COS 302 Precept 1

Princeton University

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Table of Contents

Table of Contents

Matrix Addition

Let $\boldsymbol{X}$ represent a matrix, $\boldsymbol{X_{ij}}$ denote the entry that is in the ith row and jth column of $\boldsymbol{X}$.

$(\boldsymbol{A} + \boldsymbol{B})_{ij} = A_{ij} + B_{ij}$

Matrix Multiplication

Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{B} \in \mathbb{R}^{n \times k}$
$(\boldsymbol{AB})_{ij} = \sum_{l=1}^{n} A_{il} B_{lj}$
In general, matrix multiplication is not commutative.

Properties of Matrix Multiplication

- *Associativity:*
  $\forall \boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{B} \in \mathbb{R}^{n \times p}, \boldsymbol{C} \in \mathbb{R}^{p \times q}$:
  $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
- *Distributivity:*
  $\forall \boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}, \boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^{n \times p}$,
  $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$,
  $\boldsymbol{A}(\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{AC} + \boldsymbol{AD}$
- Multiplication By Identity Matrix:
  $\forall \boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{I_m A} = \boldsymbol{A I_n} = \boldsymbol{A}$, where $\boldsymbol{I_m}$ is an $m \times m$ matrix such that it has 1s on the diagonal and 0s everywhere else. It is known as the identity matrix.

Transpose

$\boldsymbol{A}_{ij}^{T} = \boldsymbol{A}_{ji}$
If $\boldsymbol{A}^{T} = \boldsymbol{A}$, $\boldsymbol{A}$ is known as a symmetric matrix.

Matrix Transpose Properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Matrix Inverse

Definition

Consider a square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. Let matrix $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ have the property that $\boldsymbol{AB} = \boldsymbol{I_n} = \boldsymbol{BA}$. $\boldsymbol{B}$ is called the inverse of $\boldsymbol{A}$ and is denoted by $\boldsymbol{A}^{-1}$.

Matrix Inverse Properties

- $(\boldsymbol{A}+\boldsymbol{B})^{-1} \neq \boldsymbol{A}^{-1}+\boldsymbol{B}^{-1}$
- $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

Table of Contents

Systems of Linear Equations as Matrices

$$\begin{cases} x_1 + 0x_2 + 8x_3 - 4x_4 = 42 \\ 0x_1 + x_2 + 2x_3 + 12x_4 = 8 \end{cases}$$

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$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix}$$

General Approach to Finding Solutions

1. Find a particular solution to $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is a matrix, $\boldsymbol{x}$ and $\boldsymbol{b}$ are vectors.
2. Find all solutions to $\boldsymbol{Ax} = \mathbf{0}$.
3. Combine step 1 and 2 to find the general solutions

Particular Solution

A particular solution to the above system of equations is:

$$\mathbf{x} = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix}$$

General Solution

Key Idea: Adding $\mathbf{0}$ to our particular solution does not change our particular solution.

General Solution

Express third column using the first two columns

$$\begin{bmatrix} 8 \\ 2 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \left(\lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right) = \mathbf{0}$$

General Solution

Similarly, express the 4th column using the first two columns, we get:

$$\begin{bmatrix}1 & 0 & 8 & -4\\ 0 & 1 & 2 & 12\end{bmatrix}\left(\lambda_2\begin{bmatrix}-4\\12\\0\\-1\end{bmatrix}\right)=\mathbf{0}$$

General Solution

Putting everything together, we obtain solutions for the entire system:

$$\left\{ \boldsymbol{x} = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

Table of Contents

Row-Echelon Form

Definition

A matrix is in row-echelon form if:

- All rows that contain only zeros are at the bottom of the matrix; correspondingly, all rows that contain at least one nonzero element are on top of rows that contain only zeros.
- Looking at nonzero rows only, the first nonzero number from the left pivot (also called the pivot or the leading coefficient) is always strictly to the right of the pivot of the row above it.

Row-Echelon Form

Examples

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Table of Contents

Reduced Row-Echelon Form

Definition

A matrix is in reduced row-echelon form if

- It is in row-echelon form
- Every pivot(The first nonzero number from the left in each row) is 1
- The pivot is the only nonzero entry in its column.

Reduced Row-Echelon Form

Examples

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix}$$

In general, row-echelon form and reduced row-echelon form make it easier for us to determine a particular solution and the general solution. In fact, the matrix we saw in section 2 is in reduced row-echelon form.

Table of Contents

Elementary Transformations

Given a matrix $\boldsymbol{A}$, there are three elementary operations one can perform on $\boldsymbol{A}$ to transform $\boldsymbol{A}$ into reduced row-echelon form without changing the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$.

- Exchange Two Rows of a Matrix

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Given a matrix $\boldsymbol{A}$, there are three elementary operations one can perform on $\boldsymbol{A}$ to transform $\boldsymbol{A}$ into reduced row-echelon form without changing the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$.

- Exchange Two Rows of a Matrix
- Multiplication of a row with a constant $\lambda \in \mathbb{R}$, where $\lambda \neq 0$
- Addition of Two Rows

Table of Contents

Gaussian Elimination

Gaussian elimination is an algorithm that performs elementary transformations to bring a system of linear equations into reduced row-echelon form.

Gaussian Elimination

$$\begin{cases} x_1 + x_2 - x_3 = 7 \\ x_1 - x_2 + 2x_3 = 3 \\ 2x_1 + x_2 + x_3 = 9 \end{cases}$$

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The above system of equations can be represented by this augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 9 \end{array}\right]$

We will perform Gaussian Elimination on this system of equations(Open Ipython Notebook)

Invert Matrix via Gaussian Elimination

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Invert Matrix via Gaussian Elimination

Perform Gaussian Elimination on the following Augmented Matrix:

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Invert Matrix via Gaussian Elimination

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 \end{array}\right]$$

Invert Matrix via Gaussian Elimination

$$\boldsymbol{A}^{-1} = \begin{bmatrix} -1 & 2 & -2 & 2 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Justification(Optional)

Each elementary operation on $\boldsymbol{A}$ can be written as left multiplying $\boldsymbol{A}$ by a matrix. Transforming $\boldsymbol{A}$ to the identity matrix can be written as: $\boldsymbol{E_1 E_2 \cdots E_n A = I}$. This implies that $\boldsymbol{E_1 E_2 \cdots E_n A A^{-1} = I A^{-1} = A^{-1}}$, which implies that $\boldsymbol{E_1 E_2 \cdots E_n I = A^{-1}}$. This means that applying the sequence of elementary operations that transformed $\boldsymbol{A}$ to the identity matrix on $\boldsymbol{I}$ will transform $\boldsymbol{I}$ to $\boldsymbol{A^{-1}}$