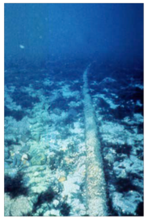
Detecting and Correcting Bit Errors

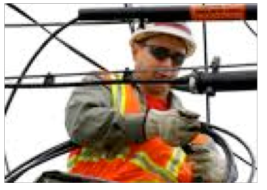
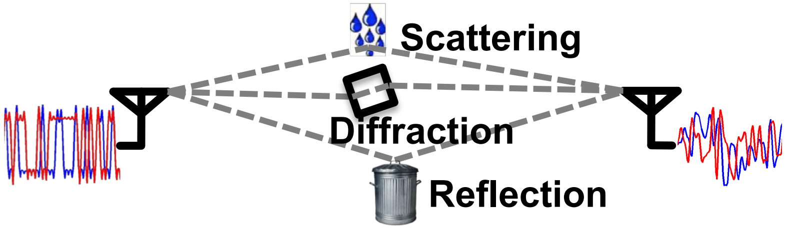
COS 463: Wireless Networks
Lecture 8

Kyle Jamieson

Bit errors on links

- Links in a network go through **hostile environments**
  - Both wired, and wireless:



  - Consequently, **errors will occur on links**
  - **Today:** ***How can we detect and correct these errors?***

- There is **limited capacity** available on any link
  - **Tradeoff** between **link utilization** & amount of **error control**

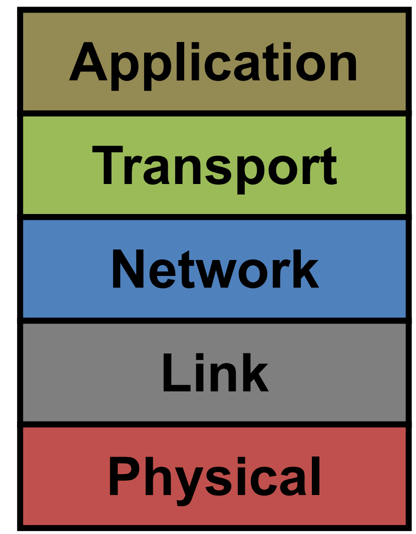
Today

1. **Error control codes**
   - **Where are codes used?**
   - Encoding and decoding **fundamentals**
   - Measuring a code's **error correcting power, overhead**
   - **Practical** error control codes
     - Parity check, Hamming block code

2. Error detection codes

Where is coding used?

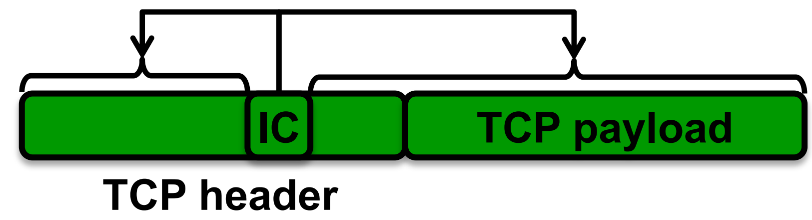
- The techniques we'll discuss today are **pervasive** throughout the internetworking stack
- Based on theory, but **broadly applicable** in practice, in other areas:
  - Hard disk drives
  - Optical media (CD, DVD, *& c.*)
  - Satellite, mobile communications



- In 463, we cover the "tip of the iceberg" of error detection and control codes

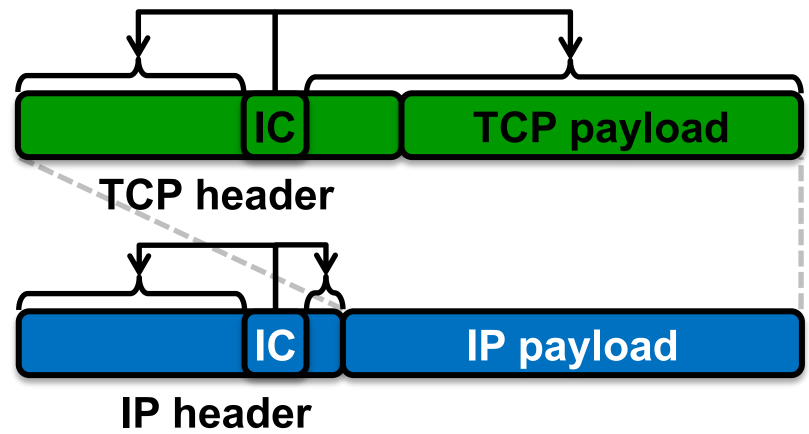
Error control in the Internet stack

- Transport layer
  - **Internet Checksum (IC)** over TCP/UDP header, data



Error control in the Internet stack

- Transport layer
  - Internet Checksum (IC) over TCP/UDP header, data
- Network layer (L3)
  - IC over IP header only



Error control in the Internet stack

- Transport layer
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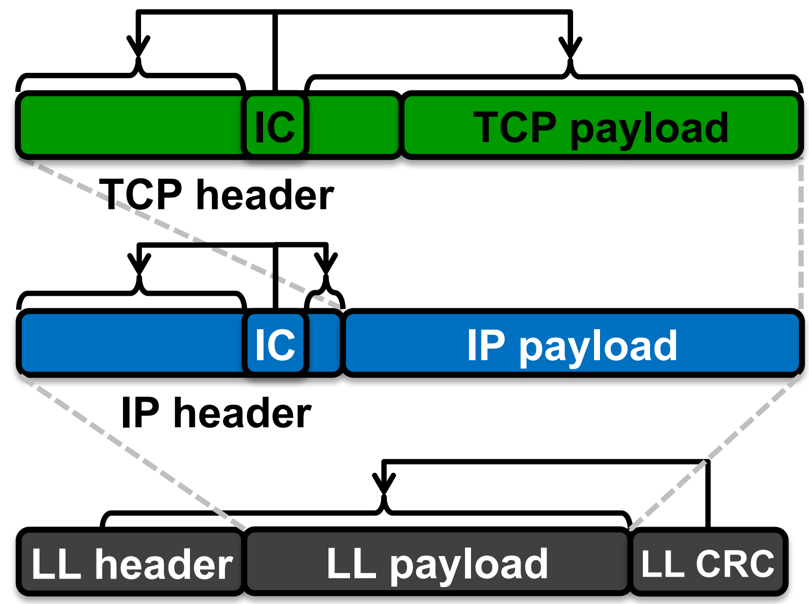
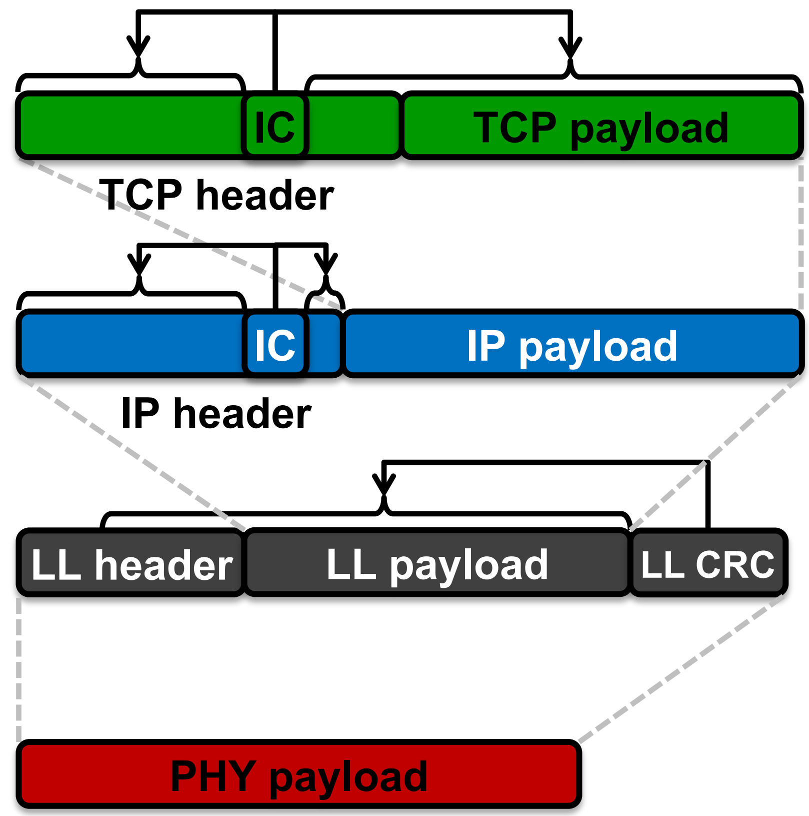
- Link layer (L2)
  - Cyclic Redundancy Check (CRC)



Error control in the Internet stack

- **Transport layer**
  - **Internet Checksum (IC)** over TCP/UDP header, data
- **Network layer (L3)**
  - **IC** over IP header only
- **Link layer (L2)**
  - **Cyclic Redundancy Check (CRC)**
- **Physical layer (PHY)**
  - **Error Control Coding (ECC)**, or
  - **Forward Error Correction (FEC)**

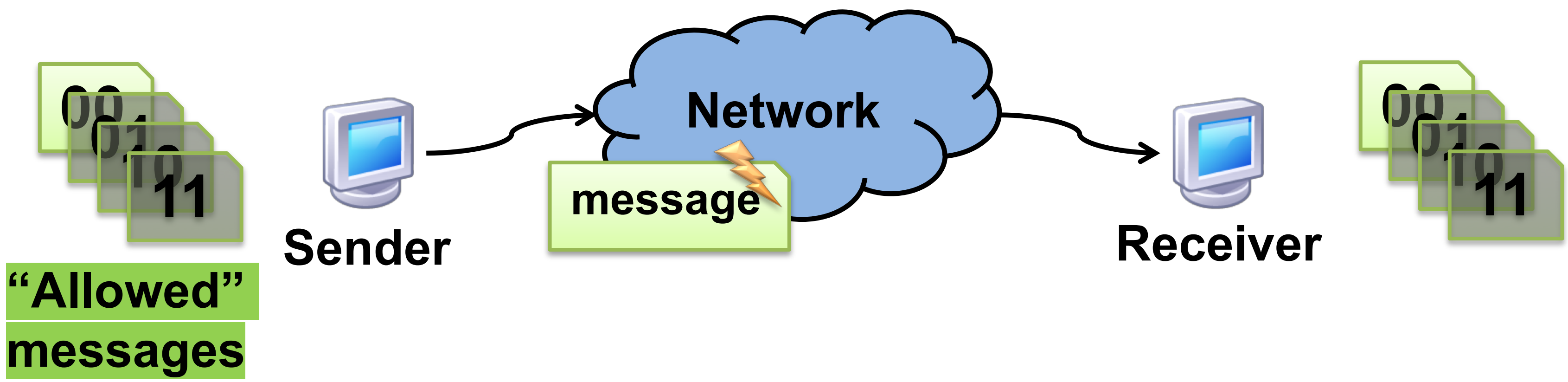


Today

1. **Error control codes**
   - Where are codes used?
   - **Encoding and decoding fundamentals**
   - Measuring a code's **error correcting power, overhead**
   - Practical error control codes
     - Parity check, Hamming block code

2. Error detection codes
   - Cyclic redundancy check (CRC)

Error control: Motivation



- *A priori,* any string of bits is an "allowed" **message**
  - Hence any **changes to the bits** (***bit errors***) the sender transmits produce "**allowed**" messages

- **Therefore without error control, receiver wouldn't know errors happened!**

Error control: Key Ideas

- **Reduce the set of “allowed” messages**
  - **Not every string of bits** is an “allowed” message
  - Receipt of a **disallowed** string of bits means that the **message was garbled** in transit over the network
- We call an allowable message (of *n* bits) a ***codeword***
  - **Not all** *n*-bit strings are codewords!
  - The remaining *n*-bit strings are **“space” between codewords**
- **Plan:** Receiver will **use that space** to both **detect** and **correct** errors in transmitted messages

Encoding and decoding

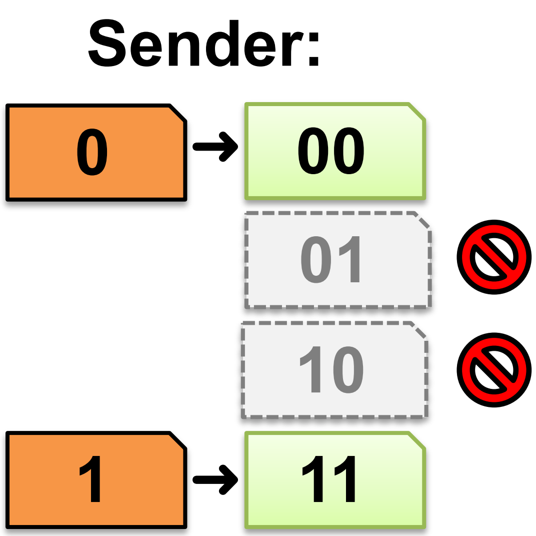
- **Problem: Not every string of bits is "allowed"**
  - But we want to be **able to send any** message!
  - *How can we send a "disallowed" message?*

- **Answer: Codes,** as a sender-receiver **protocol**
  - The sender must ***encode*** its messages ➜ codewords
  - The receiver then ***decodes*** received bits ➜ messages

- The **relationship between messages and codewords** isn't always obvious!

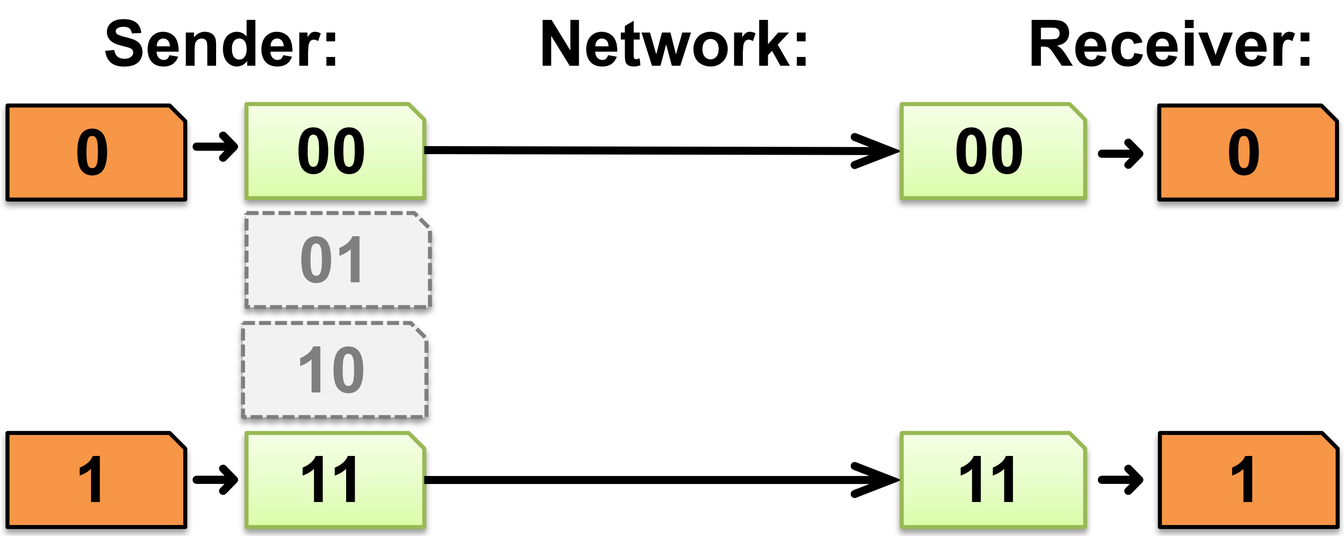
A simple error-detecting code

- Let's start simple: suppose messages are one bit long
- Take the message bit, and **repeat** it once
  - This is called a ***two-repetition code***



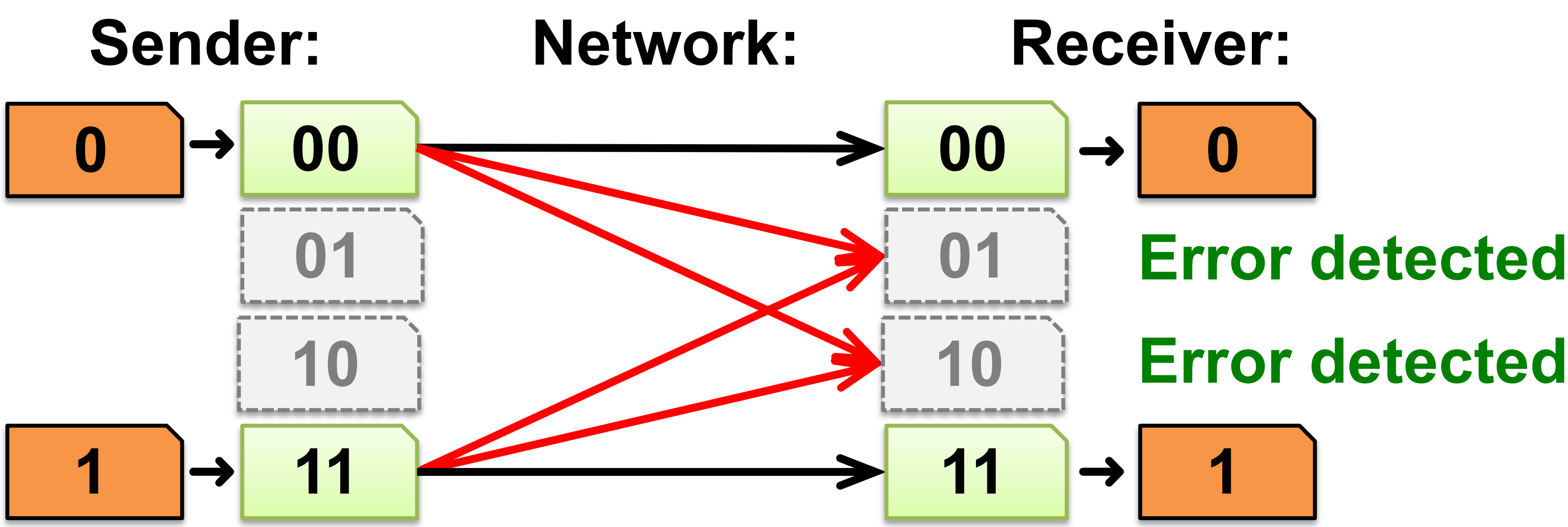
Receiving the two-repetition code

- Suppose the network causes **no bit error**
- Receiver **removes repetition** to ***correctly decode*** the message bits



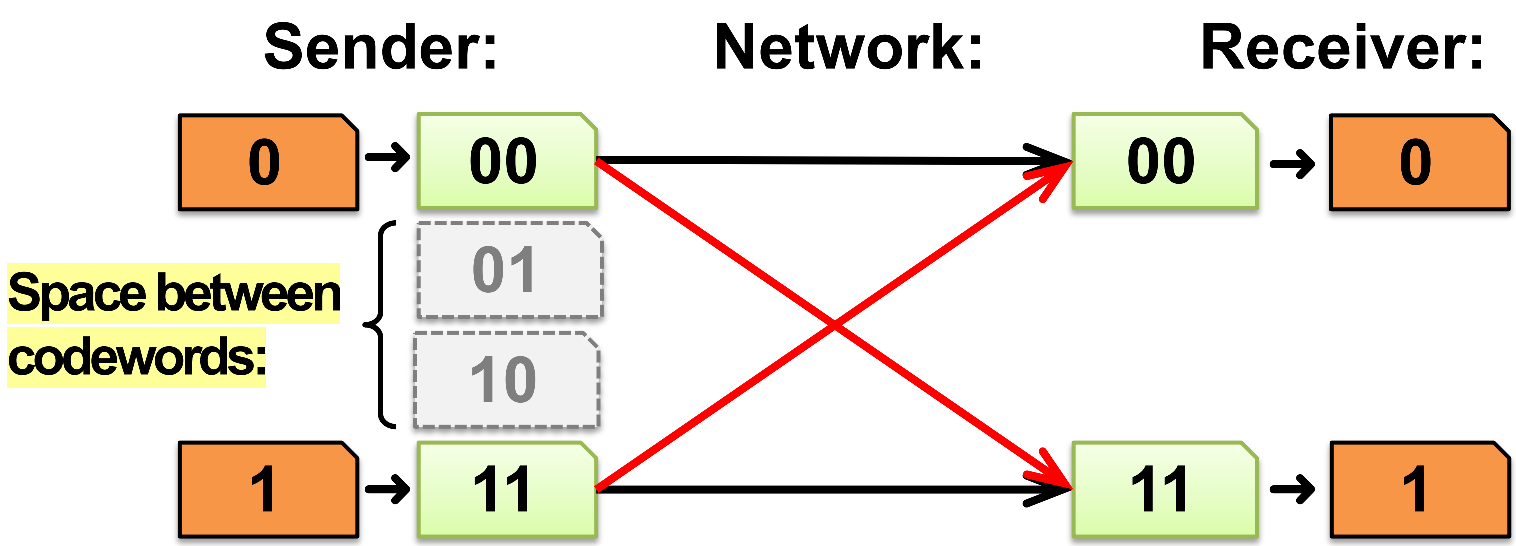
Detecting one bit error

- Suppose the network causes up to **one bit error**
- The receiver **can detect** the error:
  - It received a **non-codeword**
- Can the receiver **correct** the error?
  - **No!** The **other** codeword could have been sent as well



Reception with two bit errors

- Can receiver **detect** presence of **two bit errors?**
  - **No:** It has no way of telling which codeword was sent!
    - Enough bit errors that the sent codeword **"jumped over" the space between** codewords



Hamming distance

- Measures the **number of bit flips** to change **one codeword into another**

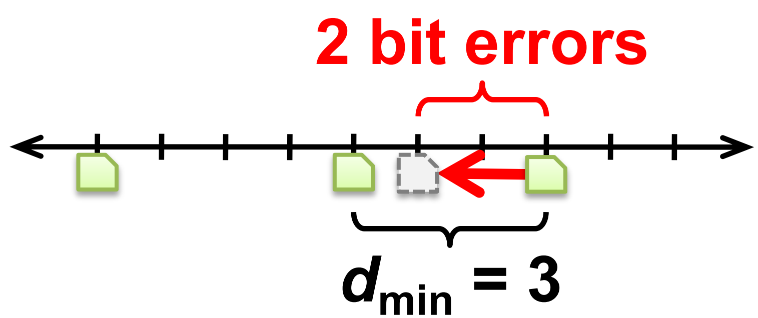
- ***Hamming distance*** between two messages m_1, m_2: The number of bit flips needed to change m_1 into m_2

- **Example:** Two bit flips needed to change codeword 00 to codeword 11, so they are Hamming distance of **two** apart:

How many bit errors can we detect?

- Suppose the **minimum Hamming distance** between **any pair** of codewords is d_{min}

- Then, we **can detect at most $d_{min} - 1$ bit errors**
  - Will land in space between codewords, as we just saw



  - Receiver will flag message as **"Error detected"**

Decoding error detecting codes

- **The receiver decodes** in a **two-step** process:

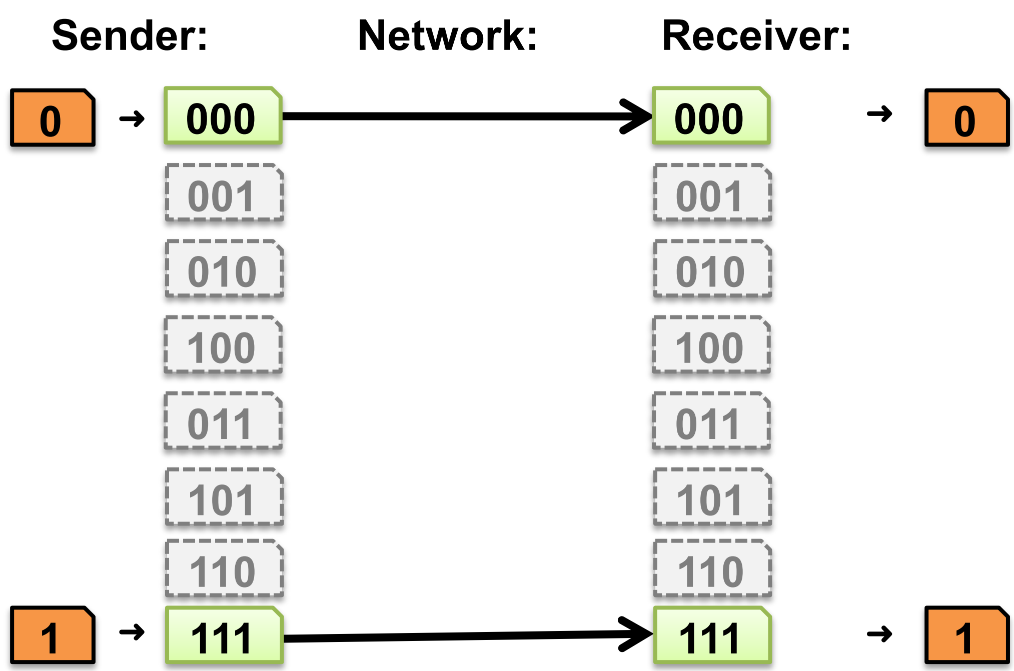
1. Map **received bits → codeword**
   - **Decoding rule:** Consider all codewords
     - **Choose** the one that **exactly matches** the received bits
     - **Return "error detected"** if none match

2. Map **codeword → source bits** and **"error detected"**
   - Use the **reverse map** of the sender

A simple error-correcting code

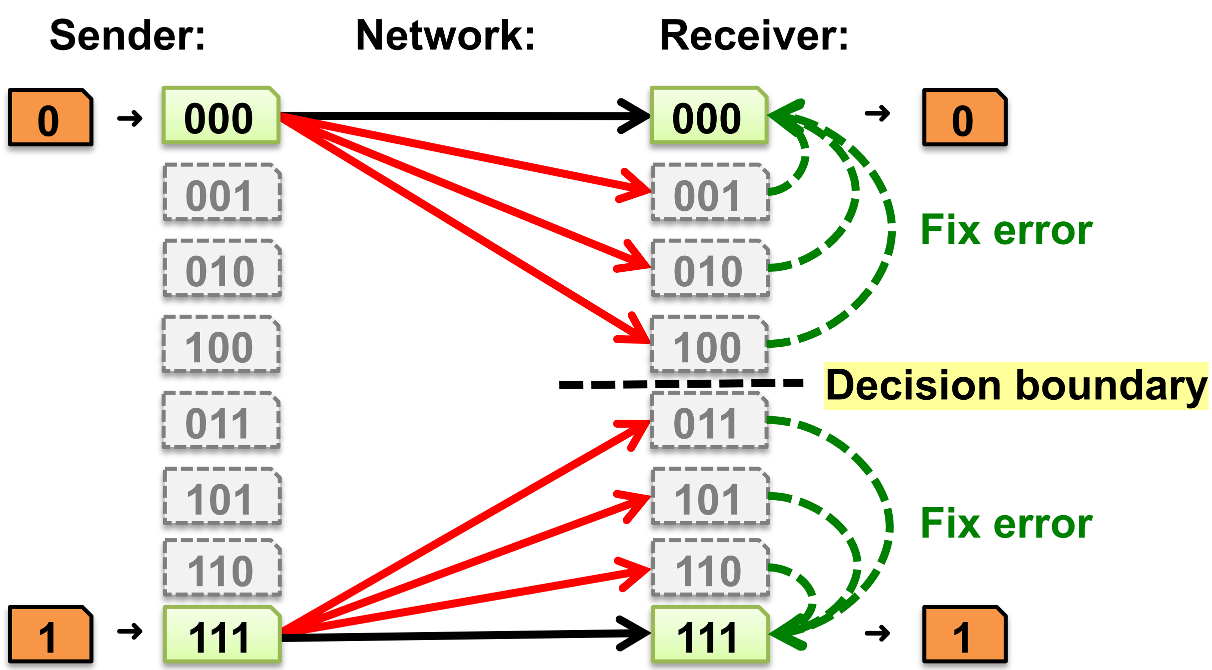
- Let's look at **a** **three-repetition** **code**

- If **no errors,** it works like the two-repetition code:



Correcting one bit error

- Receiver **chooses the closest codeword** (measured by Hamming distance) **to the received bits**
  - A ***decision boundary*** **exists** halfway between codewords



Decoding error correcting codes

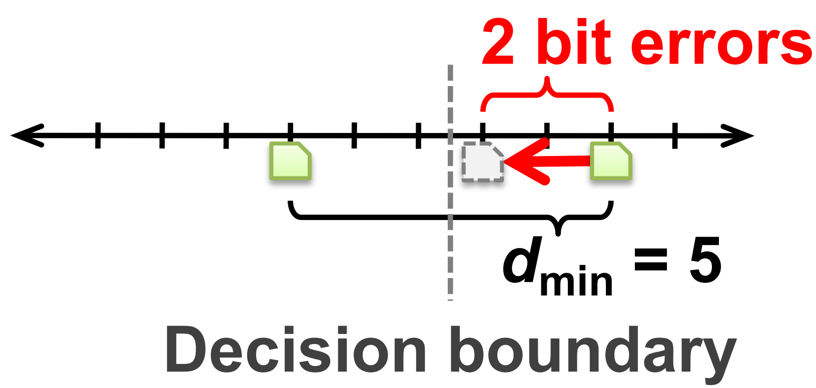
- **The receiver decodes** in a two-step process:

1. Map **received bits → codeword**
   - **Decoding rule:** Consider all codewords
     - **Choose one** with the **minimum Hamming distance** to the received bits

2. Map **codeword → source bits**
   - Use the **reverse map** of the sender

How many bit errors can we correct?

- There is $\geq d_{min}$ **Hamming distance** between any two codewords
- So we can **correct** $\leq \left\lfloor {}^{d_{\min}-1}/_{2} \right\rfloor$ **bit flips:**
  - This many bit flips can't move received bits closer to another codeword, **across the decision boundary:**



Code rate

- Suppose **codewords** of length ***n*, messages** length ***k*** ($k < n$)
- The **code rate $R = k/n$** is a fraction between 0 and 1
- So, we have a **tradeoff:**
  - **High-rate codes** (R approaching one) generally **correct fewer errors,** but **add less overhead**
  - **Low-rate codes** (R close to zero) generally **correct more errors,** but **add more overhead**

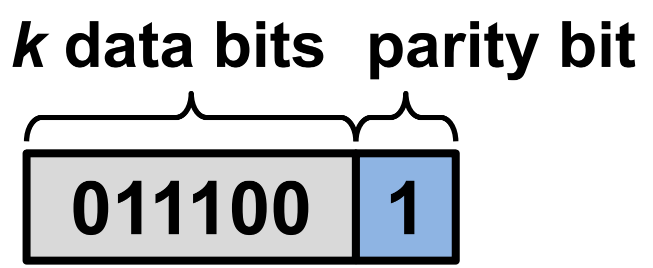
Today

1. **Error control codes**
   - Encoding and decoding fundamentals
   - Measuring a code's error correcting power
   - Measuring a code's overhead
   - **Practical error control codes**
     - **Parity check, Hamming block code**

2. Error detection codes
   - Cyclic redundancy check (CRC)

Parity bit

- Given a message of k data bits $D_1, D_2, \ldots, D_k$, append a ***parity bit P*** to make a codeword of length $n = k + 1$
  - P is the exclusive-or of the data bits:
    - $P = D_1 \oplus D_2 \oplus \cdots \oplus D_k$
  - Pick the parity bit so that **total number of 1's is even**



Checking the parity bit

- **Receiver: counts number of 1s** in received message
  - **Even:** received message is a codeword

  - **Odd:** isn't a codeword, and **error detected**
    - But receiver doesn't know where, so **can't correct**

- What about d_{min}?
  - Change one data bit → change parity bit, so $d_{min} = 2$
    - So parity bit **detects 1 bit error, corrects 0**

- Can we **detect and correct more errors,** in general?

Two-dimensional parity

- Break up data into multiple rows
  - Parity bit **across** each row (***p_i***)
  - Parity bit **down each column** (***q_i***)
  - Add a parity bit ***r*** covering row parities

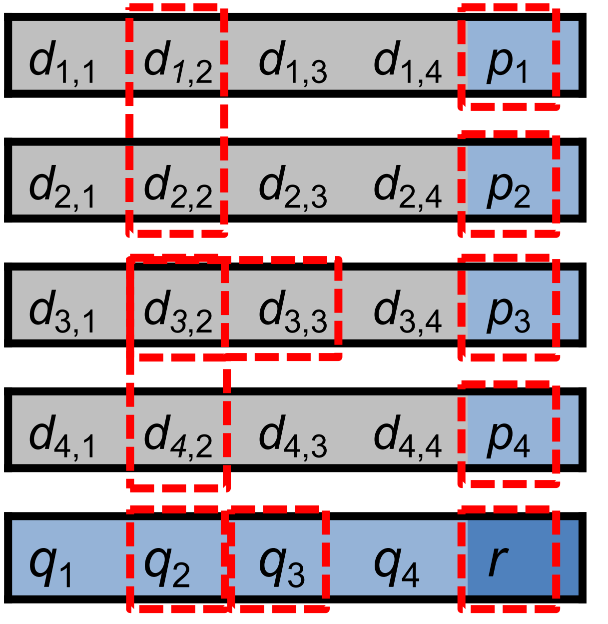
$$p_j = d_{j,1} \oplus d_{j,2} \oplus d_{j,3} \oplus d_{j,4}$$
$$q_j = d_{1,j} \oplus d_{2,j} \oplus d_{3,j} \oplus d_{4,j}$$
$$r = p_1 \oplus p_2 \oplus p_3 \oplus p_4$$

- This example has rate 16/25:

$d_{1,1}$	$d_{1,2}$	$d_{1,3}$	$d_{1,4}$	p_1
$d_{2,1}$	$d_{2,2}$	$d_{2,3}$	$d_{2,4}$	p_2
$d_{3,1}$	$d_{3,2}$	$d_{3,3}$	$d_{3,4}$	p_3
$d_{4,1}$	$d_{4,2}$	$d_{4,3}$	$d_{4,4}$	p_4
q_1	q_2	q_3	q_4	r

Two-dimensional parity: Properties

- Flip **1 data bit, 3 parity bits** flip
- Flip **2 data bits, ≥ 2 parity bits** flip
- Flip **3 data bits, ≥ 3 parity bits** flip

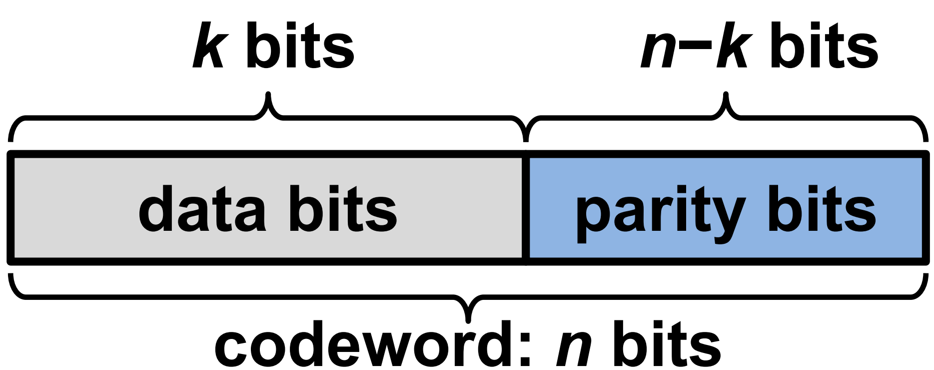


- Therefore, $\boldsymbol{d_{\mathrm{min}} = 4}$, so
  - Can detect ≤ 3 bit errors
  - Can correct single-bit errors (*how?*)

- 2-D parity detects **most** four-bit errors

Block codes

- Let's **fully generalize the parity bit** for even more error detecting/correcting power
- Split message into ***k*-bit blocks,** and **add *n*−*k* parity bits** to the end of each block:
  - This is called an **(*n*, *k*) block code**

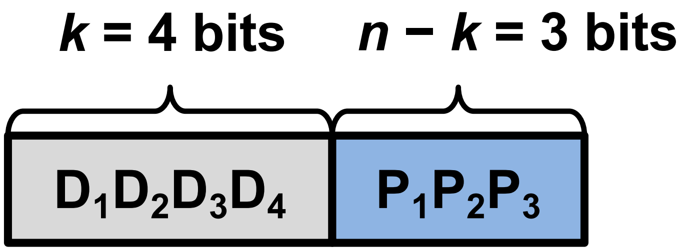


How to design a block code?

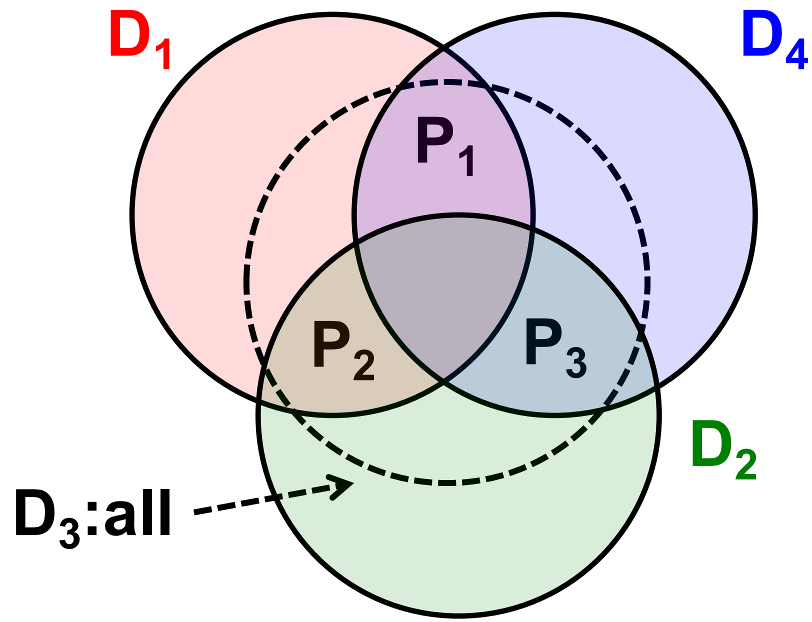
$D_1D_2D_3D_4$	P P P

- What if we **repeat the parity bit 3 × ?**
  - $P = D_1 \oplus D_2 \oplus D_3 \oplus D_4$; $R = 4/7$
  - Flip one data bit, all parity bits flip. So $\boldsymbol{d_{min} = 4}$?
    - **No!** Flip another data bit, all parity bits flip back to original values! So $\boldsymbol{d_{min} = 2}$
  - **What happened?** Parity checks either **all failed or all succeeded,** giving **no additional information**

Hamming (7, 4) code

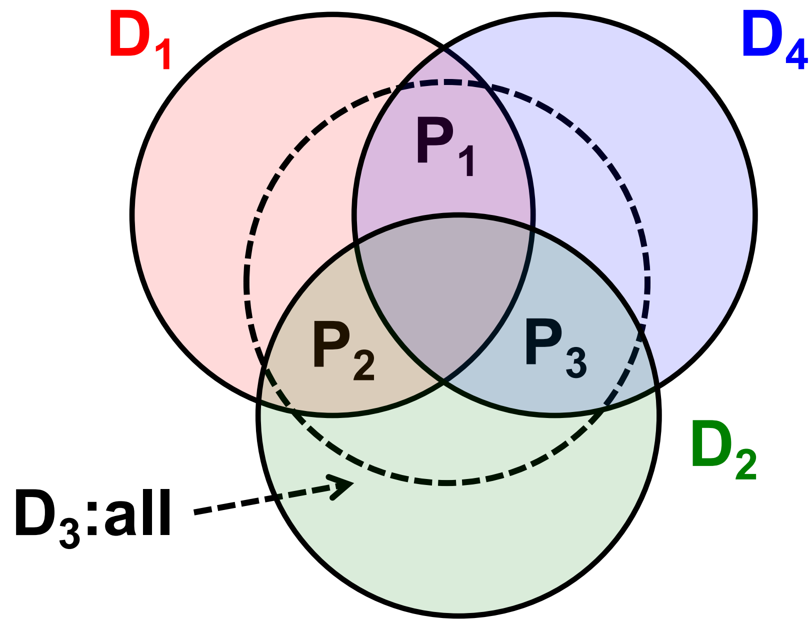


$P_1 = D_1 \oplus D_3 \oplus D_4$
$P_2 = D_1 \oplus D_2 \oplus D_3$
$P_3 = D_2 \oplus D_3 \oplus D_4$



Hamming (7, 4) code: d_{min}

- **Change one data bit,** either:
  - ⇨ Two P_i change, or
  - – Three P_i change
- Change two data bits, either:
  - – Two P_i change, or
  - ⇨ One P_i changes



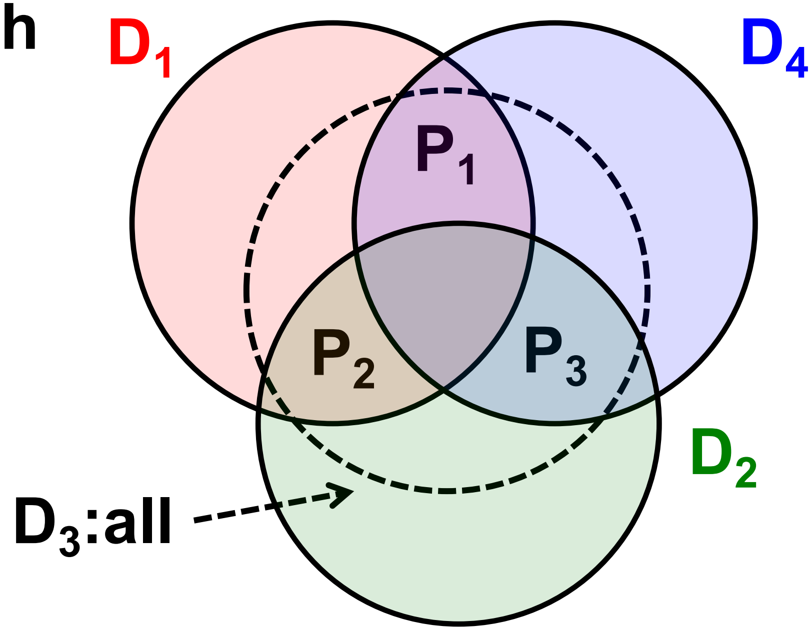
d_{min} = 3: Detect 2 bit errors, correct 1 bit error

Hamming (7, 4): Correcting One Bit Error

- **Infer which corrupt bit** from **which parity checks fail:**

- P_1 and P_2 fail ⇒ Error in **D_1**
- P_2 and P_3 fail ⇒ Error in **D_2**
- P_1, P_2, & P_3 fail ⇒ Error in **D_3**
- P_1 and P_3 fail ⇒ Error in **D_4**

- What if just **one** parity check fails?



Summary: Higher rate (R = 4/7) code correcting one bit error

Today

1. Error control codes

2. **Error detection codes**
   - **Cyclic redundancy check (CRC)**

Cyclic redundancy check (CRC)

- Represent ***k*-bit messages** as **degree $k-1$ polynomials**
  - **Each coefficient in polynomial is zero or one, *e.g.*:**

***k* = 6 bits of message**

1	0	1	1	1	0

$$M(x) = 1x^5 + 0x^4 + 1x^3 + 1x^2 + 1x + 0$$

Modulo-2 Arithmetic

- **Addition** and **subtraction** are **both** exclusive-or without carry or borrow

Multiplication example:

```
  1101
   110
 ------
  0000
 11010
110100
------
101110
```

Division example:

```
     1101
   ------
110|101110
    110
    ---
     111
     110
     ---
      011
      000
      ---
       110
       110
       ---
```

CRC at the sender

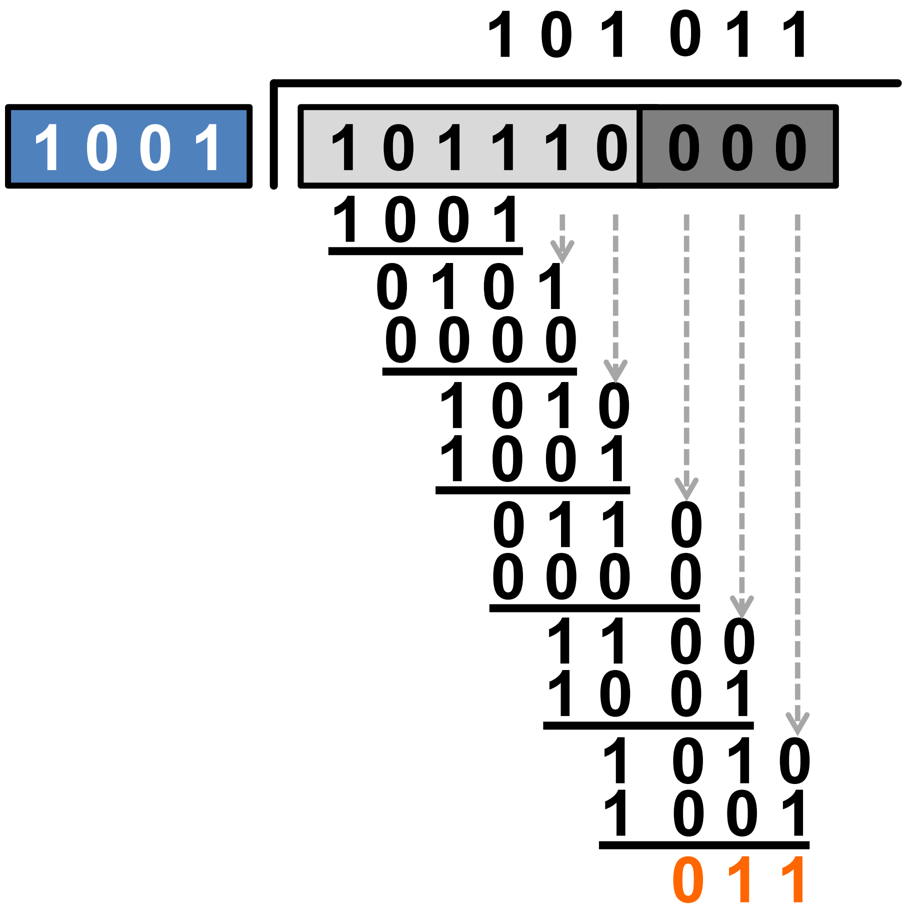
- ***M*(x)** is our **message** of length ***k***
  - ***e.g.:*** $M(x) = x^5 + x^3 + x^2 + x \ \ (k = 6)$ 　**1 0 1 1 1 0**
- Sender and receiver agree on a ***generator*** polynomial ***G*(*x*)** of degree ***g* − *1* (*i.e.*, *g* bits)**
  - ***e.g.:*** $G(x) = x^3 + 1 \ \ (g = 4)$ 　**1 0 0 1**

1. Calculate **padded message $T(x) = M(x) \cdot x^{g-1}$**
   - *i.e.*, right-pad with $g - 1$ zeroes
   - ***e.g.:*** $T(x) = \mathrm{M(x)} \cdot x^3 = x^8 + x^6 + x^5 + x^4$

1 0 1 1 1 0 0 0 0

CRC at the sender

2. **Divide** padded message $T(x)$ by generator $G(x)$
   - **The remainder $R(x)$** is the **CRC:**



$R(x) = x + 1$

CRC at the sender

3. **The sender transmits codeword $C(x) = T(x) + R(x)$**
   - *i.e.*, the sender transmits the original message with the CRC bits appended to the end

   - Continuing our example, $C(x) = x^8 + x^6 + x^5 + x^4 + x + 1$

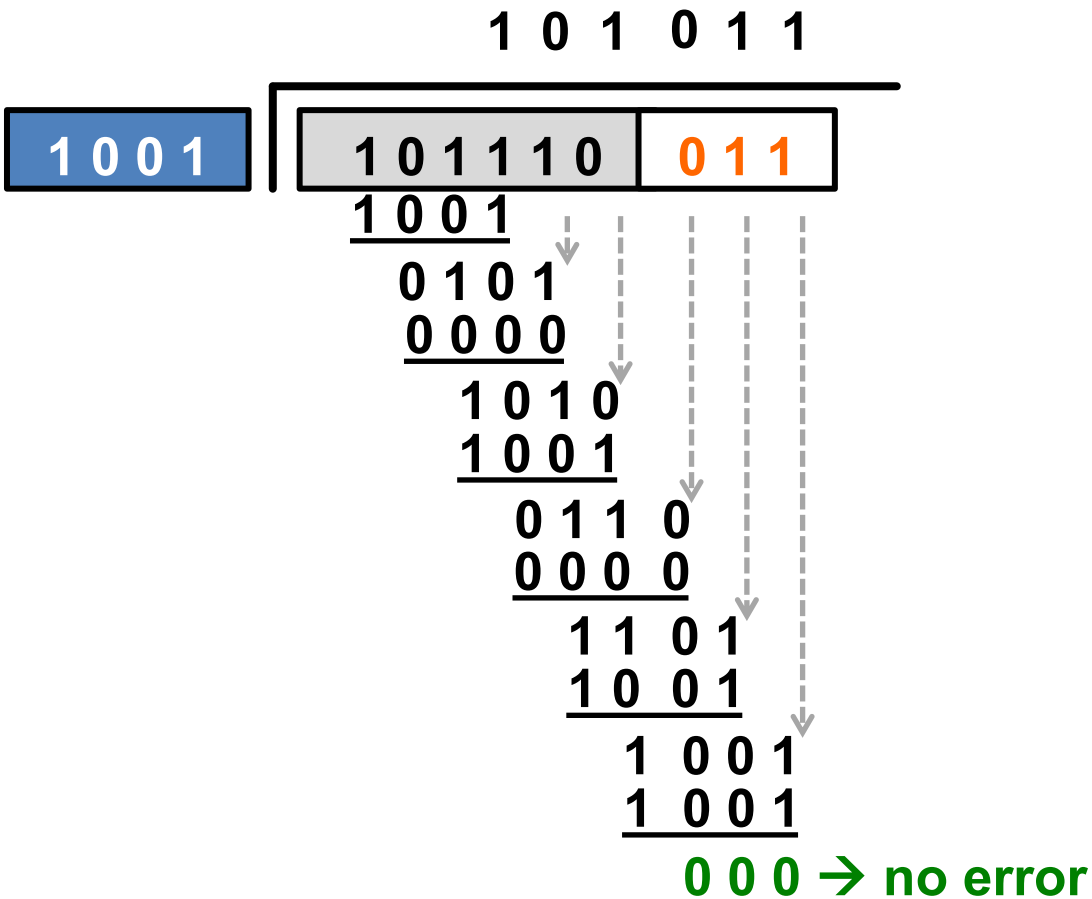
1 0 1 1 1 0	0 1 1

Properties of CRC codewords

- Remember: **Remainder** [$T(x)/G(x)$] = $R(x)$
- What happens when we divide $C(x)$ / $G(x)$?
- $C(x) = T(x) + R(x)$ so **remainder** is
  - **Remainder** [$T(x)/G(x)$] = $R(x)$, **plus**
  - **Remainder** [$R(x)/G(x)$] = $R(x)$
- Recall, **addition is exclusive-or operation**, so:
  - **Remainder** [$C(x)/G(x)$] = $R(x) + R(x) = 0$

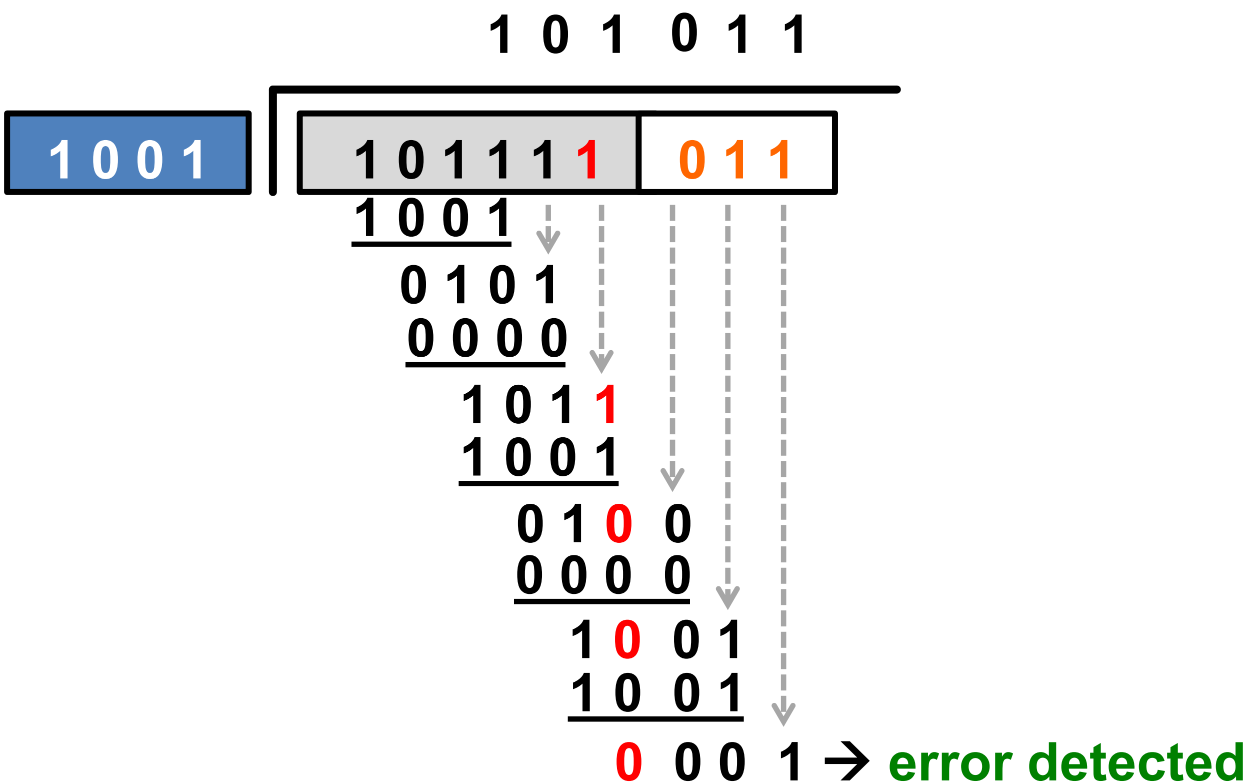
Detecting errors at the receiver

- **Receiver divides received message *C'*(*x*) by generator *G*(*x*)**
  - **If no errors occur,** remainder will be **zero**



Detecting errors at the receiver

- **Receiver divides received message *C'*(*x*) by generator *G*(*x*)**
  - **If errors occur,** remainder **may** be non-zero



Detecting errors at the receiver

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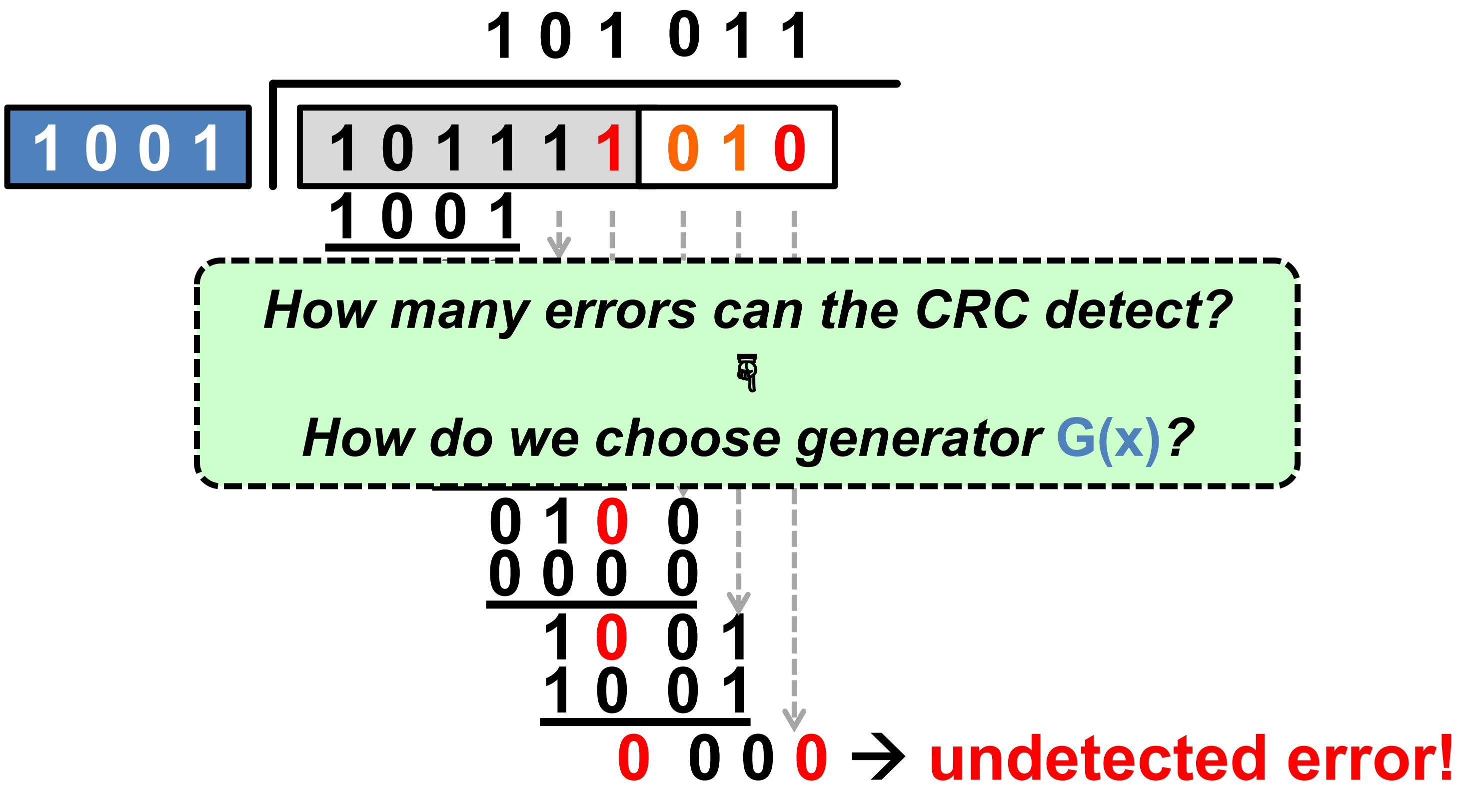


Detecting errors with the CRC

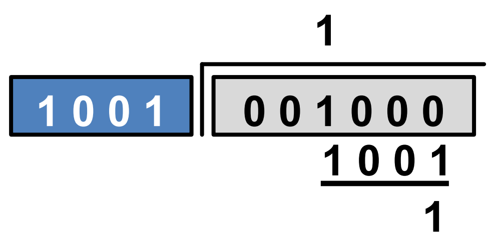
- *The* ***error polynomial*** $E(x) = C(x) + C'(x)$ is the difference between the transmitted and received codeword
  - $E(x)$ tells us **which bits the channel flipped**

- We can write the **received message $C'(x)$** in terms of $C(x)$ and $E(x)$: $C'(x) = C(x) + E(x)$, so:
  - **Remainder** [$C'(x)$ / $\mathbf{G(x)}$] = **Remainder** [$E(x)$ / $\mathbf{G(x)}$]

- When does an error go **undetected?**
  - When **Remainder** [$E(x)$ / $\mathbf{G(x)}$] = 0

Detecting single-bit errors w/CRC

- **Suppose a single-bit error** in bit-position i: $\boldsymbol{E(x) = x^i}$
  - **Choose** ***G*(*x*)** with **≥ 2** non-zero terms: x^{g-1} and 1
  - **Remainder** [$x^i / (x^{g-1} + \cdots + 1)$] **≠ 0, *e.g.*:**



- Therefore a **CRC with above choice of *G*(*x*) always detects single-bit errors** in the received message

Error detecting properties of the CRC

- The CRC will detect:

  All **single-bit errors**
  - Provided *G(x)* has two non-zero terms

  – All **burst errors** of **length ≤ $g - 1$**
  - Provided $G(x)$ begins with x^{g-1} and ends with 1
  - Similar argument to previous property

  – All **double-bit errors**
  - With conditions on the frame length and choice of $G(x)$

  – Any **odd number of errors**
  - Provided $G(x)$ contains an even number of non-zero coefficients

Error detecting code: CRC

- **Far less overhead** than error correcting codes
  - Typically **16 to 32 bits** on a **1,500 byte (12 Kbit) frame**

- **Error detecting** properties are **more complicated**

  - But in practice, **“missed” bit errors are exceedingly rare**

Next Week’s Precepts:
Midterm Review

Tuesday Topic:
Practical Wi-Fi Codes:
Convolutional Codes