COS 302 Precept 2

Princeton University

Spring 2020

Table of Contents

Table of Contents

Linear Combination

Let V be a vector space. $\mathbf{v} \in V$ is a linear combination of vectors $\mathbf{x_1}, ..., \mathbf{x_k} \in V$ if

$$\mathbf{v} = \lambda_1 \mathbf{x_1} + ... + \lambda_k \mathbf{x_k} = \sum_{i=1}^{k} \lambda_i \mathbf{x_i} \in V \tag{1}$$

The $\mathbf{0}$-vector can be trivially represented as a linear combination of k vectors as $\sum_{i=1}^{k} 0\mathbf{x_i}$. Of interest are nontrivial linear combinations where there is at least one coefficient $\lambda_i \neq 0$.

Linear (In)dependence

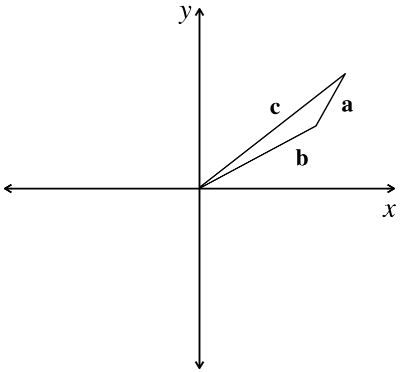
If there is at least one nontrivial linear combination of $\mathbf{x_1}, \ldots, \mathbf{x_k} \in V$ such that $\mathbf{0} = \sum_{i=1}^{k} \lambda_i \mathbf{x_i}$, then $\mathbf{x_1}, \ldots, \mathbf{x_k}$ are *linearly dependent*. Otherwise, when only the trivial solution exists, they are *linearly independent*.

Linear (In)dependence

We can think of linearly independent vectors as having no redundancy in the sense that if we remove any one of them, there will be certain vectors we can no longer represent via linear combinations.

Linear (In)dependence

Example: Consider three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ where $\mathbf{c} = \mathbf{a} + \mathbf{b}$.



These vectors are linearly dependent because $\mathbf{a} + \mathbf{b} - \mathbf{c} = \mathbf{0}$.

Some Properties

- k vectors are either linearly dependent or linearly independent.
- If any vector in a set is $\mathbf{0}$, then they are linearly dependent.
- A set of vectors is linearly dependent if and only if (iff) one of them is a linear combination of the others.

Some Properties

Gaussian elimination can be used to check if a set of vectors is linearly independent:

We construct a matrix $\mathbf{A}$ by placing each vector as a column of $\mathbf{A}$ and transform the matrix into row echelon form. Pivot columns are linearly independent of vectors to the left, while non-pivot columns can be represented as linear combinations of vectors to the left. If every column is a pivot column, all the vectors are linearly independent.

Checking Linear Independence

Example

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

We transform the corresponding matrix to reduced echelon form:

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ -3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Checking Linear Independence

Example

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & -2 \\ -3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Every column is a pivot column, so the vectors are linearly independent.

Checking Linear Independence

Now we consider a case where we have m linear combinations of k linearly independent vectors $\mathbf{b}_1, \ldots, \mathbf{b}_k$:

$$\boldsymbol{x}_1 = \sum_{i=1}^{k} \lambda_{i1} \boldsymbol{b}_i$$

$$\vdots$$

$$\boldsymbol{x}_m = \sum_{i=1}^{k} \lambda_{im} \boldsymbol{b}_i$$

Are the resulting vectors also linearly independent?

Checking Linear Independence

- With $B = [\mathbf{b}_1, ..., \mathbf{b}_k]$, we express the system as $\mathbf{x}_j = \boldsymbol{B\lambda}_j$.
- We want to find non-trivial solutions to $\sum_{j=1}^{m} \psi_j \boldsymbol{x}_j = 0$.
- We have:

$$\sum_{j=1}^{m} \psi_j \boldsymbol{x}_j = \sum_{j=1}^{m} \psi_j \boldsymbol{B\lambda}_j = \boldsymbol{B} \sum_{j=1}^{m} \psi_j \boldsymbol{\lambda}_j = \mathbf{0}.$$

Checking Linear Independence

- With $B = [\mathbf{b}_1, ..., \mathbf{b}_k]$, we express the system as $\mathbf{x}_j = B\boldsymbol{\lambda}_j$.
- We want to find non-trivial solutions to $\sum_{j=1}^{m} \psi_j \boldsymbol{x}_j = 0$.
- We have:

$$\sum_{j=1}^{m} \psi_j \boldsymbol{x}_j = \sum_{j=1}^{m} \psi_j B\boldsymbol{\lambda}_j = B\sum_{j=1}^{m} \psi_j \boldsymbol{\lambda}_j = \mathbf{0}.$$

- Because B has lin. indep. columns, the above can be true iff $\sum_{j=1}^{m} \psi_j \boldsymbol{\lambda}_j = \mathbf{0}$ (i.e., $\{\boldsymbol{\lambda}_j\}$ are also lin. indep.).

Table of Contents

Basis and Rank

In a vector space V, we can find sets of vectors $\mathcal{A}$ with a special property: *any* other vector $\mathbf{v} \in V$ can be expressed as a linear combination of vectors in $\mathcal{A}$. These are particularly useful vectors, and we characterize them in the remainder.

Generating Set and Basis

Definition: Generating Set and Span

Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and set of vectors $\mathcal{A} = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subseteq \mathcal{V}$. If every vector $\mathbf{v} \in \mathcal{V}$ can be expressed as a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_k$, then $\mathcal{A}$ is called a *generating set* of V. The set of all linear combinations of vectors in $\mathcal{A}$ is called the *span* of $\mathcal{A}$. If $\mathcal{A}$ spans the vector space V, we write $V = \text{span}[\mathcal{A}]$ or $V = \text{span}[\mathbf{x}_1, \ldots, \mathbf{x}_k]$.

Basis

Definition: Basis

Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and set of vectors $\mathcal{A} \subseteq \mathcal{V}$. A generating set $\mathcal{A}$ of V is called *minimal* if there exists *no smaller set* $\tilde{A} \subseteq \mathcal{A} \subseteq V$ that spans V. Every linearly independent generating set of V is minimal and is called a **basis of V.**

Basis example

Example

In $\mathbb{R}^3$, the *canonical* or *standard* basis is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Two other bases of $\mathbb{R}^3$ are

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ -0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}$$

Basis Non-Example

Example

The set

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix} \right\}$$

is not a generating set (and so not a basis) of $\mathbb{R}^4$. The vectors in $\mathcal{A}$ are linearly independent.

Equivalent Statements

Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\mathcal{B} \subseteq \mathcal{V}, \mathcal{B} \neq \emptyset$.
The following statements are equivalent:

- $\mathcal{B}$ is a basis of V.
- $\mathcal{B}$ is a minimal generating set.
- $\mathcal{B}$ is a maximal linearly independent set of vectors in V, i.e., adding any other vector to this set will make it linearly dependent.
- Every vector $\mathbf{x} \in \mathcal{V}$ is a linear combination of vectors from $\mathcal{B}$, and every linear combination is unique, i.e., if
$$\mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{b}_i = \sum_{i=1}^{k} \psi_i \mathbf{b}_i$$
$\lambda_i, \psi_i \in \mathbb{R}$, $\mathbf{b}_i \in \mathcal{B}$, then $\lambda_i = \psi_i$ for every i.

Remarks about Basis

- Every vector space V possesses a basis $\mathcal{B}$; there is no unique basis.
- All bases contain the same number of *basis vectors*
- The *dimension* of V is the number of basis vectors of V: intuitively, the dimension of a vector space can be thought of as the number of independent directions in this vector space.
- The dimension of a vector space is not *necessarily* the number of elements in a vector. For example,

$$V = \text{span}[\begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix}]$$

is one-dimensional.

Determining a Basis

Consider a subspace $U = \text{span}[\mathbf{x}_1, \ldots, \mathbf{x}_m] \subseteq \mathbb{R}^n$. The basis can be found by following this recipe:

1. Write the spanning vectors as columns of a matrix $\mathbf{A}$
2. Determine the row-echelon form of $\mathbf{A}$
3. The spanning vectors associated with the *pivot columns* are a basis of U

Finding Basis of a Subspace (1 of 3)

Consider a subspace U of $\mathbb{R}^5$ spanned by the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix}$$

we want to find out which of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_4$ are a basis for U. Therefore, we must solve

$$\sum_{i=1}^{4} \lambda_i \mathbf{x}_i = \mathbf{0}$$

.

Finding Basis of a Subspace (2 of 3)

We can write this as a homogeneous system of equations with

$$\begin{bmatrix} \mathbf{x}_1, & \mathbf{x}_2, & \mathbf{x}_3, & \mathbf{x}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

$$\rightsquigarrow \cdots \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding Basis of a Subspace (3 of 3)

Recall that pivot columns indicate which set of vectors is linearly independent. We can read off from the row-echelon form that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}$ is a maximally linearly independent set of vectors in U, and is therefore a basis of U.

Table of Contents

Rank

The number of linearly independent columns of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ equals the number of linearly independent rows and is called the *rank* of $\mathbf{A}$, denoted $\mathrm{rk}(\mathbf{A})$.

Example

The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

has $\mathrm{rk}(\mathbf{A}) = 2$, because $\mathbf{A}$ has two linearly independent columns / rows.

Properties of Rank 1 of 2

- $\text{rk}(\mathbf{A}) = \text{rk}(\mathbf{A}^\top)$
- The columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$ span a subspace $U \subseteq \mathbb{R}^m$ with $\dim(U) = \text{rk}(\mathbf{A})$
- The rows of $\mathbf{A} \in \mathbb{R}^{m \times n}$ span a subspace $W \subseteq \mathbb{R}^n$ with $\dim(W) = \text{rk}(\mathbf{A})$.
- For all $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{A}$ is invertible iff $\text{rk}(\mathbf{A}) = n$

Properties of Rank 2 of 2

- For $\mathbf{A} \in \mathbb{R}^{m \times n}$ the subspace of all solutions to $\mathbf{Ax} = \mathbf{0}$ has dimension $n - \text{rk}(\mathbf{A})$. This is called the kernel or null space of $\mathbf{A}$.
- A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ has **full rank** if its rank equals the largest possible rank for a matrix of the same dimensions. In other words, the rank of a full rank matrix is $\text{rk}(\mathbf{A}) = \min(m, n)$.
- A matrix is said to be *rank deficient* if it does not have full rank.