\,dx \sim \frac{1}{2} n^2$$

Ex 2. $1 + 1/2 + 1/3 + \ldots + 1/n$.

(harmonic sum)

$$\sum_{i=1}^{n} \frac{1}{i} \sim \int_{x=1}^{n} \frac{1}{x}\,dx \sim \ln n$$

Ex 3. 3-sum triple loop.

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} 1 \sim \int_{x=1}^{n} \int_{y=x+1}^{n} \int_{z=y+1}^{n} dz\,dy\,dx \sim \frac{1}{6} n^3$$

Ex 4. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

(geometric sum)

$$\int_{x=0}^{\infty} \left(\frac{1}{2}\right)^x dx = \frac{1}{\ln 2} \approx 1.4427$$

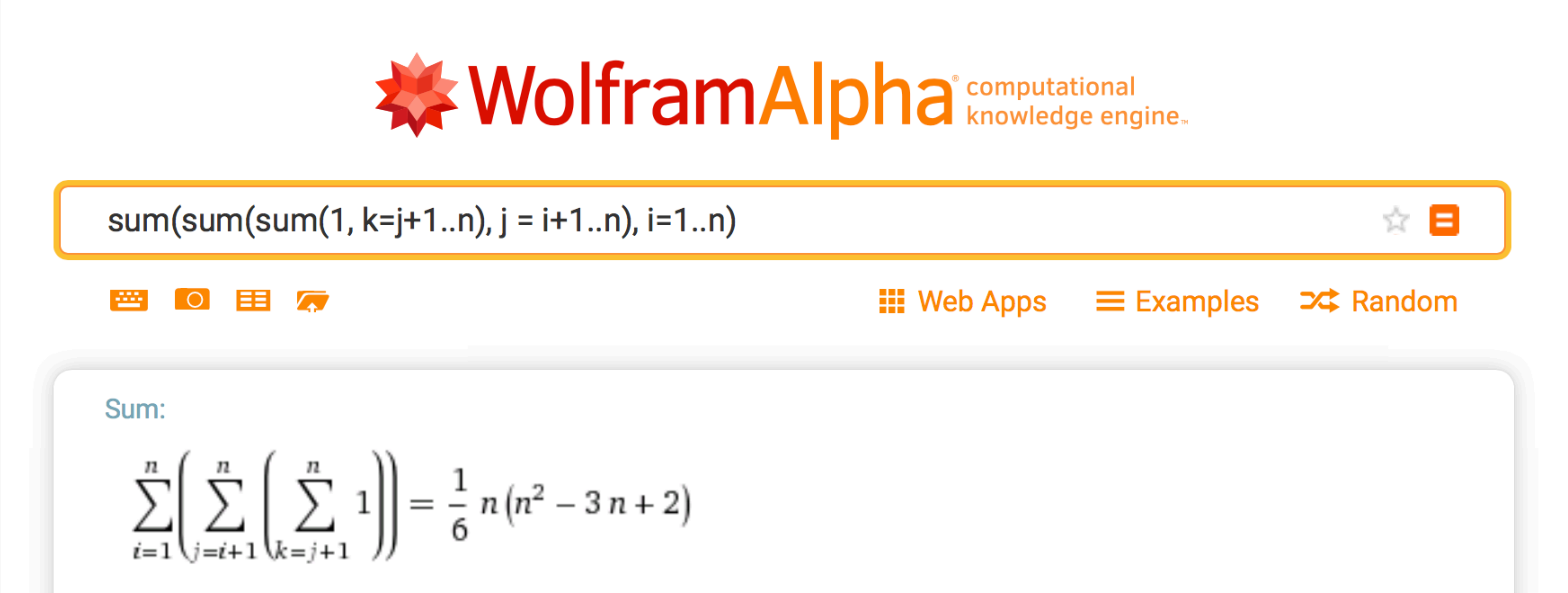
$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 2$$

integral trick doesn't always work!

Estimating a discrete sum

Q. How to estimate a discrete sum?

A3. Use Maple or Wolfram Alpha.



https://www.wolframalpha.com

How many array accesses as a function of n ?

```
int count = 0;
for (int i = 0; i < n; i++)
   for (int j = i+1; j < n; j++)
      for (int k = 1; k <= n; k = k*2)
         if (a[i] + a[j] >= a[k])
            count++;
```

A. $\sim n^2 \log_2 n$

B. $\sim 3/2 \; n^2 \log_2 n$

C. $\sim 1/2 \, n^3$

D. $\sim 3/2 \, n^3$

What is order of growth of running time as a function of n ?

```
int count = 0;
for (int i = n; i >= 1; i = i/2)
   for (int j = 1; j <= i; j++)
      count++;   ←——— "inner loop"
```

A. $\Theta(n)$

B. $\Theta(n \log n)$

C. $\Theta(n^2)$

D. $\Theta(2^n)$

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1.4 Analysis of Algorithms

- *introduction*
- *running time (experimental analysis)*
- *running time (mathematical models)*
- *memory usage*

Basics

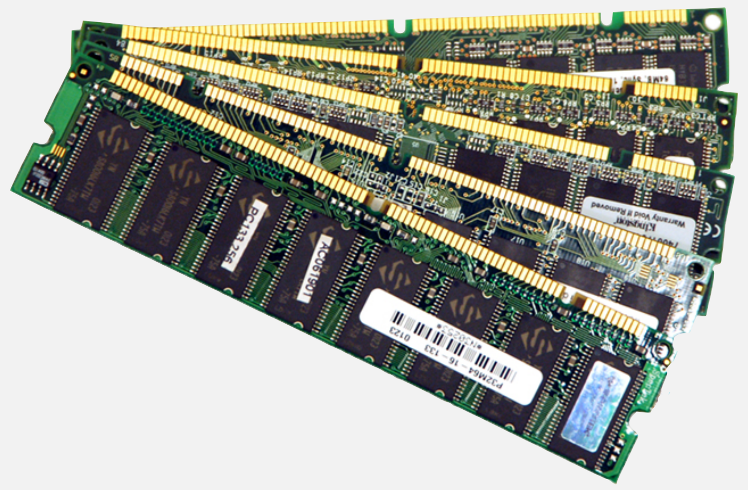
NIST most computer scientists

Bit. 0 or 1.

Byte. 8 bits.

Megabyte (MB). 1 million or 2^{20} bytes.

Gigabyte (GB). 1 billion or 2^{30} bytes.



64-bit machine. We assume a 64-bit machine with 8-byte pointers.

some JVMs "compress" ordinary object pointers to 4 bytes to avoid this cost

Typical memory usage for primitive types and arrays

type	bytes
`boolean`	1
`byte`	1
`char`	2
`int`	4
`float`	4
`long`	8
`double`	8

primitive types

type	bytes
`boolean[]`	$\boxed{1n + 24}$
`int[]`	$4n + 24$
`double[]`	$8n + \boxed{24}$

wasteful
(but $\sim 36n$ in Python 3)

array overhead = 24 byte

one-dimensional arrays (length n)

type	bytes
`boolean[][]`	$\sim 1\ n^2$
`int[][]`	$\sim 4\ n^2$
`double[][]`	$\sim 8\ n^2$

two-dimensional arrays (n-by-n)

Typical memory usage for objects in Java

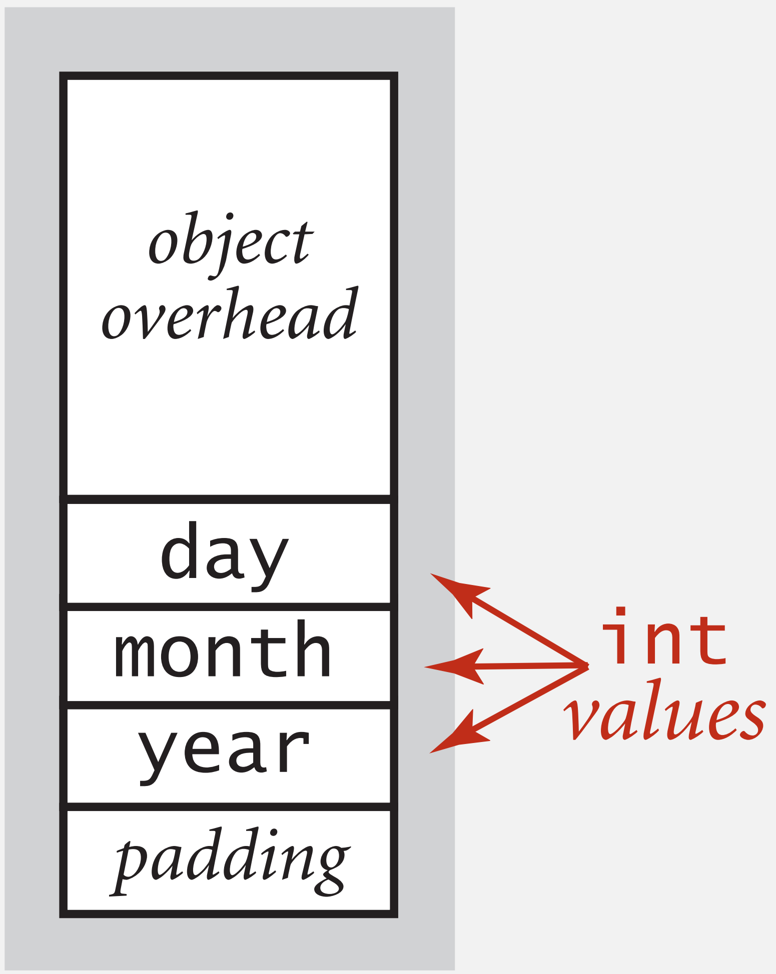
Object overhead. 16 bytes.

Reference. 8 bytes.

Padding. Memory of each object rounded up to use a multiple of 8 bytes.

Ex 1. A `Date` object uses 32 bytes of memory.

```
public class Date
{
   private int day;
   private int month;
   private int year;
...
}
```



16 bytes (object overhead)

4 bytes (int)

4 bytes (int)

4 bytes (int)

4 bytes (padding)

32 bytes

Typical memory usage summary

Total memory usage for a data type value in Java:

- Primitive type: 4 bytes for `int`, 8 bytes for `double`, ...
- Object reference: 8 bytes.
- Array: 24 bytes + memory for each array entry.
- Object: 16 bytes + memory for each instance variable.
- Padding: round up memory of each object to be a multiple of 8 bytes.

Note. Depending on application, we often count the memory for any referenced objects (recursively).

"deep memory"

How much memory does a `WeightedQuickUnionUF` use as a function of n ?

A. $\sim 4\, n$ *bytes*

B. $\sim 8\, n$ *bytes*

C. $\sim 4\, n^2$ *bytes*

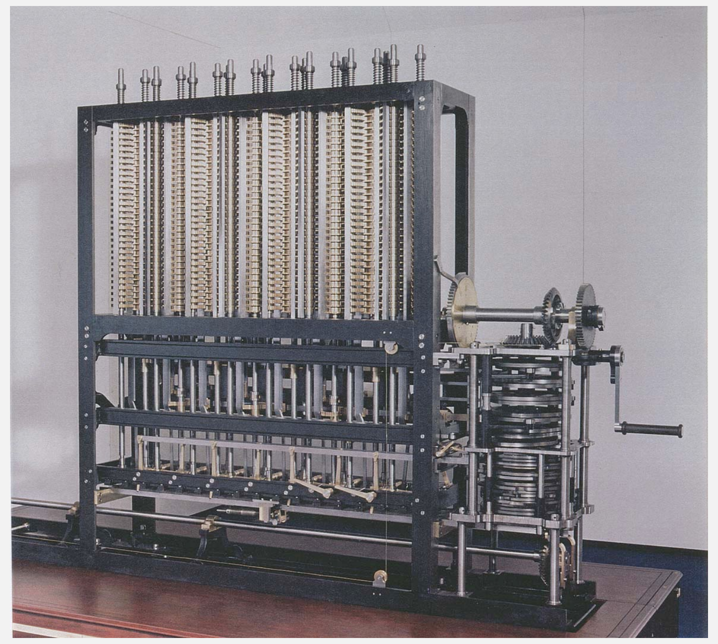
D. $\sim 8\, n^2$ *bytes*

```
public class WeightedQuickUnionUF
{
   private int[] parent;
   private int[] size;
   private int count;

   public WeightedQuickUnionUF(int n)
   {
      parent = new int[n];
      size   = new int[n];

      count = 0;
      for (int i = 0; i < n; i++)
         parent[i] = i;
      for (int i = 0; i < n; i++)
         size[i] = 1;
   }
   ...
}
```

Turning the crank: summary

Empirical analysis.

- Execute program to perform experiments.
- Assume power law.
- Formulate a hypothesis for running time.
- Model enables us to make predictions.



Mathematical analysis.

- Analyze algorithm to count frequency of operations.
- Use tilde and big-Theta notations to simplify analysis.
- Model enables us to explain behavior.

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \lceil n/2^{h+1} \rceil \, h \sim n$$

This course. Learn to use both.