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3.4 HASH TABLES

- *hash functions*
- *separate chaining*
- *linear probing*
- *context*

Symbol table implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
red–black BST	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`
hashing	n	n	n	1 †	1 †	1 †		`equals()` `hashCode()`

† under suitable technical assumptions

Q. Can we do better?

A. Yes, but with different access to the data.

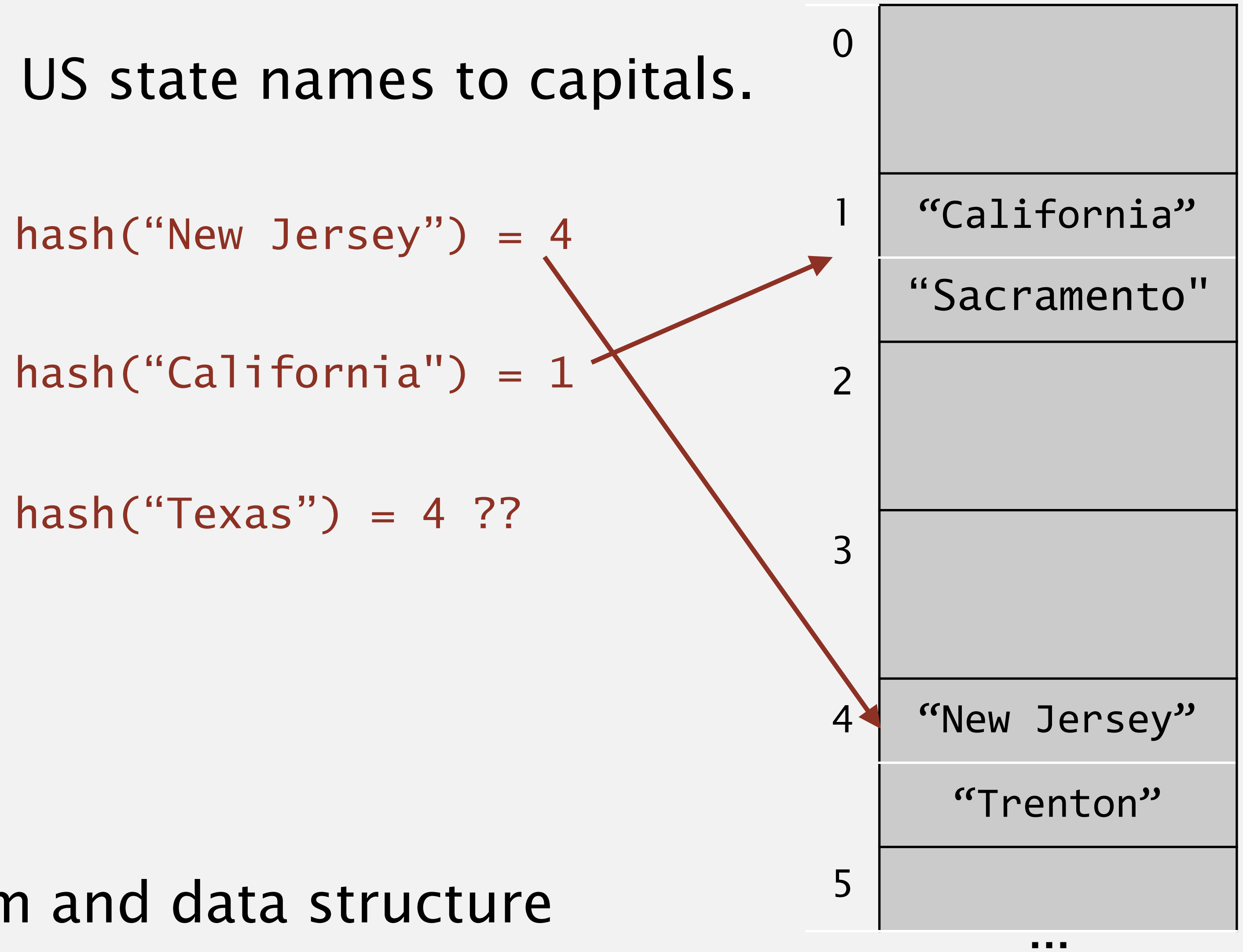
Hashing: basic plan

Save items in a key-indexed table (index is a function of the key).

Hash function. Method for computing array index from key.

Example. Symbol table that maps US state names to capitals.

Assume that `hash()` takes in a string and outputs an integer between 0 and 100.



Issues.

- Computing the hash function.
- Collision resolution: Algorithm and data structure to handle two keys that hash to the same array index.
  - Equality test: Method for checking whether two keys are equal.

Equality test

All Java classes inherit a method `equals()`.

Java requirements. For any references `x`, `y` and `z`:

- Reflexive: `x.equals(x)` is `true`.
- Symmetric: `x.equals(y)` iff `y.equals(x)`.
- Transitive: if `x.equals(y)` and `y.equals(z)`, then `x.equals(z)`.
- Non-null: `x.equals(null)` is `false`.

equivalence relation

do x and y refer to the same object?

Default implementation. `(x == y)`

Customized implementations. `Integer`, `Double`, `String`, `java.net.URL`, ...

User-defined implementations. Some care needed.

Implementing equals for user-defined types

Exercise. What are 5 additions/modifications you need to make to this code?

```
public class Date
{
   private final int month;
   private final int day;
   private final int year;
   ...

   public boolean equals(Date that)
   {

      if (this.day   != that.day  ) return false;
      if (this.month != that.month) return false;
      if (this.year  != that.year ) return false;
      return true;
   }
}
```

check that all significant fields are the same

Implementing equals for user-defined types

Seems easy, but requires some care.

typically unsafe to use equals() with inheritance (would violate symmetry)

```
public final class Date
{
   private final int month;
   private final int day;
   private final int year;
   ...

   public boolean equals(Object y)
   {
      if (y == this) return true;

      if (y == null) return false;

      if (y.getClass() != this.getClass())
         return false;

      Date that = (Date) y;
      if (this.day   != that.day   ) return false;
      if (this.month != that.month) return false;
      if (this.year  != that.year ) return false;
      return true;
   }
}
```

must be Object. Why? Experts still debate.

optimization (for reference equality)

check for null

objects must be in the same class (religion: getClass() vs. instanceof)

cast is now guaranteed to succeed

check that all significant fields are the same

Equals design

"Standard" recipe for user-defined types.

- Optimization for reference equality.
- Check against `null`.
- Check that two objects are of the same type; cast.
- Compare each significant field:
  - if field is a primitive type, use ==
  - if field is an object, use `equals()` and apply rule recursively
  - if field is an array of primitives, use `Arrays.equals()`
  - if field is an array of objects, use `Arrays.deepEquals()`

but use `Double.compare()` for `double` (to deal with –0.0 and NaN)

Tedious but necessary

Best practices.

e.g., cached Manhattan distance

- Do not use calculated fields that depend on other fields.
- Compare fields mostly likely to differ first.
- Make `compareTo()` consistent with `equals()`.

`x.equals(y)` if and only if `(x.compareTo(y) == 0)`

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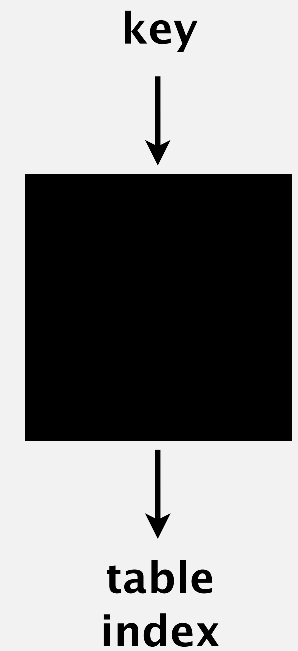
3.4 Hash Tables

- *hash functions*
- *separate chaining*
- *linear probing*
- *context*

Computing the hash function

Idealistic goal. Scramble the keys uniformly to produce a table index.

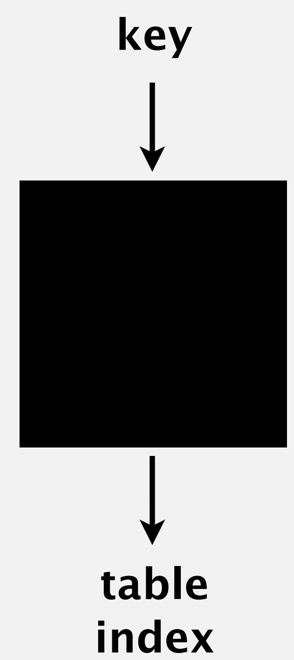


Computing the hash function

Idealistic goal. Scramble the keys uniformly to produce a table index.

- Efficiently computable.
- Each table index equally likely for each key.

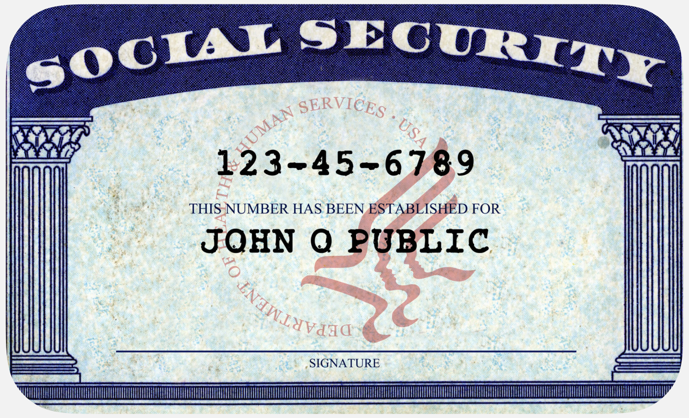
thoroughly researched problem,
still problematic in practical applications



Ex 1. Last 4 digits of Social Security number.

Ex 2. Last 4 digits of phone number.

Practical challenge. Need different approach for each key type.



Which is the last digit of your day of birth?

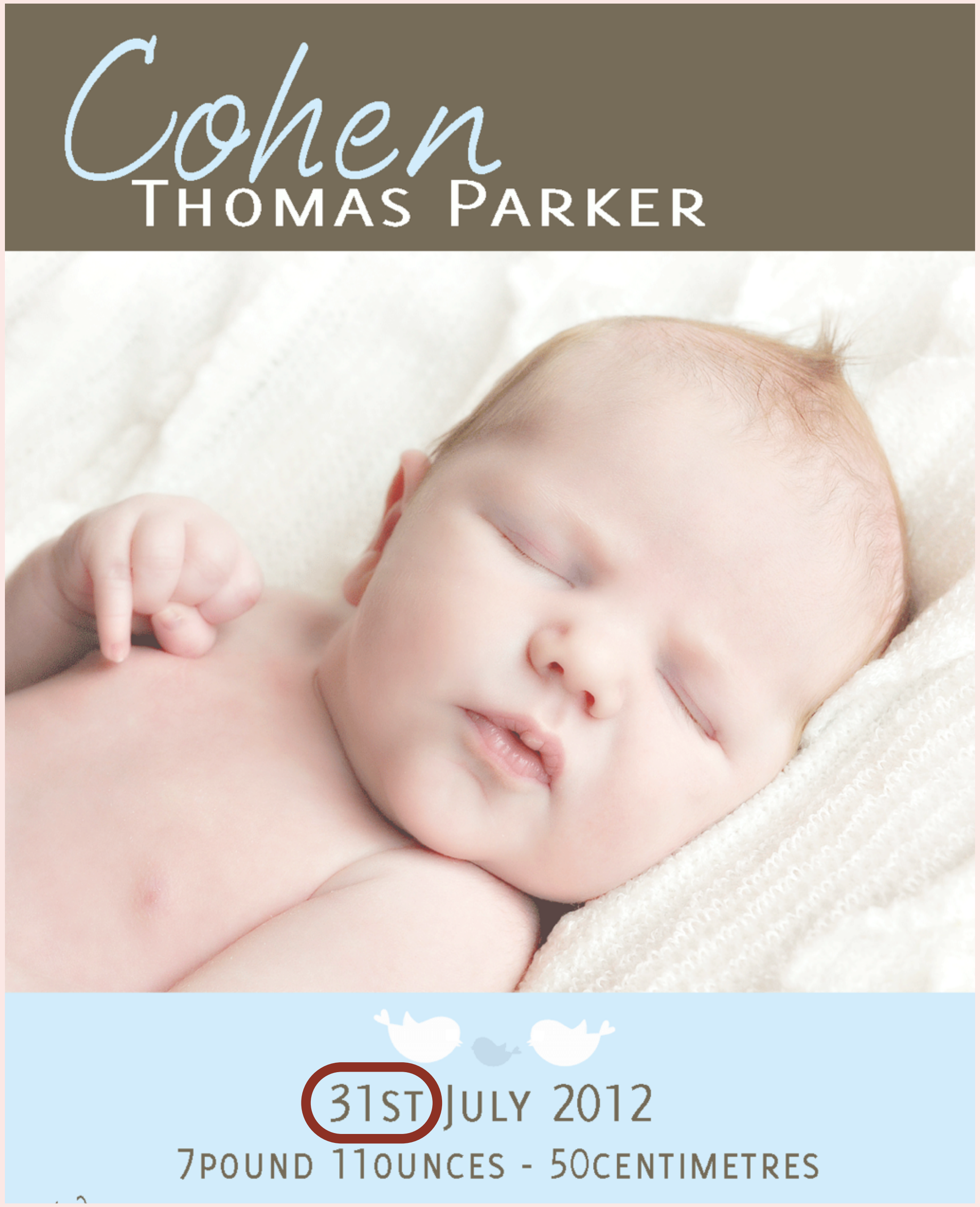
A. 0 or 1

B. 2 or 3

C. 4 or 5

D. 6 or 7

E. 8 or 9



Which is the last digit of your year of birth?

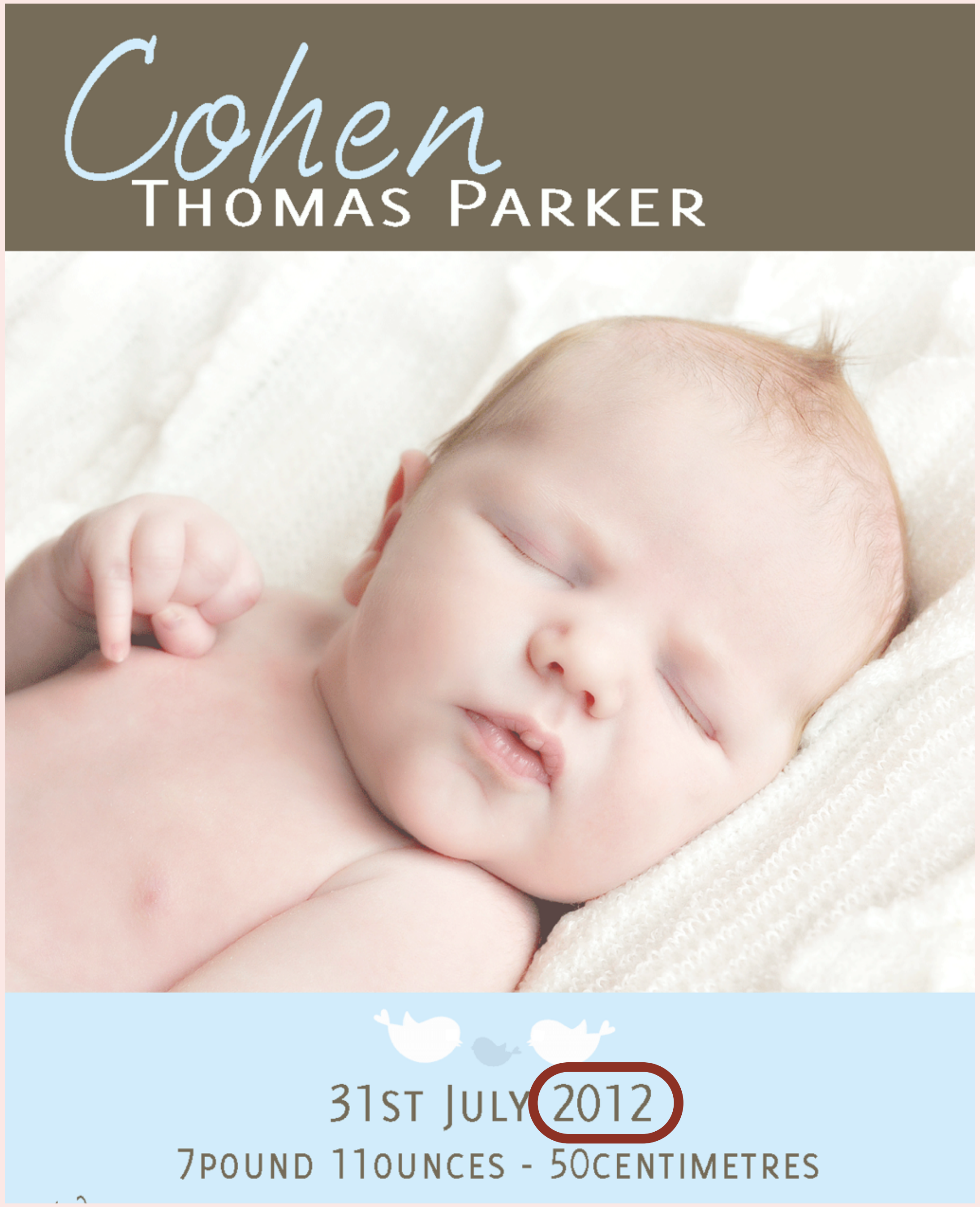
A. 0 or 1

B. 2 or 3

C. 4 or 5

D. 6 or 7

E. 8 or 9

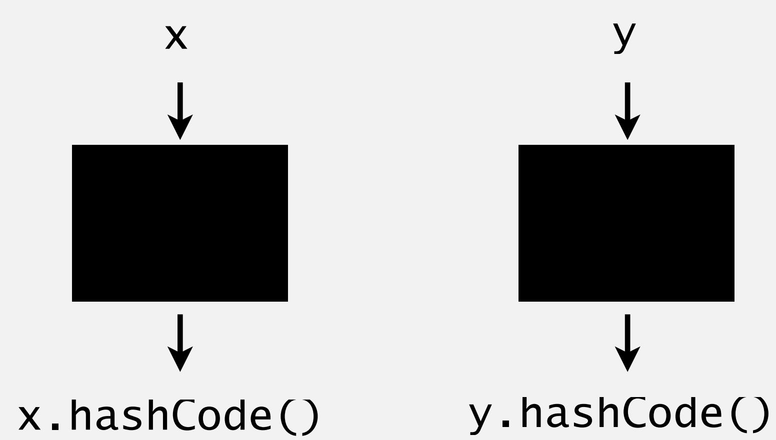


Java's hash code conventions

All Java classes inherit a method `hashCode()`, which returns a 32-bit `int`.

Requirement. If `x.equals(y)`, then `(x.hashCode() == y.hashCode())`.

Highly desirable. If `!x.equals(y)`, then `(x.hashCode() != y.hashCode())`.



Default implementation. Memory address of `x`.

Legal (but poor) implementation. Always return `17`.

Customized implementations. `Integer`, `Double`, `String`, `java.net.URL`, …

User-defined types. Users are on their own.

Implementing hash code: integers, booleans, and doubles

Java library implementations

```
public final class Integer
{
   private final int value;
   ...
   public int hashCode()
   {  return value;  }
}
```

```
public final class Boolean
{
   private final boolean value;
   ...
   public int hashCode()
   {
      if (value) return 1231;
      else       return 1237;
   }
}
```

```
public final class Double
{
   private final double value;
   ...
   public int hashCode()
   {
      long bits = doubleToLongBits(value);
      return (int) (bits ^ (bits >>> 32));
   }
}
```

convert to IEEE 64-bit representation;
xor most significant 32-bits
with least significant 32-bits

Warning: -0.0 and +0.0 have different hash codes

Implementing hash code: arrays

31x + y rule.

- Initialize hash to 1.
- Repeatedly multiply hash by 31 and add next integer in array.

```
public class Arrays
{
   ...

   public static int hashCode(int[] a) {
      if (a == null)
         return 0;                          special case for null

      int hash = 1;
      for (int i = 0; i < a.length; i++)
         hash = 31*hash + a[i];             31x + y rule
      return hash;
      }

}
```

Implementing hash code: strings

Treat a string as an array of characters.

```
public final class String
{
   private final char[] s;
   ⋮
   public int hashCode()
   {
      int hash = 0;
      for (int i = 0; i < length(); i++)
         hash = s[i] + (31 * hash);
      return hash;
   }
}
```

Initialize to 0 rather than 1

Java library implementation

char	Unicode
…	…
'a'	97
'b'	98
'c'	99
…	…

Aside: string hash collisions in Java

key	hashCode()
"Aa"	2112
"BB"	2112

key	hashCode()
"AaAaAaAa"	-540425984
"AaAaAaBB"	-540425984
"AaAaBBAa"	-540425984
"AaAaBBBB"	-540425984
"AaBBAaAa"	-540425984
"AaBBAaBB"	-540425984
"AaBBBBAa"	-540425984
"AaBBBBBB"	-540425984

key	hashCode()
"BBAaAaAa"	-540425984
"BBAaAaBB"	-540425984
"BBAaBBAa"	-540425984
"BBAaBBBB"	-540425984
"BBBBAaAa"	-540425984
"BBBBAaBB"	-540425984
"BBBBBBAa"	-540425984
"BBBBBBBB"	-540425984

2^n strings of length 2n that hash to same value!

Implementing hash code: user-defined types

```
public final class Transaction
{
   private final String  who;
   private final Date    when;
   private final double  amount;

   public Transaction(String who, Date when, double amount)
   {  /* as before */  }

   public boolean equals(Object y)
   {  /* as before */  }

   ...

   public int hashCode()
   {
      int hash = 1;
      hash = 31*hash + who.hashCode();
      hash = 31*hash + when.hashCode();
      hash = 31*hash + ((Double) amount).hashCode();
      return hash;
   }

}
```

for reference types, use hashCode()

for primitive types, use hashCode() of wrapper type

Implementing hash code: user-defined types

```
public final class Transaction
{
   private final String  who;
   private final Date    when;
   private final double  amount;

   public Transaction(String who, Date when, double amount)
   {  /* as before */  }

   public boolean equals(Object y)
   {  /* as before */  }

   ...

   public int hashCode()
   {
      return Objects.hash(who, when, amount);
   }

}
```

shorthand

Hash code design

"Standard" recipe for user-defined types.

- Combine each significant field using the $31x + y$ rule.
- Shortcut 1: use `Objects.hash()` for all fields (except arrays).
- Shortcut 2: use `Arrays.hashCode()` for primitive arrays.
- Shortcut 3: use `Arrays.deepHashCode()` for object arrays.

In practice. Recipe above works reasonably well; used in Java libraries.

In theory. Keys are bitstring; "universal" family of hash functions exist.

awkward in Java since only
one (deterministic) hashCode()

Basic rule. Need to use the whole key to compute hash code.

Hash tables: quiz 3

Which code maps hashable keys to integers between `0` and `m-1` ?

A.

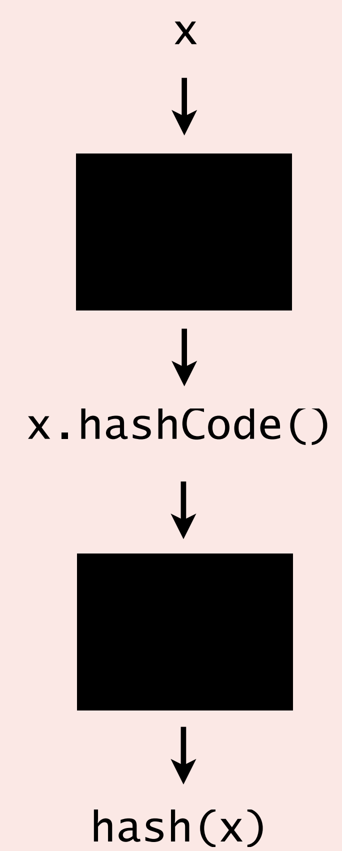
```
private int hash(Key key)
{  return key.hashCode() % m;  }
```

B.

```
private int hash(Key key)
{  return Math.abs(key.hashCode()) % m;  }
```

C. Both A and B.

D. Neither A nor B.

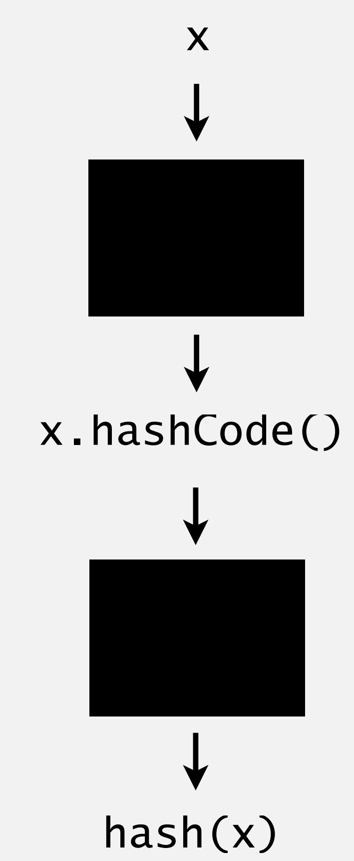


Modular hashing

Hash code. An `int` between -2^{31} and $2^{31} - 1$.

Hash function. An `int` between `0` and `m - 1` (for use as array index).

typically a prime or power of 2




```
private int hash(Key key)
{  return key.hashCode() % m;  }
```

bug

```
private int hash(Key key)
{  return Math.abs(key.hashCode()) % m;  }
```

1-in-a-billion bug

`hashCode()` of "`polygenelubricants`" is -2^{31}

```
private int hash(Key key)
{  return (key.hashCode() & 0x7fffffff) % m;  }
```

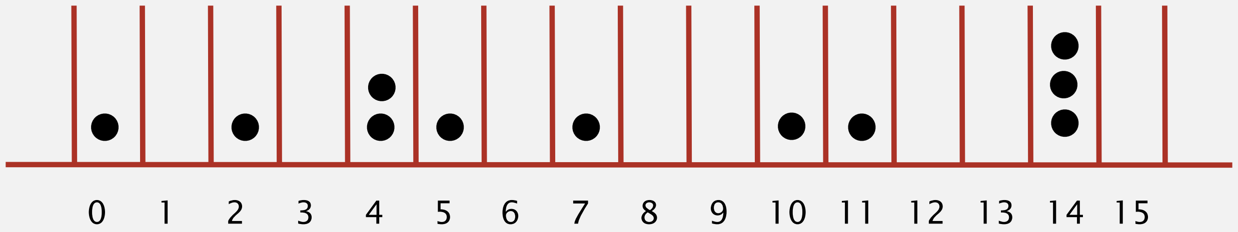
correct

if `m` is a power of 2, can use
`key.hashCode() & (m-1)`

Uniform hashing assumption

Uniform hashing assumption. Each key is equally likely to hash to an integer between 0 and $m - 1$.

Mathematical model: balls & bins. Toss n balls uniformly at random into m bins.



n = 11 balls; m = 16 bins

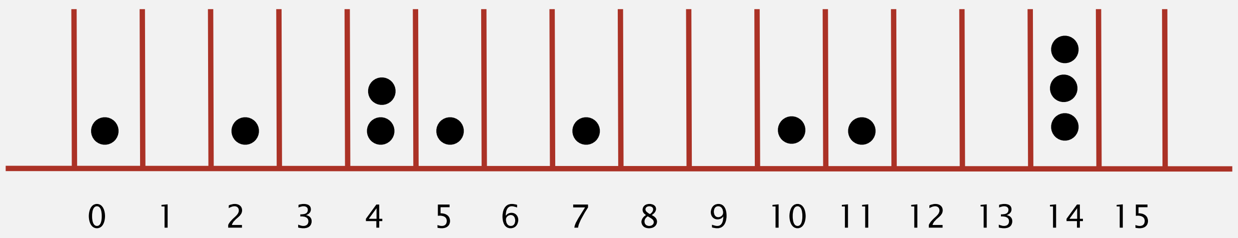
Bad news. Expect two balls in the same bin after $\sim \sqrt{\pi m / 2}$ tosses.

Birthday problem. In a random group of 23 or more people, more likely than not that two people will share the same birthday.

Uniform hashing assumption

Uniform hashing assumption. Each key is equally likely to hash to an integer between 0 and $m - 1$.

Mathematical model: balls & bins. Toss n balls uniformly at random into m bins.



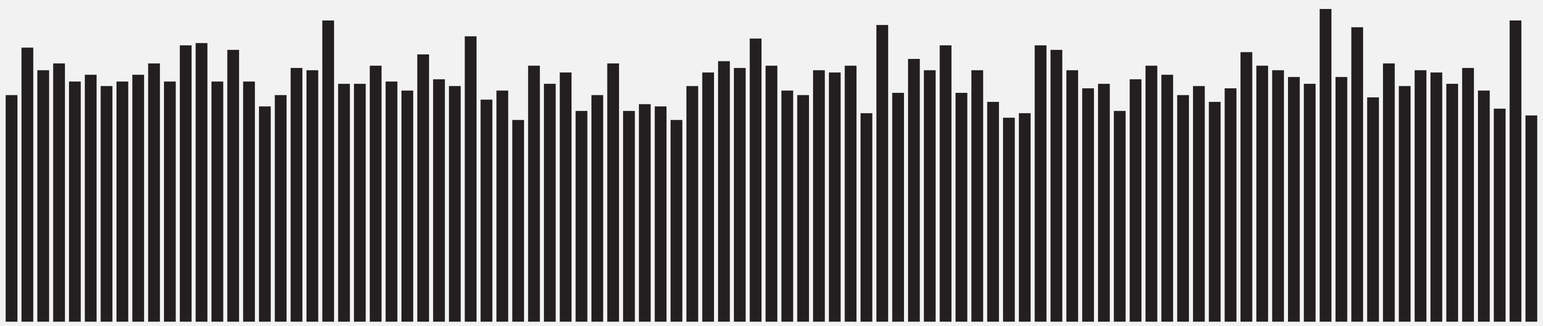
n = 11 balls; m = 16 bins

Good news: load balancing.

- When $n = m$, expect most loaded bin has $\sim \ln m / \ln \ln m$ balls.
- When $n >> m$, the number of balls in each bin is "likely" "close" to n / m.

Can be quantified and proved; see COS 340

Visual evidence.



hash value frequencies for words in Tale of Two Cities (m = 97)

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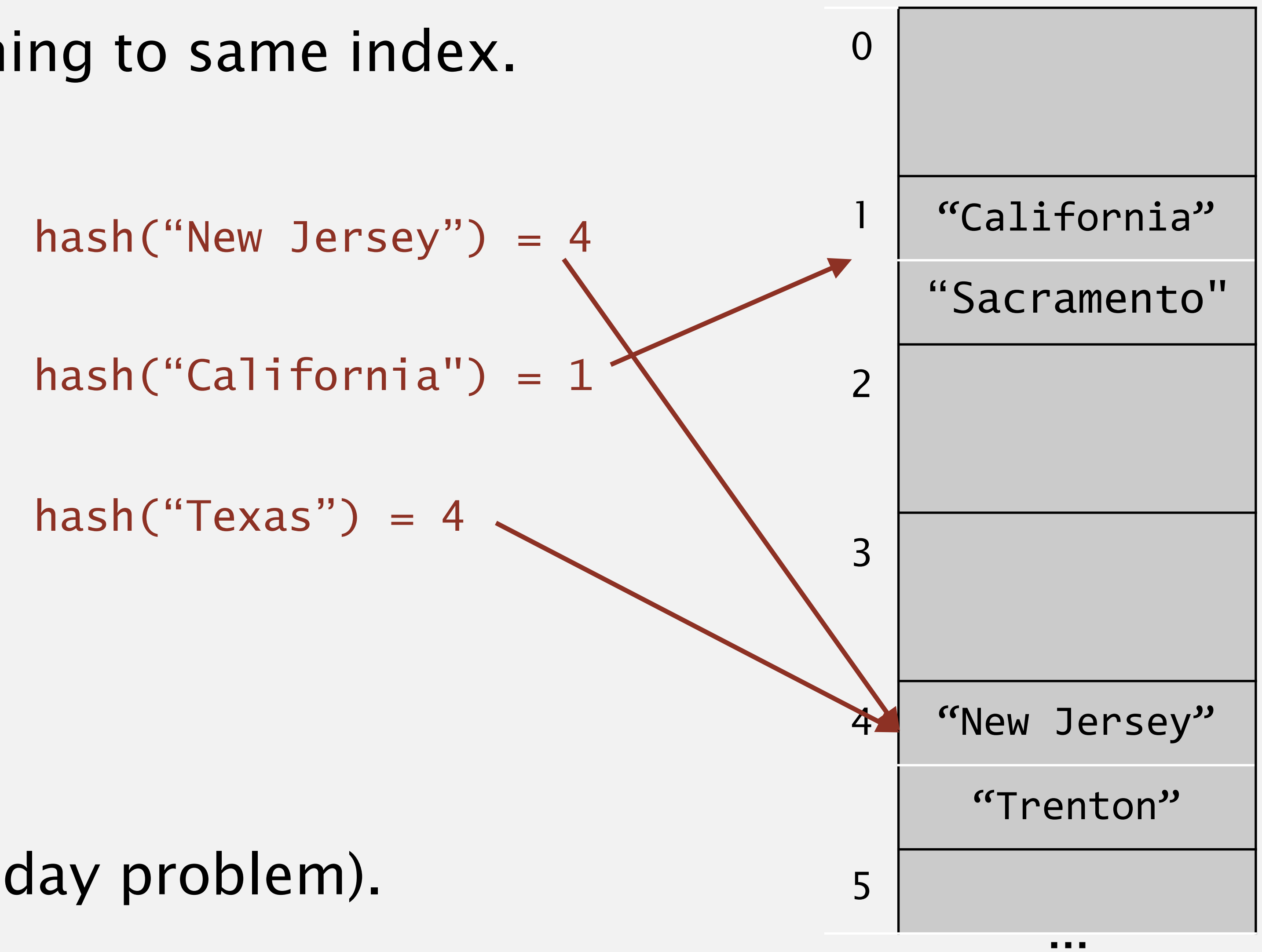
3.4 Hash Tables

- *hash functions*
- *separate chaining*
- *linear probing*
- *context*

Collisions

Collision. Two distinct keys hashing to same index.



unless you have a ridiculous (quadratic) amount of memory

Can't avoid collisions (recall birthday problem).

No index gets too many collisions (recall load balancing).

⇒ OK to do a linear search through all colliding keys.

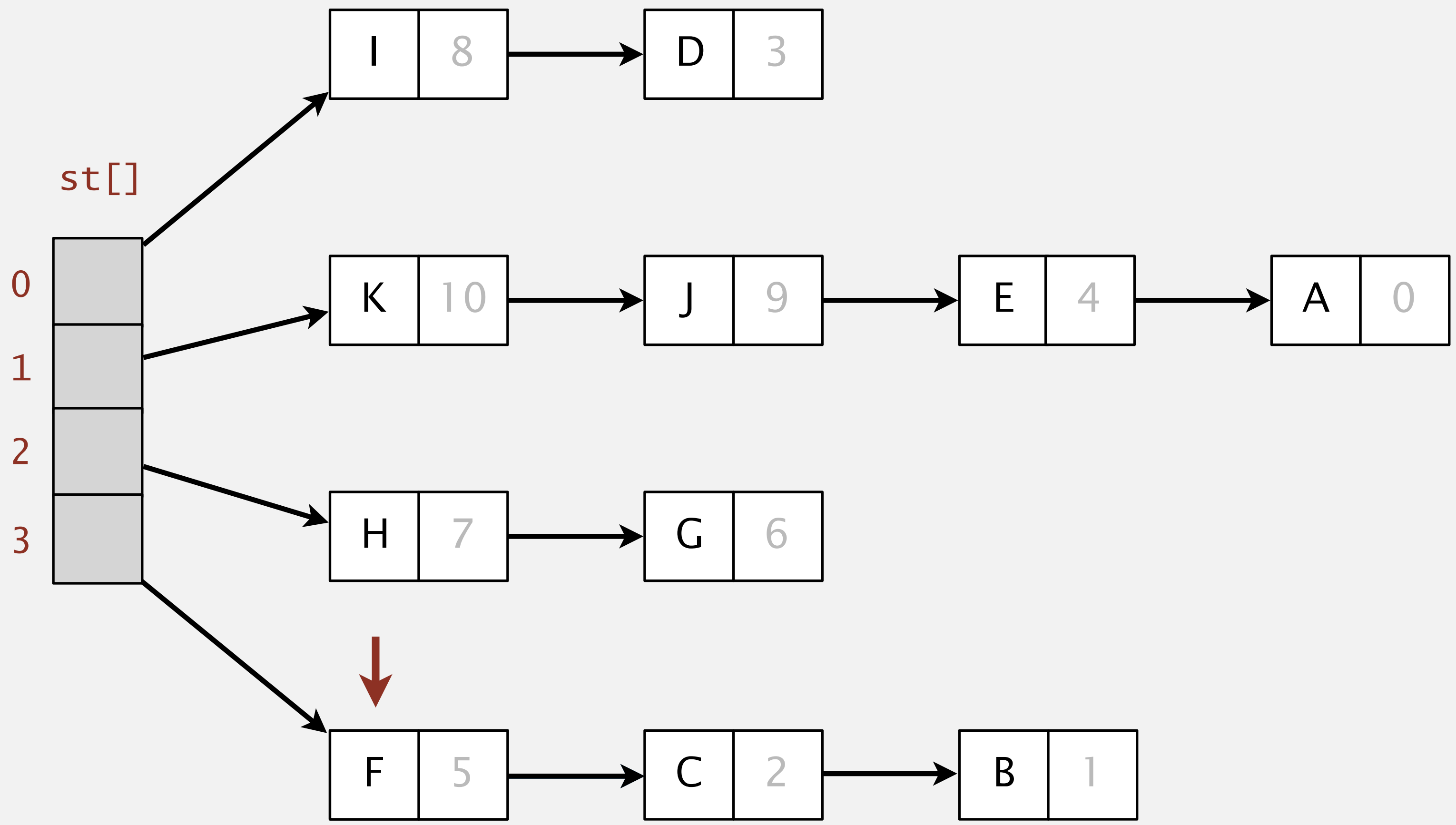
Separate-chaining symbol table

Use an array of m linked lists.

- Hash: map key to integer i between 0 and $m - 1$.
- Insert: put at front of i^{th} chain (if not already in chain).
- Search: sequential search in i^{th} chain.

put(L, 11)
hash(L) = 3

separate-chaining hash table (m = 4)



Separate-chaining symbol table

Use an array of m linked lists.

- Hash: map key to integer i between 0 and $m - 1$.
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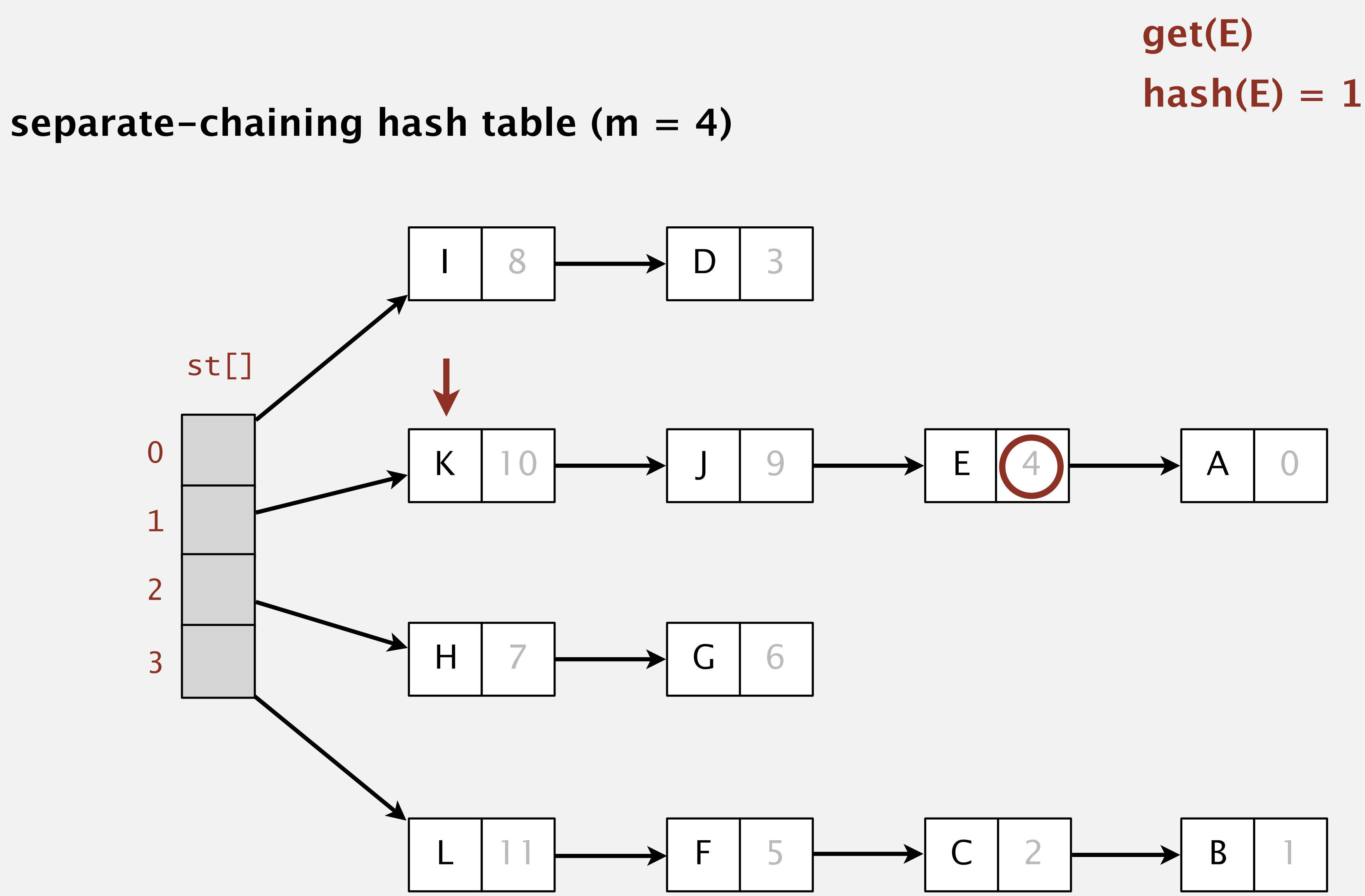


Separate-chaining symbol table: Java implementation

```
public class SeparateChainingHashST<Key, Value>
{
   private int m = 128;               // number of chains
   private Node[] st = new Node[m];   // array of chains

   private static class Node
   {
      private Object key;
      private Object val;
      private Node next;
      ...
   }

   private int hash(Key key)
   {  return (key.hashCode() & 0x7fffffff) % m;  }

   public Value get(Key key) {
      int i = hash(key);
      for (Node x = st[i]; x != null; x = x.next)
         if (key.equals(x.key)) return (Value) x.val;
      return null;
   }

}
```

array resizing code omitted

no generic array creation
(declare key and value of type Object)

Separate-chaining symbol table: Java implementation

```
public class SeparateChainingHashST<Key, Value>
{
   private int m = 128;               // number of chains
   private Node[] st = new Node[m];   // array of chains

   private static class Node
   {
      private Object key;
      private Object val;
      private Node next;
      ...
   }

   private int hash(Key key)
   {  return (key.hashCode() & 0x7fffffff) % m;  }

   public void put(Key key, Value val)
   {
      int i = hash(key);
      for (Node x = st[i]; x != null; x = x.next)
         if (key.equals(x.key)) { x.val = val; return; }
      st[i] = new Node(key, val, st[i]);
   }

}
```

Analysis of separate chaining

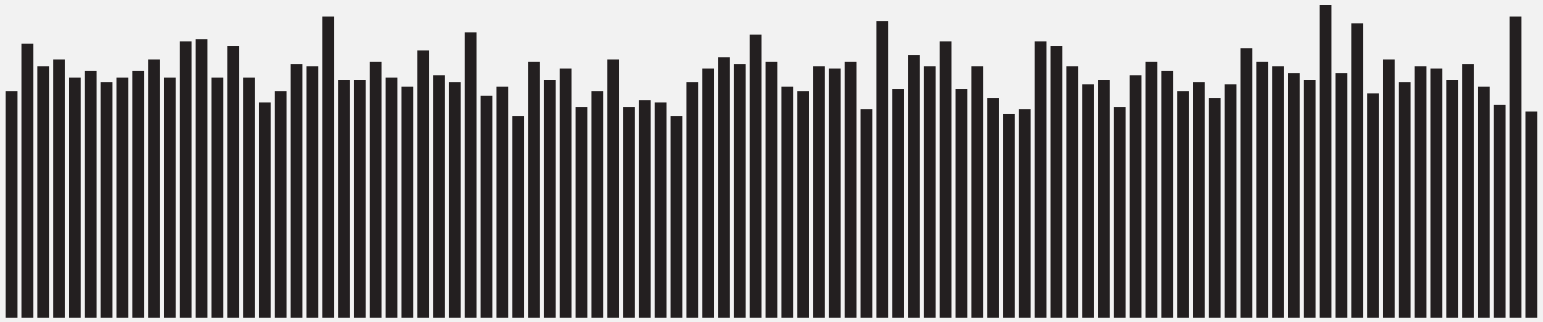
Recall. Under uniform hashing assumption, length of each chain is approximately n / m (load balancing in balls and bins model).

calls to either `equals()` or `hashCode()`

Consequence. Number of probes for search/insert is proportional to n / m.

- m too large $\Rightarrow$ too many empty chains.
- m too small $\Rightarrow$ chains too long.
- Typical choice: $m \sim \frac{1}{4} n \Rightarrow$ constant time per operation.

m times faster than sequential search



hash value frequencies for words in Tale of Two Cities (m = 97)

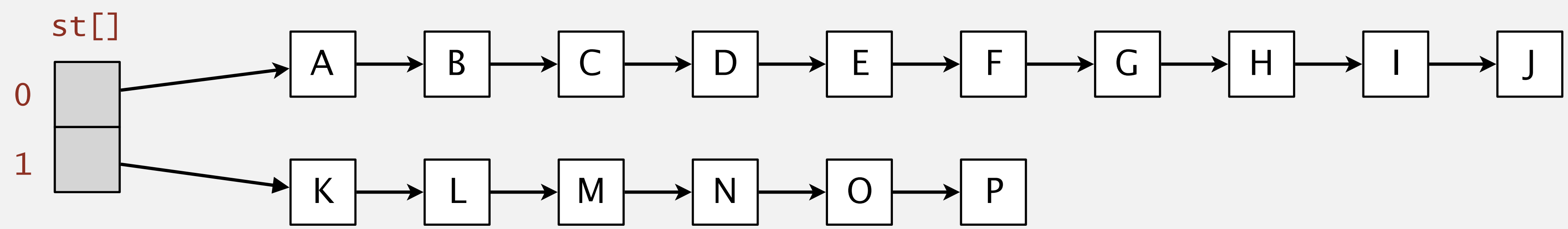
Resizing in a separate-chaining hash table

Goal. Average length of list $n \,/\, m$ = constant.

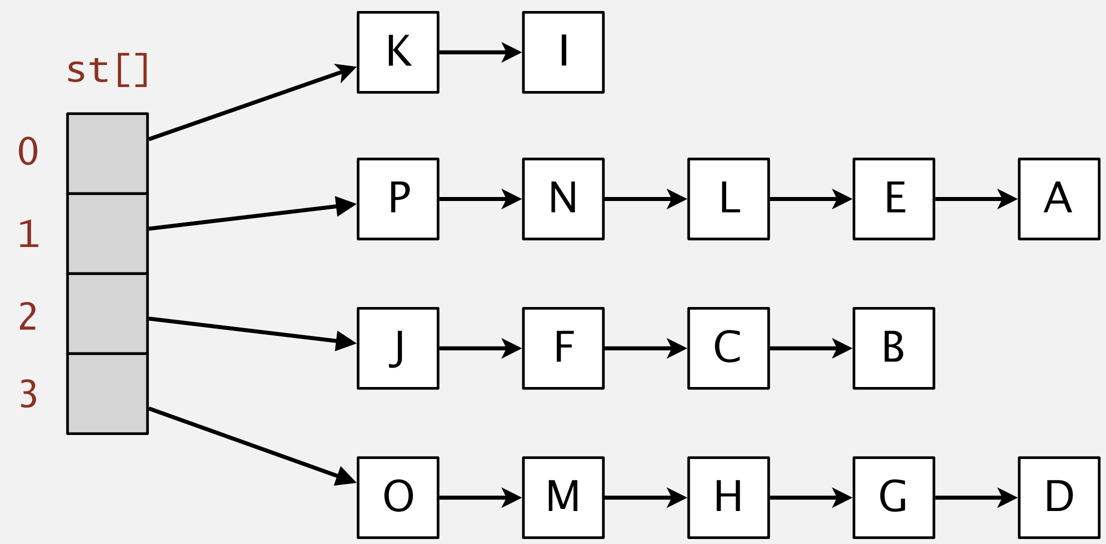
- Double length m of array when $n \,/\, m \ \geq\ 8$;
  halve length m of array when $n \,/\, m \ \leq\ 2$.
- Note: need to rehash all keys when resizing.

x.hashCode() does not change;
but hash(x) typically does

before resizing (n/m = 8)



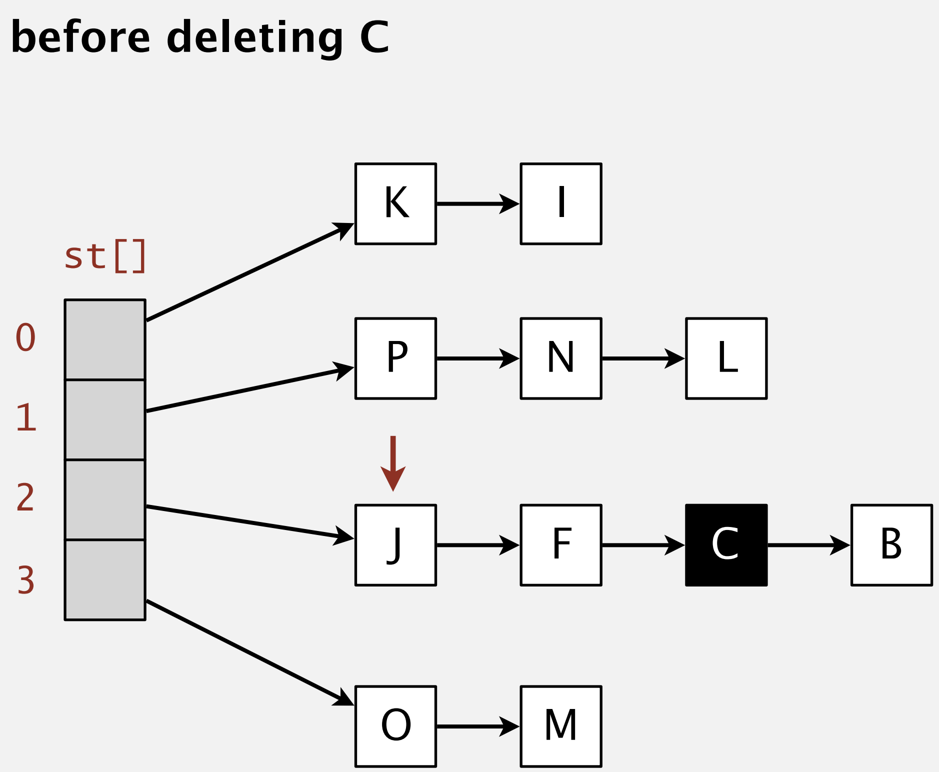
after resizing (n/m = 4)

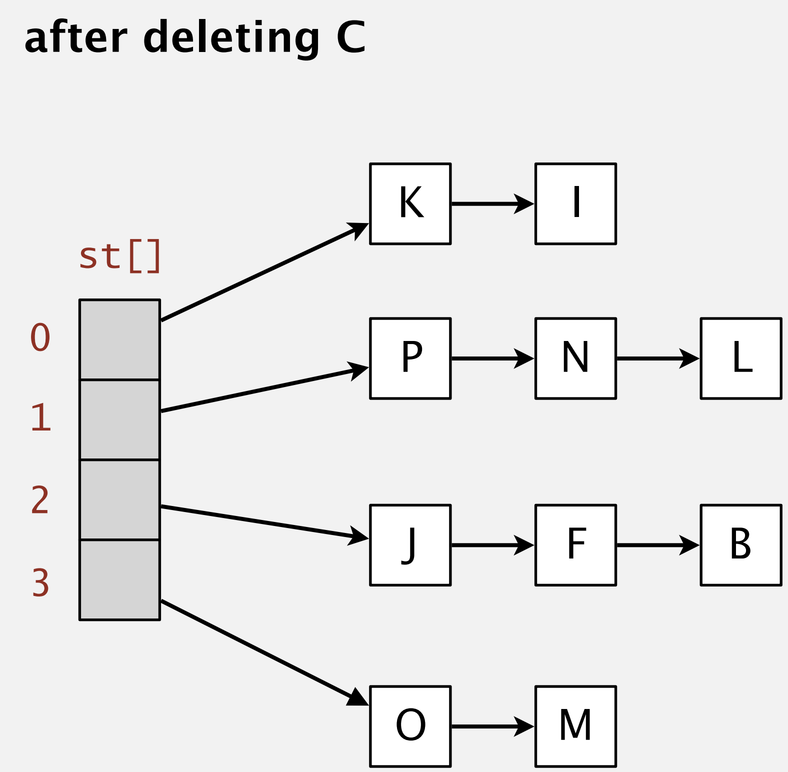


Deletion in a separate-chaining hash table

Q. How to delete a key (and its associated value)?

A. Easy: need to consider only chain containing key.





Symbol table implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
red-black BST	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`
separate chaining	n	n	n	1 †	1 †	1 †		`equals()` `hashCode()`

† under uniform hashing assumption

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3.4 Hash Tables

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- separate chaining
- *linear probing*
- context

Collision resolution: alternate approach

Open addressing.

- Maintain keys and values in two parallel arrays.
- When a new key collides, find next empty slot and put it there.

Note. If the array is full, the search doesn't terminate.

linear-probing hash table (m = 16, n =10)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
`keys[]`	P	M			A	C		H	L		E				R	X
								K								
								14								
`vals[]`	11	10			9	5		6	12		13				4	8

put(K, 14)

hash(K) = 7

Linear-probing hash table summary

Hash. Map key to integer i between 0 and $m - 1$.

Insert. Put at table index i if free; if not try $i + 1$, $i + 2$, etc.

Search. Search table index i; if occupied but no match, try $i + 1$, $i + 2$, etc.

Note. Array length m must be greater than number of key–value pairs n.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C	S	H	L		E				R	X

m = 16

Linear-probing symbol table: Java implementation

```
public class LinearProbingHashST<Key, Value>
{
   private int m = 32768;
   private Value[] vals = (Value[]) new Object[m];
   private Key[]   keys = (Key[])   new Object[m];

   private int hash(Key key)
   {  return (key.hashCode() & 0x7fffffff) % m;  }

   private void put(Key key, Value val) { /* next slide */  }

   public Value get(Key key)
   {
      for (int i = hash(key); keys[i] != null; i = (i+1) % m)
         if (key.equals(keys[i]))
            return vals[i];
      return null;
   }

}
```

array resizing code omitted

Linear-probing symbol table: Java implementation

```
public class LinearProbingHashST<Key, Value>
{
   private int m = 32768;
   private Value[] vals = (Value[]) new Object[m];
   private Key[]   keys = (Key[])   new Object[m];

   private int hash(Key key)
   {  return (key.hashCode() & 0x7fffffff) % m;  }

   private Value get(Key key) { /* prev slide */  }

   public void put(Key key, Value val)
   {
      int i;
      for (i = hash(key); keys[i] != null; i = (i+1) % m)
         if (keys[i].equals(key))
             break;
      keys[i] = key;
      vals[i] = val;
   }

}
```

Hash tables: quiz 4

Under the uniform hashing assumption, where is the next key most likely to be added in this linear-probing hash table?

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
H	I	M	N		D			A	B	E	F	G	J		K			C	L

A. Index 7

B. Index 14

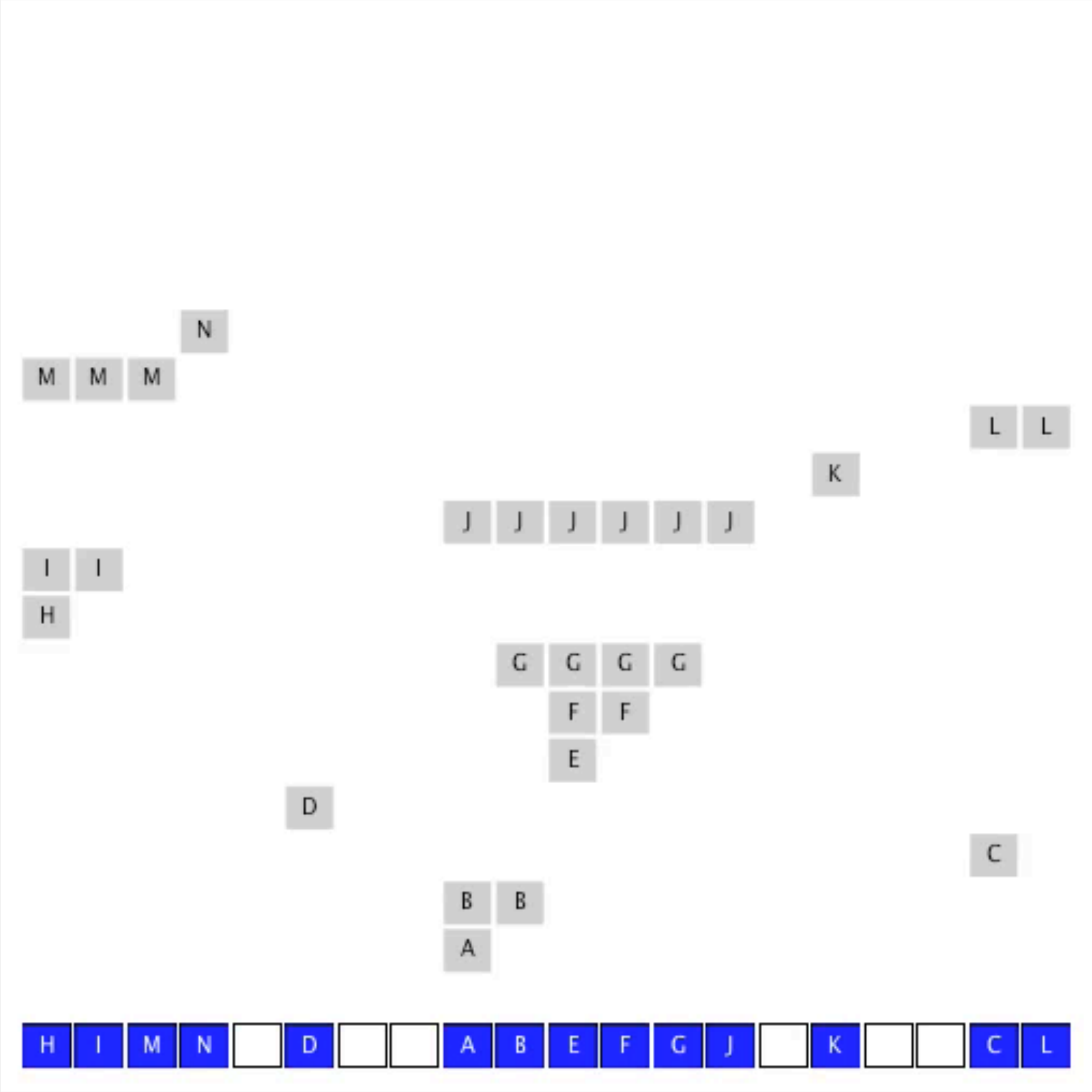
C. Index 4 or index 14

D. All open indices are equally likely

Clustering

Cluster. A contiguous block of items.

Observation. New keys likely to hash into middle of big clusters.



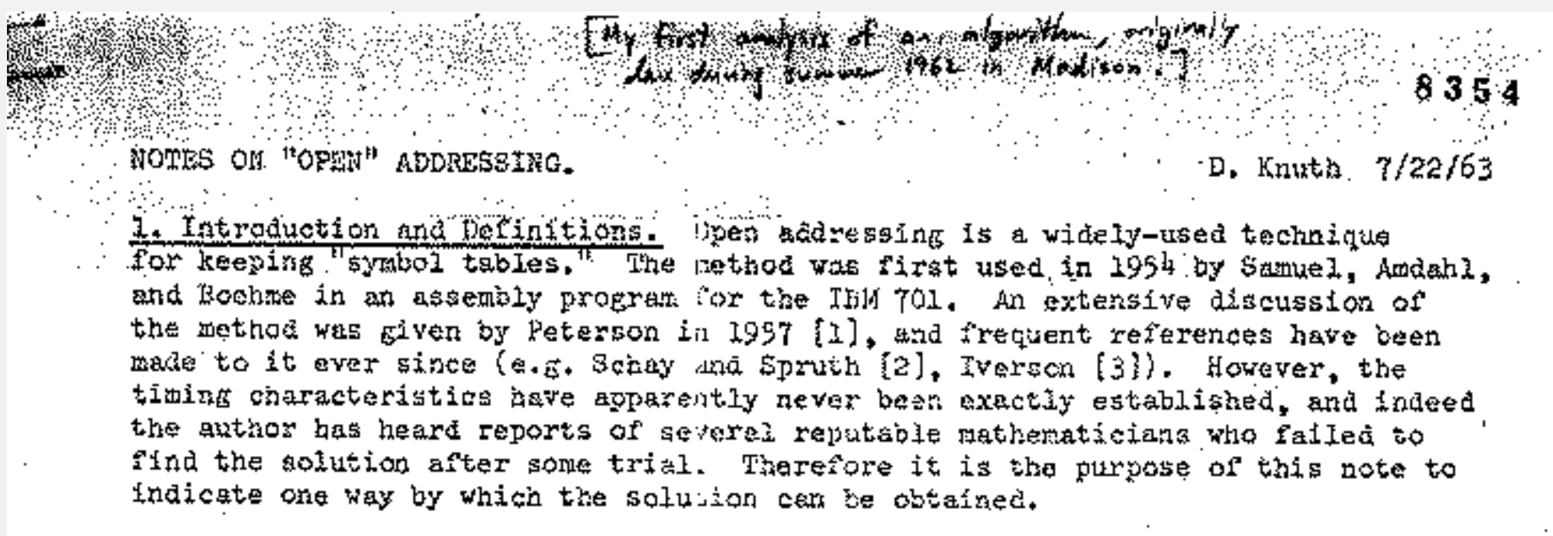
Analysis of linear probing

Proposition. Under uniform hashing assumption, the average # of probes in a linear-probing hash table of size m that contains $n = \alpha\, m$ keys is at most

$$\frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right) \qquad\qquad \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

search hit — search miss / insert

Proof. Out of scope; see:


[My first analysis of an algorithm, originally done during summer 1962 in Madison.]

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NOTES ON "OPEN" ADDRESSING. D. Knuth 7/22/63

1. Introduction and Definitions. Open addressing is a widely-used technique for keeping "symbol tables." The method was first used in 1954 by Samuel, Amdahl, and Boehme in an assembly program for the IBM 701. An extensive discussion of the method was given by Peterson in 1957 [1], and frequent references have been made to it ever since (e.g. Schay and Spruth [2], Iverson [3]). However, the timing characteristics have apparently never been exactly established, and indeed the author has heard reports of several reputable mathematicians who failed to find the solution after some trial. Therefore it is the purpose of this note to indicate one way by which the solution can be obtained.

Parameters.

- m too large $\Rightarrow$ too many empty array entries.
- m too small $\Rightarrow$ search time blows up.
- Typical choice: $\alpha = n / m \sim ½$.

probes for search hit is about 3/2
probes for search miss is about 5/2

Resizing in a linear-probing hash table

Goal. Fullness of array ("load factor") $n / m \le \frac{1}{2}$.

- Double length of array m when $n / m \ge \frac{1}{2}$.
- Halve length of array m when $n / m \le \frac{1}{8}$.
- Need to rehash all keys when resizing.

before resizing

	0	1	2	3	4	5	6	7
`keys[]`		E	S			R	A	
`vals[]`		1	0			3	2	

after resizing

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
`keys[]`					A		S				E				R	
`vals[]`					2		0				1				3	

Deletion in a linear-probing hash table

Q. How to delete a key (and its associated value)?

A. Requires some care: can't simply delete array entries.

before deleting S

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C	S	H	L		E				R	X
vals[]	10	9			8	4	0	5	11		12				3	7

after deleting S ?

doesn't work, e.g., if hash(H) = 4

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C		H	L		E				R	X
vals[]	10	9			8	4		5	11		12				3	7

ST implementations: summary

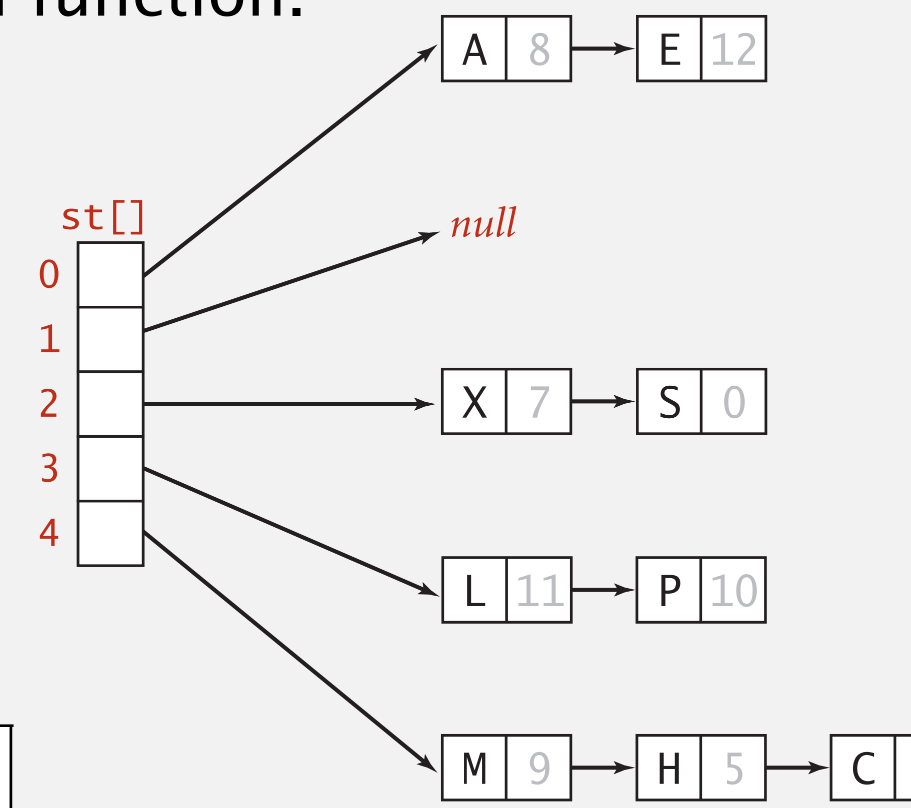
implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
red-black BST	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`
separate chaining	n	n	n	1 †	1 †	1 †		`equals()` `hashCode()`
linear probing	n	n	n	1 †	1 †	1 †		`equals()` `hashCode()`

† under uniform hashing assumption

Separate chaining vs. linear probing

Separate chaining.

- Performance degrades gracefully as number of keys increases.
- Clustering less sensitive to poorly-designed hash function.
  - Potentially fewer probes.



Linear probing.

- Less wasted space.
- Better cache performance (locality).

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
keys[]	P	M			A	C	S	H	L		E				R	X
vals[]	10	9			8	4	0	5	11		12				3	7

Hash tables vs. balanced search trees

Hash tables.

- Simpler to code.
- No effective alternative for unordered keys.
- Faster for simple keys (a few arithmetic ops versus $\log n$ compares).

Balanced search trees.

- Stronger performance guarantee.
- Support for ordered ST operations.
- Easier to implement `compareTo()` than `hashCode()`.

Java system includes both.

- Balanced search trees: `java.util.TreeMap`, `java.util.TreeSet`. ← red–black BST
- Hash tables: `java.util.HashMap`, `java.util.IdentityHashMap`.

separate chaining (`java.util.HashMap`)

linear probing (`java.util.IdentityHashMap`)