COS320: Compiling Techniques

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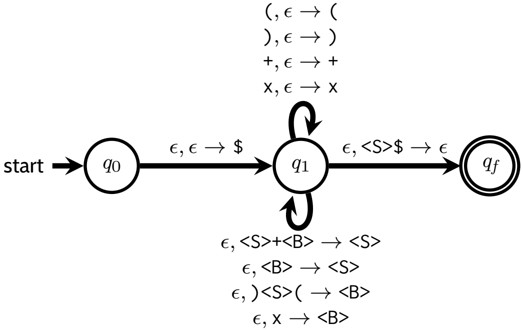
Parsing III: LR parsing

Bottom-up parsing

- Stack holds a word in $(N \cup \Sigma)^*$ from which it is possible to derive the part of the input string that has been consumed
- At any time, may read a letter from input string and push it on top of the stack
- At any time, may non-deterministically choose a rule $A ::= \gamma_1 ... \gamma_n$ and apply it in reverse: pop $\gamma_n ... \gamma_1$ off the top of the stack, and push A.
- Accept when stack just contains start non-terminal

<S> ::= <B>+<S> | <B>

<B> ::= (<S>) | x



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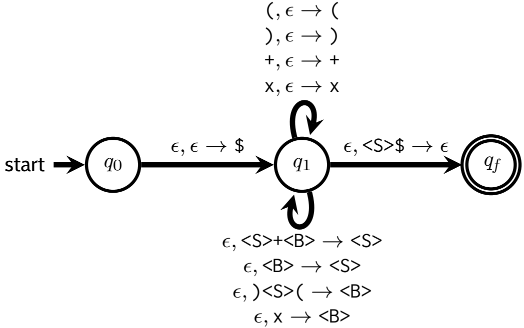


State	**Stack**	**Input**
q_0	ϵ	(x+x)+x
q_1	$	(x+x)+x
q_1	($	x+x)+x
q_1	x($	+x)+x
q_1	<B>($	+x)+x
q_1	+<B>($	x)+x
q_1	x+<B>($	)+x
q_1	<B>+<B>($	)+x
q_1	<S>+<B>($	)+x
q_1	<S>($	)+x
q_1	)<S>($	+x
q_1	<B>$	+x
q_1	+<B>$	x
q_1	x+<B>$	ϵ
q_1	<B>+<B>$	ϵ
q_1	<S>+<B>$	ϵ
q_1	<S>$	ϵ
q_f	ϵ	ϵ

LL vs LR

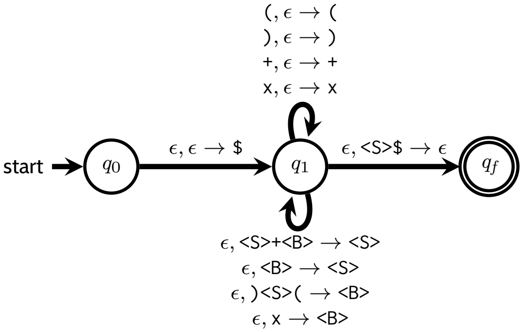
- LL parsers are top-down, LR parsers are bottom-up
- Easier to write LR grammars
  - Every LL(k) grammar is also LR(k), but not vice versa.
  - No need to eliminate left (or right) recursion
  - No need to left-factor
- Harder to write LR parsers
  - But parser generators will do it for us!

Bottom-up PDA has two kinds of actions:

- *Shift*: move lookahead token to the top of the stack
- *Reduce*: remove $\gamma_n, \ldots, \gamma_1$ from the top of the stack, replace with A (where $A ::= \gamma_1 \ldots \gamma_n$ is a rule of the grammar)
- Just as for LL parsing, the trick is to resolve non-determinism.
  - When should the parser shift?
  - When should the parser reduce?

<S> ::= <B>+<S> | <B>

<B> ::= (<S>) | x



Determinizing the bottom-up PDA

- **Intuition**: reduce greedily
  - If any reduce action applies, then apply it
    - Actually, a bit more nuanced: only apply reduction action if it is "relevant" (can eventually lead to the input word being accepted)
  - If no reduce action applies, then shift
- Can use the states of the PDA to implement greedy strategy
  - State tracks top few symbols of the stack

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  - If no reduce action applies, then shift
- Can use the states of the PDA to implement greedy strategy
  - State tracks top few symbols of the stack
- **Challenge**: after applying reduce action, need to re-compute the state
- **Solution**: use the stack to store *states*
  - Shift reads current state off the top of the stack, then pushes the next state
  - Reduce $A ::= \gamma_1, \ldots \gamma_n$ pops last n states, then proceeds from $(n-1)$th state as if A had been read

Warm-up: LR(0) parsing

<S> ::= (<L>) | x

<L> ::= <S> | <L>;<S>

- $LR(0)$ = LR with 0-symbol lookahead
- An **LR(0) item** of a grammar $G = (N, \Sigma, R, S)$ is of the form $A ::= \gamma_1 \dots \gamma_i \bullet \gamma_{i+1} \dots \gamma_n$, where $A ::= \gamma_1 \cdots \gamma_n$ is a rule of G
  - $\gamma_1 \dots \gamma_i$ derives part of the word that has already been read
  - $\gamma_{i+1} \dots \gamma_n$ derives part of the word that remains to be read
  - LR(0) items $\sim$ states of an NFA that determines when a reduction applies to the top of the stack
- LR(0) items for the above grammar:
  - <S> ::= ●(<L>), <S> ::= (●<L>), <S> ::= (<L>●), <S> ::= (<L>)●,
  - <S> ::= ●x, <S> ::= x●,
  - <L> ::= ●<S>, <L> ::= <S>●,
  - <L> ::= ●<L>;<S>, <L> ::= <L>●;<S>, <L> ::= <L>;●<S>, <L> ::= <L>;<S>●,

closure and goto

- For any set of items I, define closure(I) to be the least set of items such that
  - closure(I) contains I
  - If closure(I) contains an item of the form $A ::= \alpha \bullet B\beta$ where A is a non-terminal, then closure(I) contains $B ::= \bullet\gamma$ for all $B ::= \gamma \in R$
- **closure**(I) saturates I with all items that may be relevant to reducing via I
  - E.g., **closure**({<S> ::= (•<L>)}) =
    {<S> ::= (•<L>), <L> ::= •<S>, <L> ::= •<L>;<S>, <S> ::= •(<L>)<S> ::= •x}
  - Part of the not-quite greedy strategy: don't try to reduce using all rules all the time, track only a relevant subset

closure and goto

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- **closure**(I) saturates I with all items that may be relevant to reducing via I
  - E.g., **closure**$(\{\texttt{<S> ::= (}\bullet\texttt{<L>)}\}) =$ $\{\texttt{<S> ::= (}\bullet\texttt{<L>)}, \texttt{<L> ::= }\bullet\texttt{<S>}, \texttt{<L> ::= }\bullet\texttt{<L>;<S>}, \texttt{<S> ::= }\bullet\texttt{(<L>)<S> ::= }\bullet\texttt{x}\}$
  - Part of the not-quite greedy strategy: don't try to reduce using all rules all the time, track only a relevant subset
- For any item set I, and (terminal or non-terminal) symbol $\gamma \in N \cup \Sigma$ define **goto**$(I, \gamma) =$ **closure**$(\{A ::= \alpha\gamma \bullet \beta \mid A ::= \alpha \bullet \gamma\beta \in I\})$
  - I.e., **goto**(I, γ) is the result of "moving $\bullet$ across γ"
  - E.g., **goto**(**closure**$(\{\texttt{<S> ::= (}\bullet\texttt{<L>)}\}, \texttt{<L>})) = \{\texttt{<S> ::= (<L>}\bullet\texttt{)}, \texttt{<L> ::= <L>}\bullet\texttt{;<S>}, \}$

Mechanical construction of LR(0) parsers

1. Add a new production $S' ::= S\$$ to the grammar.
   - S' is new start symbol
   - $\$$ marks end of the stack
2. Construct transitions as follows: for each closed item set I,
   - For each item of the form $A ::= \gamma_1 ... \gamma_n \bullet$ in I, add *reduce* transition

$$\epsilon, IJ_1 ... J_{n-1}K \to K'K, \text{ where } K' = \textbf{goto}(K, A)$$

   - For each item of the form $A ::= \gamma \bullet a\beta$ in I with $a \in \Sigma$, add a *shift* transition

$$a, I \to I'I \text{ where } I' = \textbf{goto}(I, a)$$

Resulting automaton is deterministic $\iff$ grammar is LR(0)

Example run

<S> ::= (<L>) | x

<L> ::= <S> | <L>;<S>

- Input word: `(x;x)$`
- Stack:

```
<S'> ::= ●<S>$,
<S>  ::= ●(<L>),
<S>  ::= ●x
```

- Action: shift

Example run

<S> ::= (<L>) | x

<L> ::= <S> | <L>;<S>

- Input word: (x;x)$
- Stack:

  { <S'> ::= ●<S>$,
  <S> ::= ●(<L>),
  <S> ::= ●x }

  { <S> ::= (●<L>)
  <L> ::= ●<S>
  <L> ::= ●<L>;<S>
  <S> ::= ●(<L>)
  <S> ::= ●x }

- Action: shift

Example run

$$\langle S\rangle ::= (\langle L\rangle) \mid x$$

$$\langle L\rangle ::= \langle S\rangle \mid \langle L\rangle ; \langle S\rangle$$

- Input word: (x ; x) $
- Stack:

$$\left\{\begin{array}{l} \langle S'\rangle ::= \bullet\langle S\rangle\$, \\ \langle S\rangle ::= \bullet(\langle L\rangle), \\ \langle S\rangle ::= \bullet x \end{array}\right\} \left\{\begin{array}{l} \langle S\rangle ::= (\bullet\langle L\rangle) \\ \langle L\rangle ::= \bullet\langle S\rangle \\ \langle L\rangle ::= \bullet\langle L\rangle;\langle S\rangle \\ \langle S\rangle ::= \bullet(\langle L\rangle) \\ \langle S\rangle ::= \bullet x \end{array}\right\} \left\{ \langle S\rangle ::= x\bullet \right\}$$

- Action: reduce

Example run

$$<S> ::= (<L>) \mid x$$

$$<L> ::= <S> \mid <L>;<S>$$

- Input word: (x;x)$
- Stack:

$$\left\{\begin{array}{l} <S'> ::= \bullet<S>\$, \\ <S> ::= \bullet(<L>), \\ <S> ::= \bullet x \end{array}\right\} \left\{\begin{array}{l} <S> ::= (\bullet<L>) \\ <L> ::= \bullet<S> \\ <L> ::= \bullet<L>;<S> \\ <S> ::= \bullet(<L>) \\ <S> ::= \bullet x \end{array}\right\} \left\{ <L> ::= <S>\bullet \right\}$$

- Action: reduce

Example run

<S> ::= (<L>) | x

<L> ::= <S> | <L>;<S>

- Input word: (x;x)$
- Stack:

$$\left\{\begin{array}{l} <S'> ::= \bullet<S>\$, \\ <S> ::= \bullet(<L>), \\ <S> ::= \bullet x \end{array}\right\} \left\{\begin{array}{l} <S> ::= (\bullet<L>) \\ <L> ::= \bullet<S> \\ <L> ::= \bullet<L>;<S> \\ <S> ::= \bullet(<L>) \\ <S> ::= \bullet x \end{array}\right\} \left\{\begin{array}{l} <S> ::= (<L>\bullet) \\ <L> ::= <L>\bullet;<S> \end{array}\right\}$$

- Action: shift

Example run

$$\langle S\rangle ::= (\langle L\rangle) \mid x$$

$$\langle L\rangle ::= \langle S\rangle \mid \langle L\rangle ; \langle S\rangle$$

- Input word: $(x;x)\$$
- Stack:

$$\left\{\begin{array}{l} \langle S'\rangle ::= \bullet\langle S\rangle\$, \\ \langle S\rangle ::= \bullet(\langle L\rangle), \\ \langle S\rangle ::= \bullet x \end{array}\right\}
\left\{\begin{array}{l} \langle S\rangle ::= (\bullet\langle L\rangle) \\ \langle L\rangle ::= \bullet\langle S\rangle \\ \langle L\rangle ::= \bullet\langle L\rangle;\langle S\rangle \\ \langle S\rangle ::= \bullet(\langle L\rangle) \\ \langle S\rangle ::= \bullet x \end{array}\right\}
\left\{\begin{array}{l} \langle S\rangle ::= (\langle L\rangle\bullet) \\ \langle L\rangle ::= \langle L\rangle\bullet;\langle S\rangle \end{array}\right\}
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- Action: shift

Example run

<S> ::= (<L>) | x

<L> ::= <S> | <L>;<S>

- Input word: (x;x)$
- Stack:

{ <S'> ::= ●<S>$, <S> ::= ●(<L>), <S> ::= ●x }
{ <S> ::= (●<L>) <L> ::= ●<S> <L> ::= ●<L>;<S> <S> ::= ●(<L>) <S> ::= ●x }
{ <S> ::= (<L>●) <L> ::= <L>●;<S> }
{ <L> ::= <L>;●<S> <S> ::= ●(<L>), <S> ::= ●x }
{ <S> ::= x● }

- Action: reduce

Example run

$$\langle S\rangle ::= (\langle L\rangle) \mid x$$

$$\langle L\rangle ::= \langle S\rangle \mid \langle L\rangle;\langle S\rangle$$

- Input word: (x;x)$
- Stack:

$$\left\{\begin{array}{l} \langle S'\rangle ::= \bullet\langle S\rangle\$, \\ \langle S\rangle ::= \bullet(\langle L\rangle), \\ \langle S\rangle ::= \bullet x \end{array}\right\} \left\{\begin{array}{l} \langle S\rangle ::= (\bullet\langle L\rangle) \\ \langle L\rangle ::= \bullet\langle S\rangle \\ \langle L\rangle ::= \bullet\langle L\rangle;\langle S\rangle \\ \langle S\rangle ::= \bullet(\langle L\rangle) \\ \langle S\rangle ::= \bullet x \end{array}\right\} \left\{\begin{array}{l} \langle S\rangle ::= (\langle L\rangle\bullet) \\ \langle L\rangle ::= \langle L\rangle\bullet;\langle S\rangle \end{array}\right\} \left\{\begin{array}{l} \langle L\rangle ::= \langle L\rangle;\bullet\langle S\rangle \\ \langle S\rangle ::= \bullet(\langle L\rangle), \\ \langle S\rangle ::= \bullet x \end{array}\right\} \left\{ \langle L\rangle ::= \langle L\rangle;\langle S\rangle\bullet \right\}$$

- Action: reduce

Example run

<S> ::= (<L>) | x

<L> ::= <S> | <L>;<S>

- Input word: (x;x)$
- Stack:

{ <S'> ::= ●<S>$, <S> ::= ●(<L>), <S> ::= ●x }

{ <S> ::= (●<L>), <L> ::= ●<S>, <L> ::= ●<L>;<S>, <S> ::= ●(<L>), <S> ::= ●x }

{ <S> ::= (<L>●), <L> ::= <L>●;<S> }

- Action: shift

Example run

<S> ::= (<L>) | x

<L> ::= <S> | <L>;<S>

- Input word: (x ; x) $
- Stack:

{ <S'> ::= ●<S>$, <S> ::= ●(<L>), <S> ::= ●x }
{ <S> ::= (●<L>) <L> ::= ●<S> <L> ::= ●<L>;<S> <S> ::= ●(<L>) <S> ::= ●x }
{ <S> ::= (<L>●) <L> ::= <L>●;<S> }
{ <S> ::= (<L>)● }

- Action: reduce

Example run

$$\langle S\rangle ::= (\langle L\rangle) \mid x$$

$$\langle L\rangle ::= \langle S\rangle \mid \langle L\rangle ; \langle S\rangle$$

- Input word: (x ; x) $
- Stack:

$$\left\{\begin{array}{l} \texttt{<S'> ::= } \bullet\texttt{<S>\$,} \\ \texttt{<S> ::= } \bullet\texttt{(<L>),} \\ \texttt{<S> ::= } \bullet\texttt{x} \end{array}\right\} \left\{ \texttt{<S'> ::= <S>}\bullet\texttt{\$} \right\}$$

- Action: shift

Example run

<S> ::= (<L>) | x

<L> ::= <S> | <L>;<S>

- Input word: (x ; x) $
- Stack:

$$\left\{\begin{array}{l} \text{<S'> ::= }\bullet\text{<S>\$,} \\ \text{<S> ::= }\bullet\text{(<L>),} \\ \text{<S> ::= }\bullet\text{x} \end{array}\right\} \left\{\text{<S'> ::= <S>}\bullet\text{\$}\right\} \left\{\text{<S'> ::= <S>\$}\bullet\right\}$$

- Action: accept

Conflicts

- Recall: Automaton is deterministic $\iff$ grammar is LR(0)
- Observe: for LR(0) grammars, each closed set of items is either a *reduce* state or a *shift* state
  - Reduce state has exactly one item, and it's of the form $\{A ::= \gamma\bullet\}$
  - Shift state has *no* items of the form $A ::= \gamma\bullet$
- Reduce/reduce conflict: state has two or more items of the form $A ::= \gamma\bullet$ (choice of reduction is non-deterministic!)
- Reduce/reduce conflict: state has an item of the form $A ::= \gamma\bullet$ *and* one of the form $A ::= \gamma \bullet a\beta$ (choice of whether to shift or reduce is non-deterministic!)

Simple LR (SLR)

- Simple LR is a straight-forward extension of LR(0) with a lookahead token
- **Idea**: proceed exactly as LR(0), but eliminate (some) conflicts using lookahead token
  - For each item of the form $A ::= \gamma_1 ... \gamma_n \bullet$ in I, add *reduce* transition

$$\epsilon, IJ_1 ... J_{n-1}K \to K' K, \text{ where } K' = \mathbf{goto}(K, A)$$

  with any lookahead token in **follow**(A)

- Example: the following grammar is SLR, but not LR(0)

$$\text{<S>} ::= \text{<T>b}$$
$$\text{<T>} ::= \text{a<T>} \mid \epsilon$$

Consider: **closure**({<S'> ::= •<S>$}) contains T ::= •.

LR(1) parser construction

- An **LR(1) item** of a grammar $G = (N, \Sigma, R, S)$ is of the form $(A ::= \gamma_1 \ldots \gamma_i \bullet \gamma_{i+1} \ldots \gamma_n, a)$, where $A ::= \gamma_1 \cdots \gamma_n$ is a rule of G and $a \in \Sigma$
  - $\gamma_1 \ldots \gamma_i$ derives part of the word that has already been read
  - $\gamma_{i+1} \ldots \gamma_n$ derives part of the word that remains to be read
  - a is a lookahead symbol
- For any set of items I, define **closure**(I) to be the least set of items such that
  - **closure**(I) contains I
  - If **closure**(I) contains an item of the form $(A ::= \alpha \bullet B\beta, a)$ where B is a non-terminal, then **closure**(I) contains $(B ::= \bullet\gamma, b)$ for all $B ::= \gamma \in R$ and all $b \in$ **first**(βa).
- Construct PDA as in LR(0)

LALR(1)

- LR(1) transition tables can be very large
- LALR(1) (“lookahead LR(1)”) make transition table smaller by merging states that are identical except for lookahead
- Merging states can create reduce/reduce conflicts. Say that a grammar is LALR(1) if this merging *doesn’t* create conflicts.

Summary of parsing

- For any k, $LL(k)$ grammars are $LR(k)$
- SLR grammars are $LALR(1)$ are $LR(1)$
- In terms of *language expressivity*, there is an SLR (and therefore LALR(1) and LR(1) grammar for any context-free language that can be accepted by a deterministic pushdown automaton).
- Not every deterministic context free language is LL(k): $\{a^n b^n : n \in \mathbb{N}\} \cup \{a^n c^n : n \in \mathbb{N}\}$ is DCFL but not LL(k) for any k.[1]

[1] John C. Beatty, *Two iteration theorems for the LL(k) Languages*