Final exam information

Final exam.

- Check out between 4:30pm on 5/18 and 4:30pm on 5/20.
- 3 hours to complete.
- Gradescope platform.

Rules.

including associated
readings and assignments
(but no serious Java programming)

- Honor Code.
- Closed book, closed note.
- Strongly emphasizes material since the midterm.
- Electronic devices are prohibited (except to take the exam).
- 8.5-by-11 cheatsheet (both sides, in your own handwriting).

see course website
for exam archive

Final exam preparation.

- Practice by doing old exams.
- Ask questions via Zoom office hours or Ed.

see course website for schedule

https://algs4.cs.princeton.edu

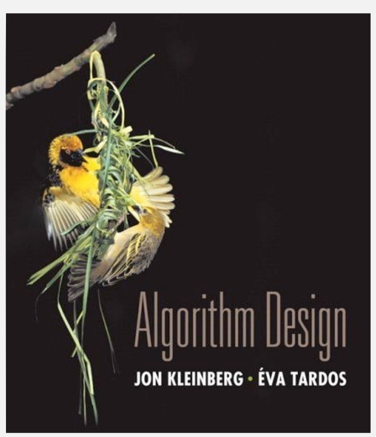
ALGORITHM DESIGN

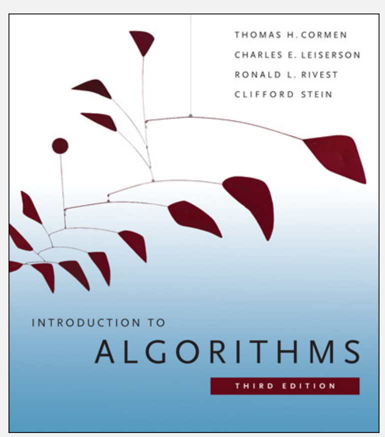
- *analysis of algorithms*
- *greedy*
- *network flow*
- *dynamic programming*
- *divide-and-conquer*
- *randomized algorithms*

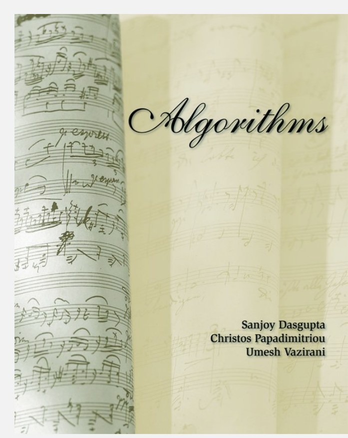
Algorithm design

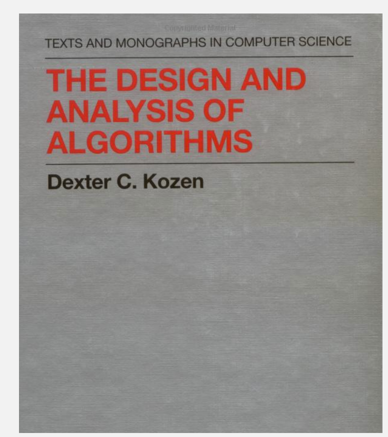
Algorithm design patterns.

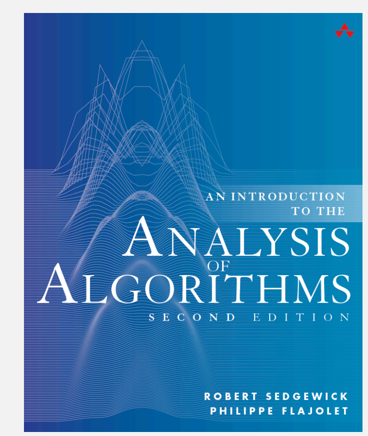
- Analysis of algorithms.
- Greedy.
- Network flow.
- Dynamic programming.
- Divide-and-conquer.
- Randomized algorithms.











Want more? See COS 340, COS 343, COS 423, COS 445, COS 451, COS 488, ...

Interview questions

Apple Computer

facebook

Cisco Systems

Google

Nintendo

IBM

Jane Street

Adobe

Morgan Stanley

NETFLIX

RSA Security

DE Shaw & Co

ORACLE

PANDORA

Akamai

YAHOO!

amazon.com

Microsoft

PIXAR Animation Studios

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Algorithm Design

- *analysis of algorithms*
- *greedy*
- *network flow*
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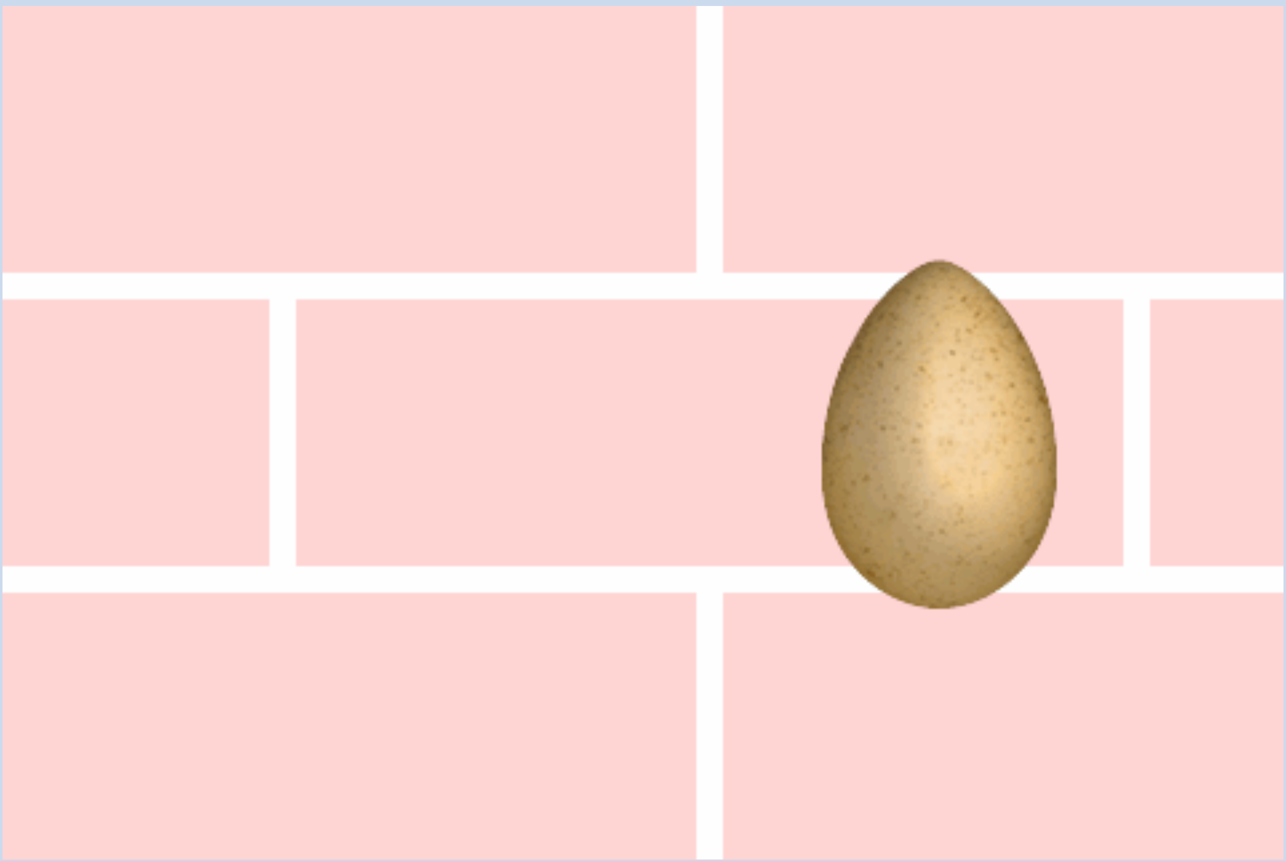
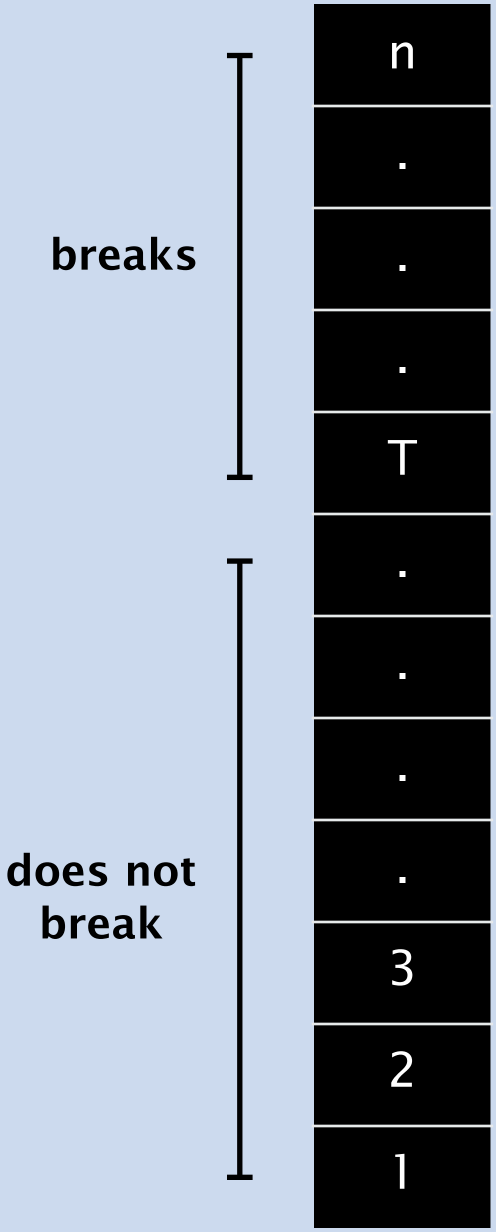
- *divide-and-conquer*
- *randomized algorithms*

Egg drop

Goal. Find T using fewest number of tosses.

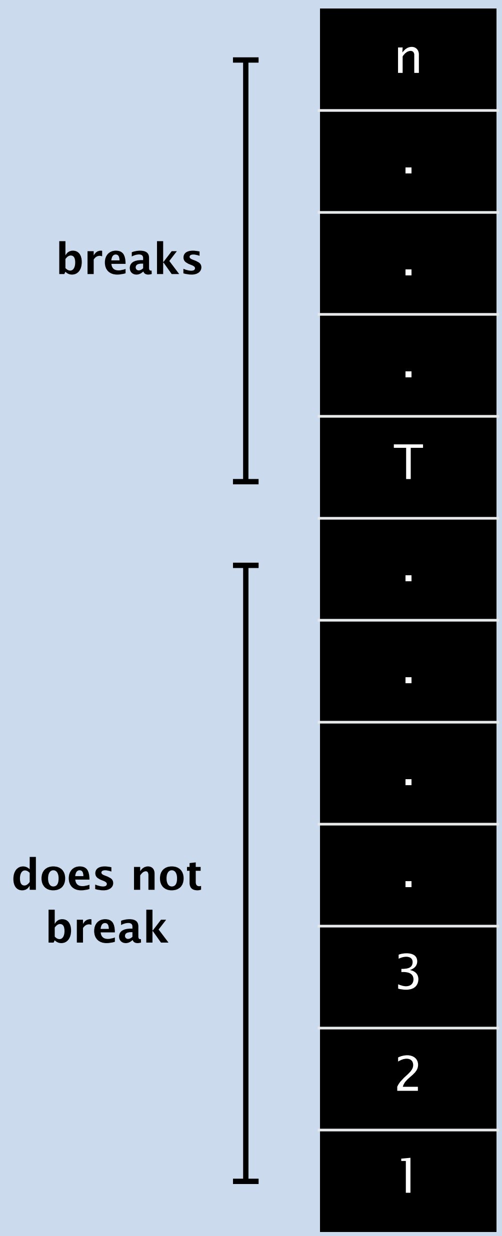




Egg drop

Goal. Find T using fewest number of tosses.

Variant 0. 1 egg.
Variant 1. ∞ eggs.
Variant 2. 2 eggs.



Egg drop

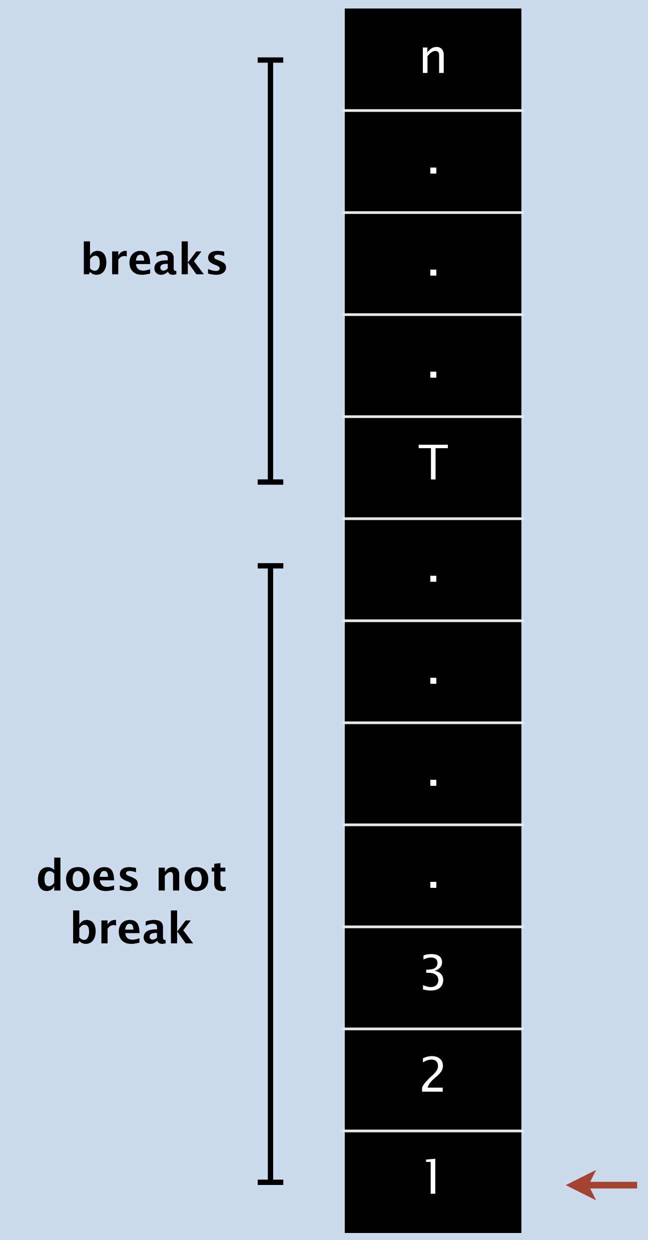
Goal. Find T using fewest number of tosses.

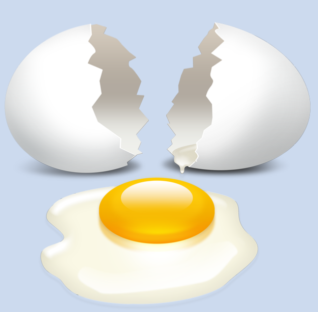
Variant 0. 1 egg.

Solution. Use sequential search: drop on floors $1, 2, 3, \ldots, T$ until egg breaks.

Analysis. 1 egg and T tosses.

running time depends upon
a parameter that you don't no a priori





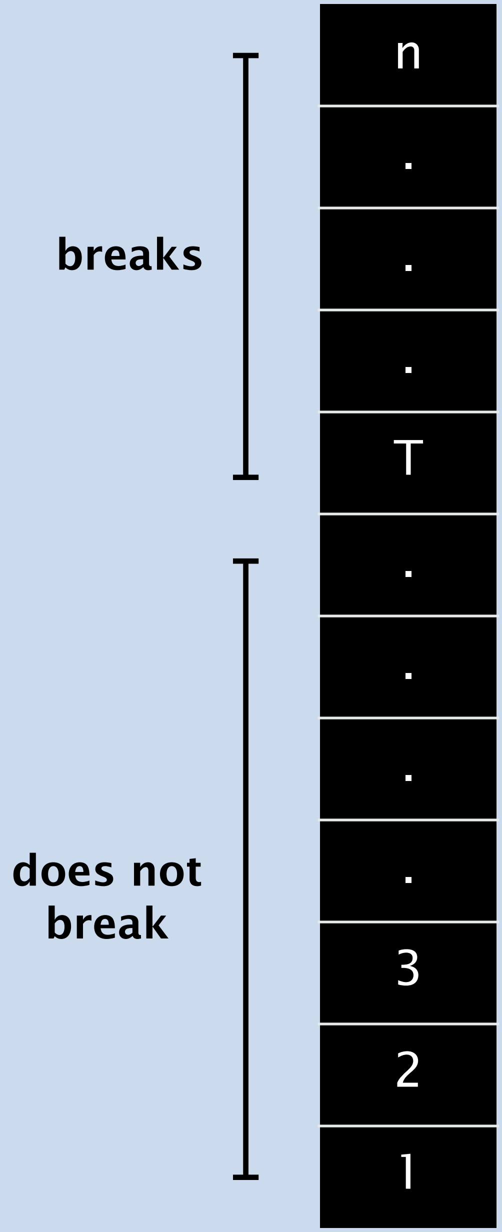
Egg drop

Goal. Find T using fewest number of tosses.

Variant 1. ∞ eggs.

Solution. Binary search for T.

- Initialize $[\text{lo}, \ hi] = [0, \ n]$.
- Repeat until length of interval is 1:
  - drop on floor $mid = (lo + hi) / 2$.
  - if it breaks, update $hi = mid$.
  - if it doesn't break, update $lo = mid$.



Analysis. $\sim \log_2 n$ eggs, $\sim \log_2 n$ tosses.

Suppose T is much smaller than n.
Can you guarantee $\Theta(\log T)$ tosses?

Egg drop

Goal. Find T using fewest number of tosses.

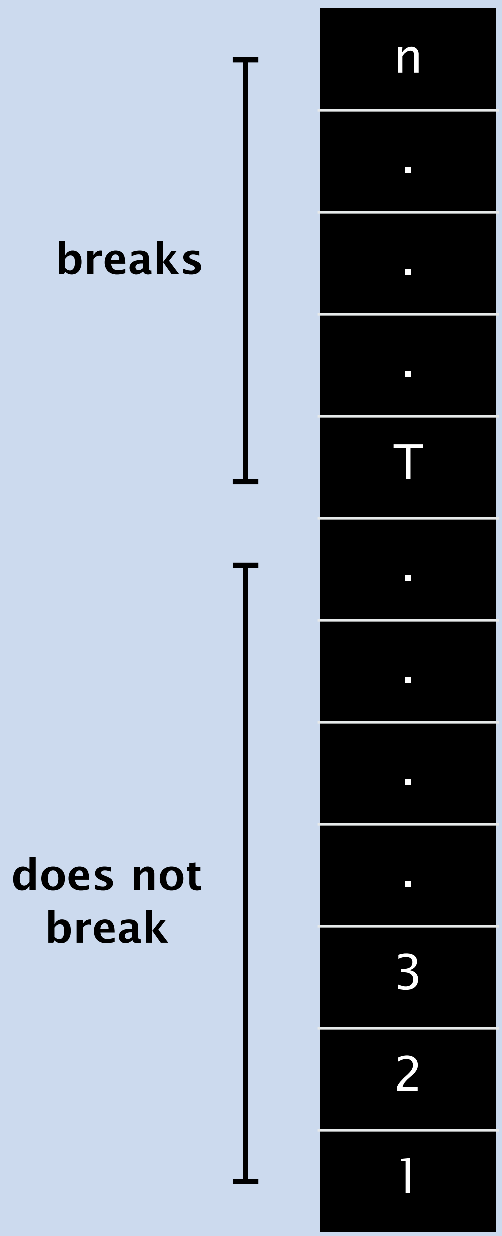
Variant 1′. ∞ eggs and $\Theta(\log T)$ tosses.

Solution. Use repeated doubling; then binary search.

- Drop on floors $1, 2, 4, 8, 16, \ldots, x$ to find a floor x such that $T \leq x < 2T$.
- Binary search in interval $[\frac{1}{2}x, x]$.

Analysis. $\sim \log_2 T$ eggs, $\sim 2 \log_2 T$ tosses.

- Repeated doubling: 1 egg and $1 + \log_2 x$ tosses.
- Binary search: $\sim \log_2 x$ eggs and $\sim \log_2 x$ tosses.
- Recall: $T \leq x < 2T$.



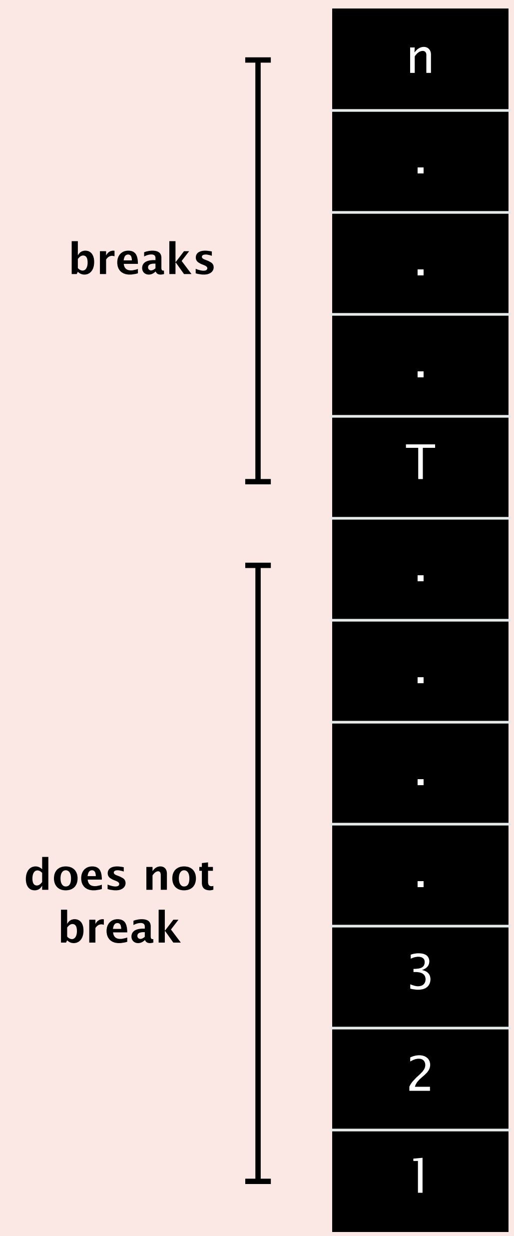
Algorithm design: quiz 1

Goal. Find T using fewest number of tosses.

Variant 2. 2 eggs.

In worst case, how many tosses needed as a function of n?

- **A.** $\Theta(1)$
- **B.** $\Theta(\log n)$
- **C.** $\Theta(\sqrt{n}\,)$
- **D.** $\Theta(n)$



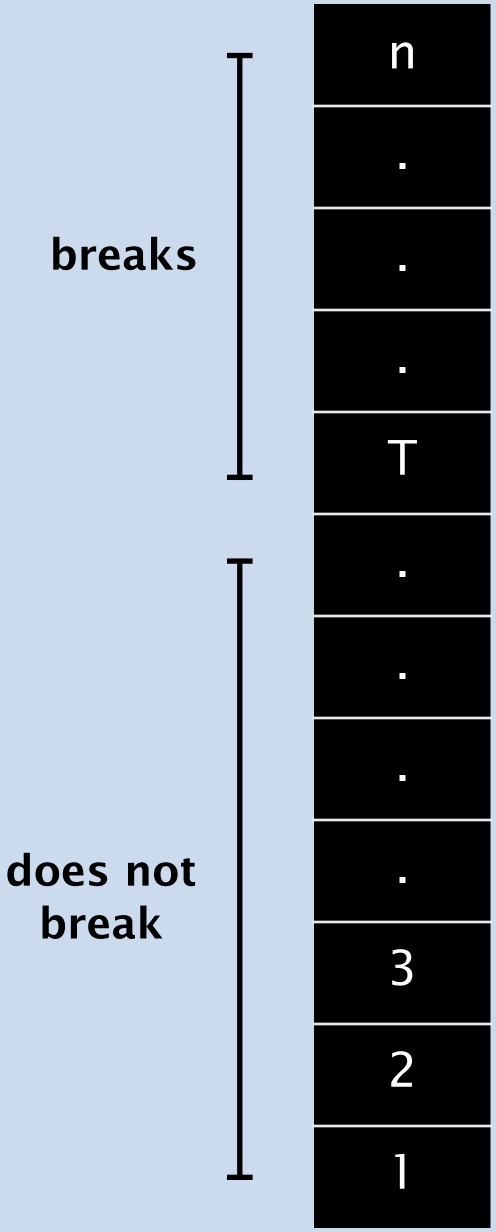
Egg drop (asymmetric search)

Goal. Find T using fewest number of tosses.

Variant 2. 2 eggs.

Solution. Use gridding; then sequential search.

- Toss at floors $\sqrt{n},\ 2\sqrt{n},\ 3\sqrt{n},\ \ldots$
  until first egg breaks, say at floor $c\sqrt{n}$.
- Sequential search in interval $\left[\, c\sqrt{n} - \sqrt{n},\ c\sqrt{n}\,\right]$.



Analysis. At most $2\sqrt{n}$ tosses.

- First egg: ≤ $\sqrt{n}$ tosses.
- Second egg: ≤ $\sqrt{n}$ tosses.

Signing bonus 1. Use 2 eggs and at most $\sqrt{2n}$ tosses.

Signing bonus 2. Use 3 eggs and at most $3\,n^{1/3}$ tosses.

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ALGORITHM DESIGN

- *analysis of algorithms*
- *greedy*
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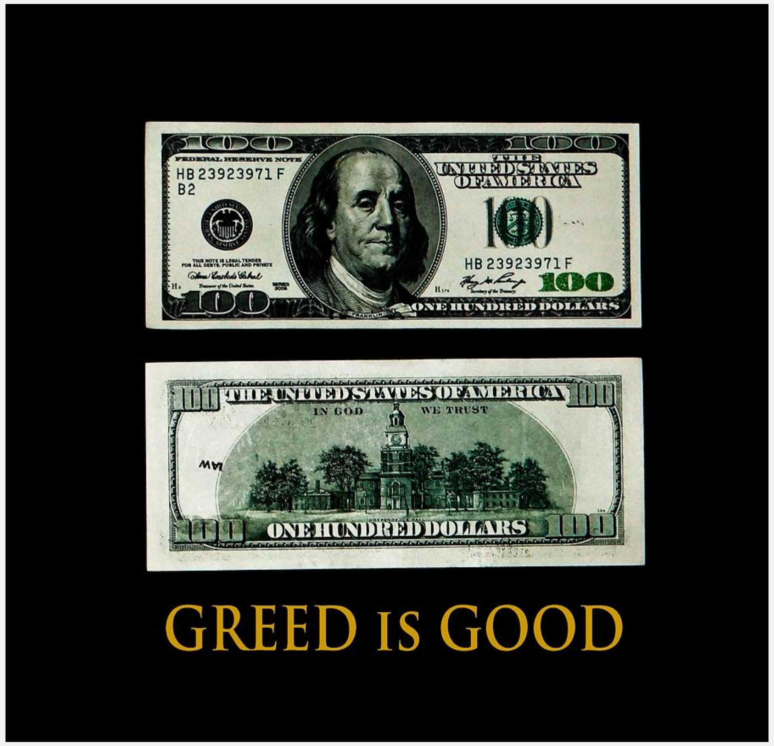
- *dynamic programming*
- *divide-and-conquer*
- *randomized algorithms*

Greedy algorithms

Make locally optimal choices at each step.

Familiar examples.

- Huffman coding.
- Prim's algorithm.
- Kruskal's algorithm.
- Dijkstra's algorithm.



More classic examples.

- U.S. coin changing.
- Activity scheduling.
- Gale–Shapley stable marriage.
- ...

Caveat. Greedy algorithm rarely leads to globally optimal solution.
(but is often used anyway, especially for intractable problems)

Document search

Given a document that is a sequence of n words, and a query that is a sequence of m words, find the smallest range in the document that includes the m query words (in the same order).

Ex. Query = "textbook programming computer"

```
This book is intended to survey the most important computer
algorithms in use today, and to teach fundamental techniques
to the growing number of people in need of knowing them. It
is intended for use as a textbook for a second course in
computer science, after students have acquired basic
programming skills and familiarity with computer systems.
The book also may be useful for self-study or as a reference
for people engaged in the development of computer systems or
applications programs, since it contains implementations of
useful algorithms and detailed information on performance
characteristics and clients.
```

19 words

Document search

Given a document that is a sequence of n words, and a query that is a sequence of m words, find the smallest range in the document that includes the m query words (in the same order).

Solution.

- For each query word, maintain a queue of positions where it occurs.

how?

textbook: 130 200 250 345

programming: 100 180 184 300 315

computer: 120 150 190 290 320 400

Document search

Given a document that is a sequence of n words, and a query that is a sequence of m words, find the smallest range in the document that includes the m query words (in the same order).

Solution.

- For each query word, maintain a queue of positions where it occurs.
- Maintain smallest range containing m query words, with first word at i.

range: 130 to ???

textbook: 130 200 250 345

programming: 100 180 184 300 315

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range: 200 to 320

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range: 250 to 320

textbook: 130 200 250 345

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computer: 120 150 190 290 320 400

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programming: 100 180 184 300 315

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Algorithm design: quiz 2

What is running time as a function of the number of words n in the input and the number of words m in the query?

Assume m ≤ n and that each word is at most, say, 20 characters.

A. $\Theta(\log n)$

B. $\Theta(n)$

C. $\Theta(n \log n)$

D. $\Theta(n^2)$

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Algorithm Design

- analysis of algorithms
- greedy
- *network flow*
- dynamic programming
- divide-and-conquer
- randomized algorithms

Network flow

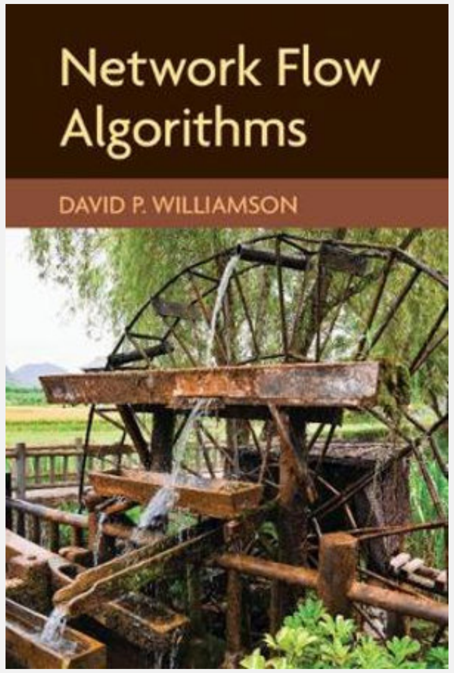
Classic problems on graphs and digraphs.

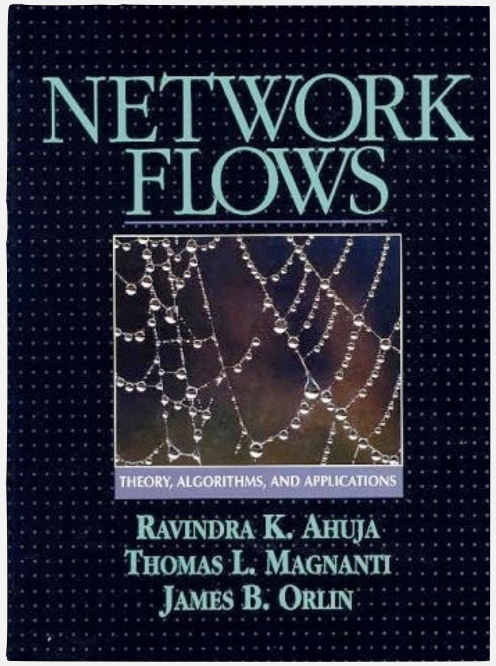
Familiar examples.

- Shortest paths.
- Bipartite matching.
- Maxflow and mincut.
- Minimum spanning tree.

Other classic examples.

- Minimum-cost arborescence.
- Non-bipartite matching.
- Assignment problem.
- Minimum-cost flow.
- ...





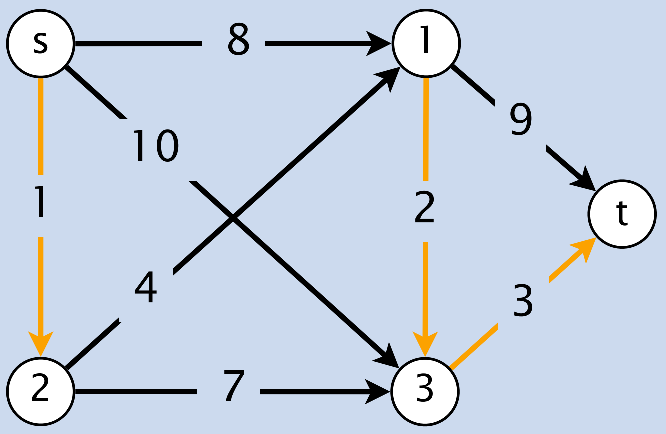
"reduction"

Applications. Many many problems can be modeled using network flow.

Shortest path with orange and black edges

Goal. Given a digraph, where each edge has a positive weight and is orange or black, find shortest path from s to t that uses at most k orange edges.

G



k = 0: s→1→t (17)

k = 1: s→3→t (13)

k = 2: s→2→3→t (11)

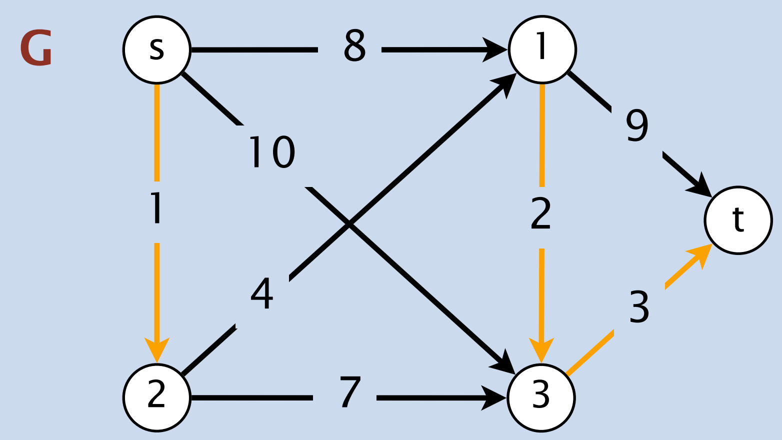
k = 3: s→2→1→3→t (10)

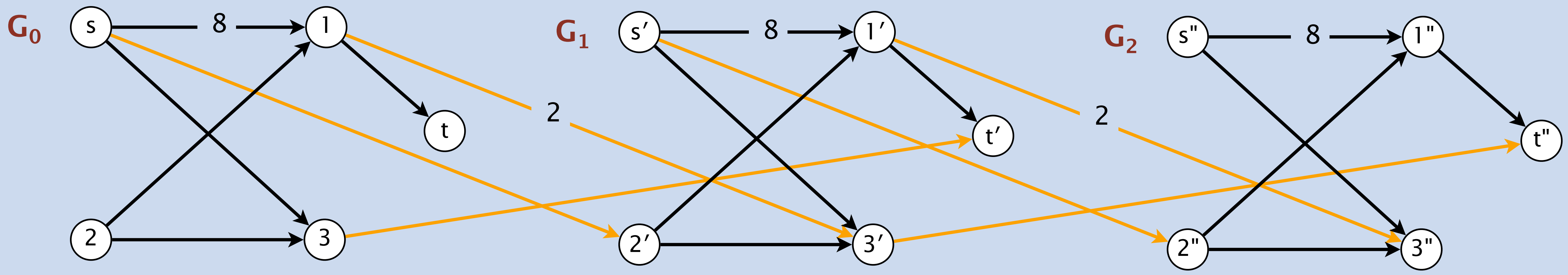
Shortest path with orange and black edges

Goal. Given a digraph, where each edge has a positive weight and is orange or black, find shortest path from s to t that uses at most k orange edges.

Solution. Create $k+1$ copies of the digraph $G_0, G_1, \ldots, G_k$. For each edge $v \to w$

- Black: add edge from vertex v in graph G_i to vertex w in G_i.
- Orange: add edge from vertex v in graph G_i to vertex w in G_{i+1}.





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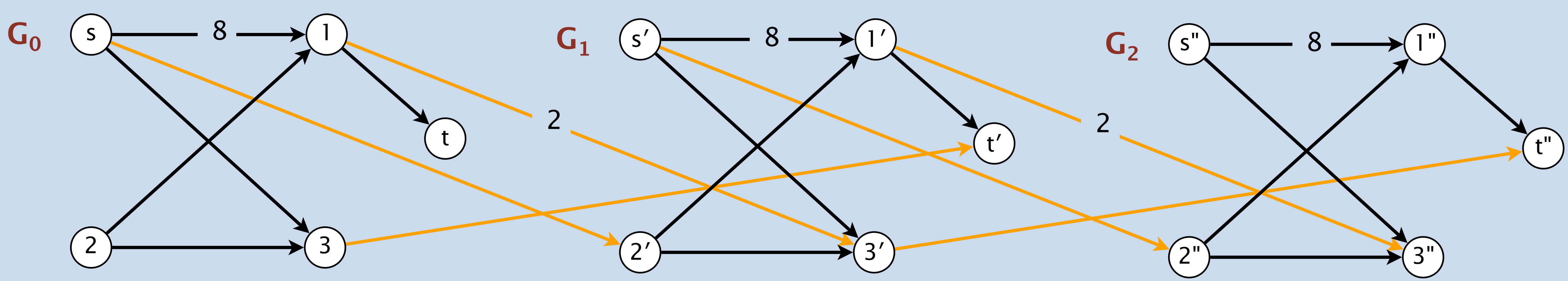
- Black: add edge from vertex v in graph G_i to vertex w in G_i.
- Orange: add edge from vertex v in graph G_i to vertex w in G_{i+1}.
- Find shortest path from s to every copy of t (and choose best).



Algorithm design: quiz 3

What is worst-case running time of algorithm as a function of k, number of vertices V, and number of edges E? Assume E ≥ V.

A. $\Theta(E \log V)$

B. $\Theta(k\, E)$

C. $\Theta(k\, E \log V)$

D. $\Theta(k^2\, E \log V)$

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Dynamic programming

- Break up problem into a series of overlapping subproblems.
- Build up solutions to larger and larger subproblems.
  (caching solutions to subproblems in a table for later reuse)

THE THEORY OF DYNAMIC PROGRAMMING

RICHARD BELLMAN

Familiar examples.

- Shortest paths in DAGs.
- Seam carving.
- Bellman–Ford.

More classic examples.

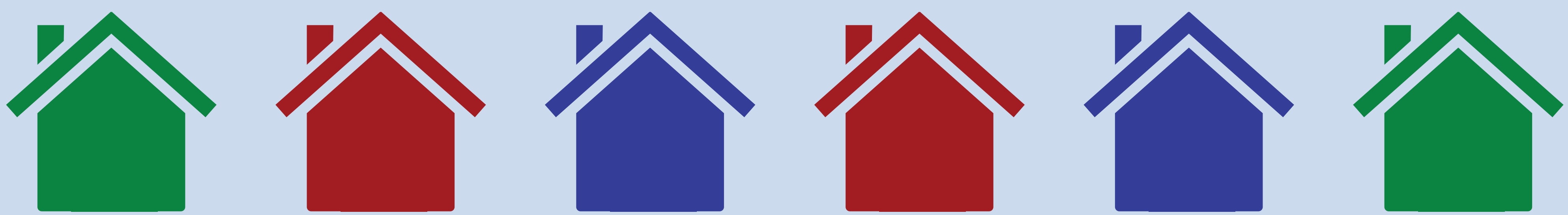
- Unix diff.
- Viterbi algorithm for hidden Markov models.
- Smith–Waterman for DNA sequence alignment.
- CKY algorithm for parsing context-free grammars.

  ...

House coloring problem

Goal. Paint a row of n houses red, green, or blue so that

- No two adjacent houses have the same color.
- Minimize total cost, where $cost(i, color)$ is cost to paint i given color.



	A	B	C	D	E	F
	7	6	7	8	9	20
	3	8	9	22	12	8
	16	10	4	2	5	7

cost to paint house i the given color
(3 + 6 + 4 + 8 + 5 + 8 = 34)

House coloring problem

Goal. Paint a row of n houses red, green, or blue so that

- No two adjacent houses have the same color.
- Minimize total cost, where $cost(i, color)$ is cost to paint i given color.

Subproblems.

- $R[i]$ = min cost to paint houses $1, \ldots, i$ with i red.
- $G[i]$ = min cost to paint houses $1, \ldots, i$ with i green.
- $B[i]$ = min cost to paint houses $1, \ldots, i$ with i blue.
- Optimal cost = $\min \{ R[n], G[n], B[n] \}$.

Dynamic programming recurrence.

- $R[i+1] = cost(i+1, red) + \min \{ G[i], B[i] \}$
- $G[i+1] = cost(i+1, green) + \min \{ B[i], R[i] \}$
- $B[i+1] = cost(i+1, blue) + \min \{ R[i], G[i] \}$

overlapping subproblems

Algorithm design: quiz 4

What is running time of algorithm as a function of n?

- **A.** $\Theta(n)$
- **B.** $\Theta(n \log n)$
- **C.** $\Theta(n \log^2 n)$
- **D.** $\Theta(n^2)$

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Algorithm Design

- *analysis of algorithms*
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- *network flow*
- *dynamic programming*
- *divide-and-conquer*
- *randomized algorithms*

Divide and conquer

- Break up problem into two or more independent subproblems.
- Solve each subproblem recursively.
- Combine solutions to subproblems to form solution to original problem.

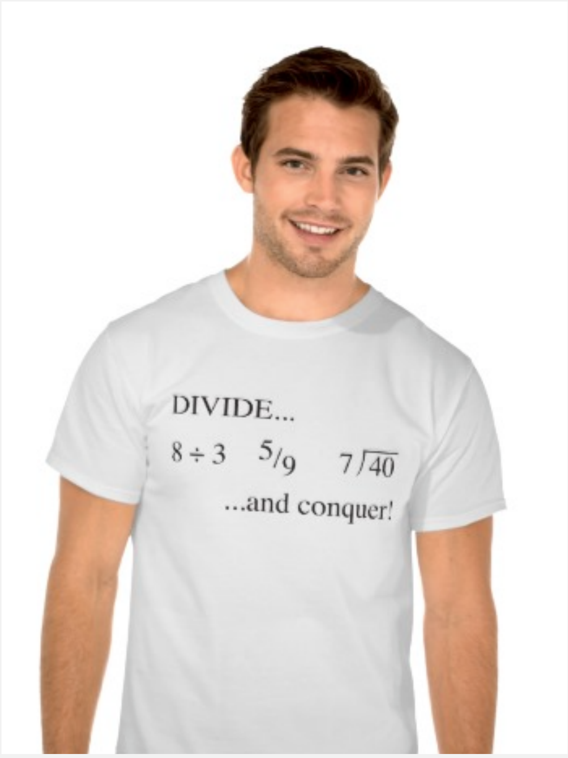
Familiar examples.

- Mergesort.
- Quicksort.

More classic examples.

- Closest pair.
- Convolution and FFT.
- Matrix multiplication.
- Integer multiplication.

...



needs to take COS 226?

Prototypical usage. Turn brute-force $\Theta(n^2)$ algorithm into $\Theta(n \log n)$ one.

Personalized recommendations

Music site tries to match your song preferences with others.

- Your ranking of songs: $0, 1, \ldots, n-1$.
- My ranking of songs: $a_0, a_1, \ldots, a_{n-1}$.
- Music site consults database to find people with similar tastes.

Kendall-tau distance. Number of inversions between two rankings.

Inversion. Songs i and j are inverted if $i < j$, but $a_i > a_j$.

	A	B	C	D	E	F	G	H
you	0	1	2	3	4	5	6	7
me	0	2	3	1	4	5	7	6

3 inversions: 2–1, 3–1, 7–6

Counting inversions

Problem. Given a permutation of length n, count the number of inversions.

0	2	3	1	4	5	7	6

3 inversions: 2–1, 3–1, 7–6

Brute-force n^2 algorithm. For each $i < j$, check if $a_i > a_j$.

A bit better. Run insertion sort; return number of exchanges.

Goal. $n \log n$ time (or better).

Counting inversions: divide-and-conquer

input	0	4	3	7	9	1	5	8	2	6	
count inversions in left subarray	0	4	3	7	9	1	5	8	2	6	**1**
count inversions in right subarray	0	4	3	7	9	1	5	8	2	6	**3**
count inversions with one element in each subarray	0	4	3	7	9	1	5	8	2	6	**13**
output	0	4	3	7	9	1	5	8	2	6	**1 + 3 + 13 = 17**

Left subarray inversions: 4–3

Right subarray inversions: 5–2 8–2 8–6

Inversions with one element in each subarray: 3–1 3–2 4–1 4–2 7–1 7–2 7–5 7–6 9–1 9–2 9–5 9–6 9–8

this step seems to require n^2 time

Counting inversions: divide-and-conquer

input	0	4	3	7	9	1	5	8	2	6	
count inversions in left subarray and sort	0	3	4	7	9	1	5	8	2	6	**1**
count inversions in right subarray and sort	0	3	4	7	9	1	2	5	6	8	**3**
count inversions with one element in each sorted subarray	0	3	4	7	9	1	2	5	6	8	**13**
and merge into sorted whole	0	1	2	3	4	5	6	7	8	9	**1 + 3 + 13 = 17**

Algorithm design: quiz 5

What is running time of algorithm as a function of n?

A. $\Theta(n)$

B. $\Theta(n \log n)$

C. $\Theta(n \log^2 n)$

D. $\Theta(n^2)$

https://algs4.cs.princeton.edu

Algorithm Design

- analysis of algorithms
- greedy
- network flow
- dynamic programming
- divide-and-conquer
- *randomized algorithms*

Randomized algorithms

Algorithm whose performance (or output) depends on the results of random coin flips.

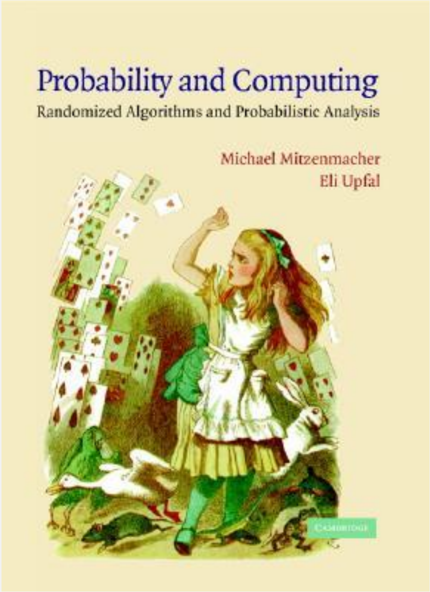
Familiar examples.

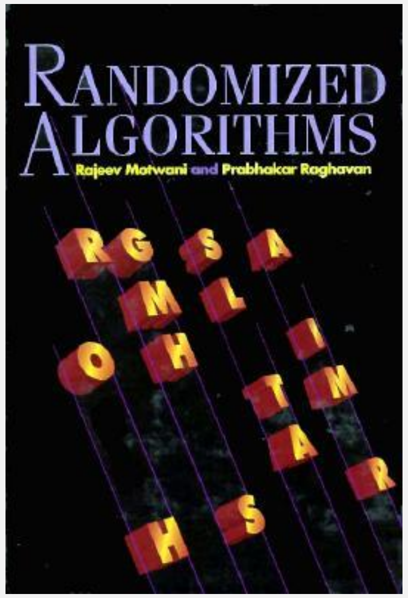
- Quicksort.
- Quickselect.

More classic examples.

- Rabin–Karp substring search.
- Miller–Rabin primality testing.
- Polynomial identity testing.
- Volume of convex body.
- Universal hashing.
- Global min cut.

  ...





Nuts and bolts

Problem. A disorganized carpenter has a mixed pile of n nuts and n bolts.

- The goal is to find the corresponding pairs of nuts and bolts.
- Each nut fits exactly one bolt and each bolt fits exactly one nut.
- By fitting a nut and a bolt together, the carpenter can see which one is bigger (but cannot directly compare either two nuts or two bolts).

Brute-force algorithm. Compare each bolt to each nut: $\Theta(n^2)$ compares.

Challenge. Design an algorithm that makes $O(n \log n)$ compares.

Nuts and bolts

Shuffle. Shuffle the nuts and bolts.

bolts	5	3	6	0	9	1	4	8	2	7
nuts	7′	2′	8′	1′	5′	9′	4′	0′	6′	3′

Partition.

- Pick leftmost bolt i and compare against all nuts; divide nuts smaller than i from those that are larger than i.
- Let i' be the nut that matches bolt i. Compare i' against all bolts; divide bolts smaller than i' from those that are larger than i'.

	smaller bolts						larger bolts			
bolts	3	0	1	4	2	5	6	9	8	7
nuts	2′	1′	4′	0′	3′	5′	7′	8′	9′	6′
	smaller nuts						larger nuts			

Divide-and-conquer. Recursively solve two subproblems.

Algorithm design: quiz 6

What is the expected running time of algorithm as a function of n?

- **A.** $\Theta(n)$
- **B.** $\Theta(n \log n)$
- **C.** $\Theta(n \log^2 n)$
- **D.** $\Theta(n^2)$

Nuts and bolts

Hiring bonus. Algorithm that takes $O(n \log n)$ time in the worst case.

Chapter 27

Matching Nuts and Bolts in $O(n \log n)$ Time

(Extended Abstract)

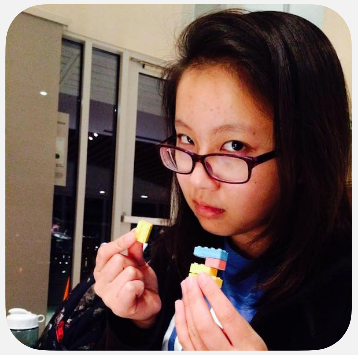
János Komlós [1,4] Yuan Ma [2] Endre Szemerédi [3,4]

Abstract

Given a set of n nuts of distinct widths and a set of n bolts such that each nut corresponds to a unique bolt of the same width, how should we match every nut with its corresponding bolt by comparing nuts with bolts (no comparison is allowed between two nuts or between two bolts)? The problem can be naturally viewed as a variant of the classic sorting problem as follows. Given two lists of n numbers each such that one list is a permutation of the other, how should we sort the lists by comparisons only between numbers in different lists? We give an $O(n \log n)$-time deterministic algorithm for the problem. This is optimal up to a constant factor and answers an open question posed by Alon, Blum, Fiat, Kannan, Naor, and Ostrovsky [3]. Moreover, when copies of nuts and bolts are allowed, our algorithm runs in optimal $O(\log n)$ time on n processors in Valiant's parallel comparison tree model. Our algorithm is based on the AKS sorting algorithm with substantial modifications.

Credits

Faculty lead preceptors and graduate student AIs.

Undergraduate graders and lab TAs. Apply to be one next semester!

Ed tech. Several developed here at Princeton!

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A farewell video (from P04, Fall 2018)

A final thought

“ Algorithms and data structures are love.

Algorithms and data structures are life. ”

— anonymous COS 226 student