COS 522, February 3, 2000
Due: February 17, 2000

Homework Set 1

Let $\vec{x}$ denote $(x_1, x_2, \cdots, x_n) \in R^n$. A *homogeneous linear function* in R^n is a function of the form $\sum_{1\le i\le n} a_i x_i$ where a_i are real numbers. Let $g, h_1, h_2, \cdots, h_t$ be homogeneous linear functions in R^n, and $G = \{\vec{x} \,|\, g(\vec{x}) > 0\}$, $H = \{\vec{x} \,|\, h_1(\vec{x}) > 0, \cdots, h_t(\vec{x}) > 0\}$. In class, we stated:

Farkas' Lemma (as given in class) If $\emptyset \neq H \subseteq G$, then there exist non-negative real numbers $\lambda_1, \lambda_2, \cdots, \lambda_t$ such that $g(\vec{x}) = \sum_{1\le i\le t} \lambda_i h_i(\vec{x})$.

We now give a more general version. A *linear function* in R^n is a function of the form $\sum_{1\le i\le n} a_i x_i - c$ where a_i, c are real numbers. Let $g, h_1, h_2, \cdots, h_t$ be linear functions in R^n, and $G = \{\vec{x} \,|\, g(\vec{x}) > 0\}$, $H = \{\vec{x} \,|\, h_1(\vec{x}) > 0, \cdots, h_t(\vec{x}) > 0\}$. The following is a version of the Farkas' Lemma which might be useful for solving problems in this homework set.

Farkas' Lemma II If $\emptyset \neq H \subseteq G$, then there exist non-negative real numbers $\lambda_1, \lambda_2, \cdots, \lambda_t, c$ such that $g(\vec{x}) = \sum_{1\le i\le t} \lambda_i h_i(\vec{x}) + c$.

Problem 1 Prove Farkas' Lemma II using the version of Farkas' Lemma given in class.

Problem 2 Let K_n be the set of all points $\vec{x} = (x_1, x_2, \cdots, x_n) \in R^n$ such that $\sum_{i\in A} x_i \neq 1$ for all $A \subseteq \{1, 2, \cdots, n\}$. Prove that $\beta_0(K_n) \geq 2^{\Omega(n^2)}$.

Problem 3 Given an input of n distinct real numbers $x_1, x_2, \cdots, x_n$, we would like to output i where x_i is the second largest of these n numbers. Show that any linear decision tree algorithm T for this problem must have at least $n \cdot 2^{n-2}$ leaves. Prove that the linear decision tree complexity for this problem is at least $n - 2 + \lceil \log_2 n \rceil$.

Problem 4 Given an input of 9 distinct real numbers $x_1, x_2, \cdots, x_9$, we would like to find a *mediocre* element, ie, i such that the $4 \leq rank(x_i) \leq 6$. Let $C_1(P)$ and $C_2(P)$ denote respectively the linear decision tree and the quadratic decision tree complexity for this problem. Show that $C_1(P) \geq 6$. Show that $2 \leq C_2(P) \leq 8$.

Problem 5 As discussed in class, the *Collinearity Problem* is to decide whether an input of n points on the plane contains three points lying on a line. Show that there exists no linear decision tree algorithm for this problem.

Let $S \subseteq R^n$ and $a, b \in S$. A *path in S* from a to b is a continuous map p from the real interval $[0, 1]$ into S such that $p(0) = a$ and $p(1) = b$. Write $a \,\#\, b$ if there exists a path in S from a to b. Clearly $\#$ is an equivalence relation on S. Each equivalence class is a called a *path-connected component*, and $\beta_0(S)$ is defined as the number of path-conncected components.

Problem 6 Let $1 \leq k \leq n$. Let $S \subseteq R^n$, and

$$V = \{(x_1, x_2, \cdots, x_k) \mid \text{ there exist } x_{k+1}, \cdots, x_n \text{ such that } (x_1, x_2, \cdots, x_n) \in S\}.$$

Give a rigorous proof (as if you were in a Calculus class) that $\beta_0(V) \leq \beta_0(S)$.