COS 226 **Algorithms and Data Structures** **Spring 2020**

Midterm Solutions

1. **Initialization.** Don't forget to do this.

2. **Memory.**

   (a) `isEmpty()`, `addFront()`, `removeFront()`, `addBack()`

   To implement `size()`, `removeBack()`, *and* `sample()`, *you would have to traverse the singly linked list, from* `first` *to* `last`*. The challenge with implementing* `removeBack()` *efficiently is updating the* `last` *pointer.*

   (b) $\sim 32n$

   Each `Node` object uses 32 bytes of memory and there are n nodes.

   - 16 bytes of object overhead
   - 8 bytes for `Node` reference
   - 8 bytes for `double` item

3. **Five sorting algorithms.**

   (3.1) *insertion sort after 16 iterations*

   (3.2) *heapsort after heap construction phase and putting 6 keys into place*

   (3.3) *selection sort after 12 iterations*

   (3.4) *mergesort just before the last call to* `merge()`

   (3.5) *quicksort after first partitioning step*

4. **Analysis of algorithms.**

   (4.1) $\sim 2n^2$

   Selection sort always makes $\sim \frac{1}{2}m^2$ *compares to sort an array of length* m*.Here* $m = 2n$*.*

   (4.2) $\sim n^2$

   In each of the first n iterations (except the first), there is one compare (and no exchange). In each of the last n iterations (except the last), there are n compares and n exchanges.

   (4.3) $\sim n \log_2 n$

   Mergesort requires $\frac{1}{2} n \log_2 n$ *compares to sort a sorted array of length* n*. Thus, mergesort makes* $\frac{1}{2} n \log_2 n$ *compares to sort the left subarray (of length* n*) and* $\frac{1}{2} n \log_2 n$ *compares to sort the right subarray (of length* n*). Finally, it makes* n *compares to merge the two subarrays together.*

   (4.4) Timsort

   Timsort is optimized for situations when an array has a small number of non-increasing (or strictly decreasing) runs. In this case, there are only two runs (the first n *elements containing the value* n*, and the last* n *elements containing the integers 1 to* n*). So, Timsort will run in linear time on staircase arrays.*

(4.5) $O(n^3), O(n^4), \Theta(n^3)$

Big O and big Theta notations discard both lower-order terms and the leading coefficient. The main difference is that big O notation includes functions that grow more slowly. So, $O(n^4)$ *includes not only functions like* $2n^4$ *and* $\frac{1}{2}n^4$*, but also* $3n^3$ *and* $5n^2$*.*

5. **Level-order traversal.**

B F H J L M A

```
public Iterable<Key> levelOrder() {
     Queue<Key> keys = new Queue<Key>();
     Queue<Node> queue = new Queue<Node>();
     queue.enqueue(root);
     while (!queue.isEmpty()) {
         Node x = queue.dequeue();
         if (x != null) {
             keys.enqueue(x.key);
             queue.enqueue(x.left);
             queue.enqueue(x.right);
         }
     }
     return keys;
 }
```

6. **Hash tables.**

(6.1) B D F

(6.2) B E

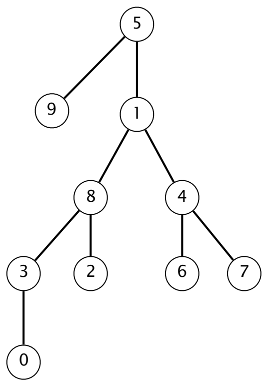
(6.3) D G

(6.4) C E

7. **Data structures.**

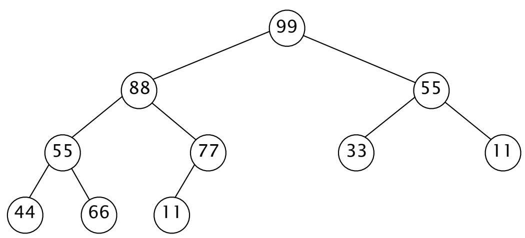
(7.1) could not arise

The height of the tree is 4. However, the height of any weighted quick-union tree on n elements is at most $\log_2 n$. Note that $\log_2 10 < \log_2 16 = 4$, so the height must be strictly less than 4.



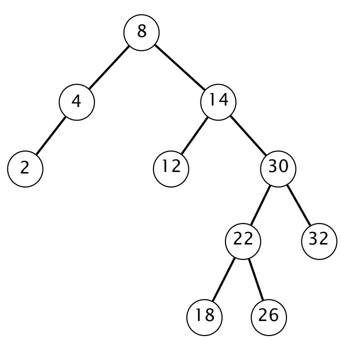
(7.2) could not arise

The corresponding binary tree is not heap-ordered because 55 is greater than 66.



(7.3) could arise

Here is the BST with the given level-order traversal.



(7.4) could not arise

Perfect black balance is not satisfied. The path from the root to the right null link of 8 has only 2 black links (including the null link) but all other paths from the root to null links have 3 black links.

(7.5) could arise

It's a valid kd-tree. It could have arisen by inserting the points in a variety of orders, including level order: $(6,7),(1,4),(8,5),(4,2),(2,8),(0,9),(3,6)$.

8. **Problem identification.**

8.1 Possible

This can be done with mergesort, as discussed in lecture.

8.2 Possible

This can be done with 3-way quicksort. The number of 3-way partitioning steps equals the number of distinct keys. Each partitioning step makes at most n *compares.*

8.3 Impossible

This would violate the sorting lower bound. We could insert the n *keys; then delete-max the* n *keys to get them in sorted order. This would give us a compare-based sorting algorithm that makes* $\Theta(n \log \log n)$ *compares in the worst case.*

8.4 Possible

You could use binary search directly. Or you could compose an algorithm by combining operations that we've seen in the course. For example, if k *is not in the array, then the predecessor is the floor (which we saw how to compute using binary search). If* k *is in the array, then you could search for the first occurrence of* k *and return the previous key (which you did on the Autocomplete assignment using binary search).*

8.5 Impossible

This would violate the sorting lower bound. We could insert the n *keys into a BST; then we could perform an inorder traversal to get them in sorted order. Since performing an inorder traversal doesn't require any key compares, this would give us a compare-based sorting algorithm that makes* $\Theta(n)$ *compares in the worst case.*

8.6 Impossible

There may be $\Theta(n^2)$ *pairs that intersect, so it will take* $\Theta(n^2)$ *time to collect them in a list.*

9. **Design question.**

    9.1 true

    9.2 true

    9.3 The main idea is to use *binary search* to find the adjacent inversion, maintaining a subarray $a[lo..hi]$ for which (lo, hi) is an inversion: $lo < hi$ and $a[lo] > a[hi]$.

    - Initialize $lo \leftarrow p$ and $hi \leftarrow q$
    - Terminate the loop when $hi = lo + 1$, in which case (lo, hi) is an adjacent inversion.
    - Otherwise,
        - Set $mid = (lo + hi)/2$.
        - If $a[mid] > a[hi]$, then update $lo \leftarrow mid$.
          This guarantees $a[lo] > a[hi]$.
        - If $a[mid] \leq a[hi]$, then update $hi \leftarrow mid$.
          This guarantees $a[lo] > a[hi]$ because $a[lo]$ stays the same and $a[hi]$ does not increase.

    Here's the corresponding Java code.

```
int lo = p, hi = q;
while (hi > lo + 1) {
    int mid = lo + (hi - lo) / 2;
    if (a[mid] > a[hi]) lo = mid;
    else hi = mid;
}
```

Here's a symmetric version that compares `a[mid]` *to* `a[lo]`*.*

```
int lo = p, hi = q;
while (hi > lo + 1) {
    int mid = lo + (hi - lo) / 2;
    if (a[lo] > a[mid]) hi = mid;
    else lo = mid;
}
```

Here's another version that does two compares per iteration of the `while` *loop. The second compare is unnecessary because, if the first compare fails, then it must be the case that* `a[mid] >= a[lo] > a[hi]`*.*

```
int lo = p, hi = q;
while (hi > lo + 1) {
    int mid = lo + (hi - lo) / 2;
    if (a[lo] > a[mid]) hi = mid;
    else if (a[mid] > a[hi]) lo = mid;
}
```