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3.3 BALANCED SEARCH TREES

- *2–3 search trees*
- *red–black BSTs*
- *B-trees (see book or videos)*

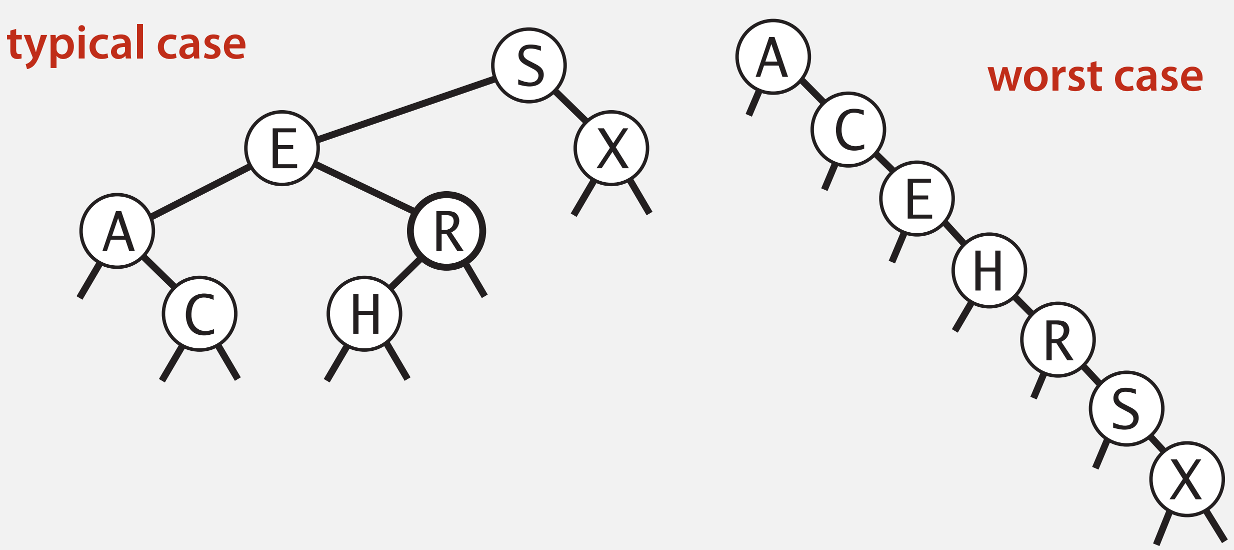
The story so far

Symbol table / dictionary / map is a fundamental data type

Naive implementations (arrays/linked lists) are way too slow.

(Binary) search trees work well in the average case, but can grow too tall (imbalanced) in the worst case

How to balance search trees?



application	purpose of search	key	value
dictionary	find definition	word	definition
book index	find relevant pages	term	list of page numbers
file share	find song to download	name of song	computer ID
financial account	process transactions	account number	transaction details
web search	find relevant web pages	keyword	list of page names
compiler	find properties of variables	variable name	type and value
routing table	route Internet packets	destination	best route
DNS	find IP address	domain name	IP address
reverse DNS	find domain name	IP address	domain name
genomics	find markers	DNA string	known positions
file system	find file on disk	filename	location on disk

Symbol table review

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
goal	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`

Challenge. Guarantee performance.

optimized for teaching and coding;
introduced to the world in this course!

This lecture. 2–3 trees and left-leaning red–black BSTs.

co-invented by Bob Sedgewick

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3.3 Balanced Search Trees

- *2–3 search trees*
- *red–black BSTs*
- *B-trees*

2-3 tree

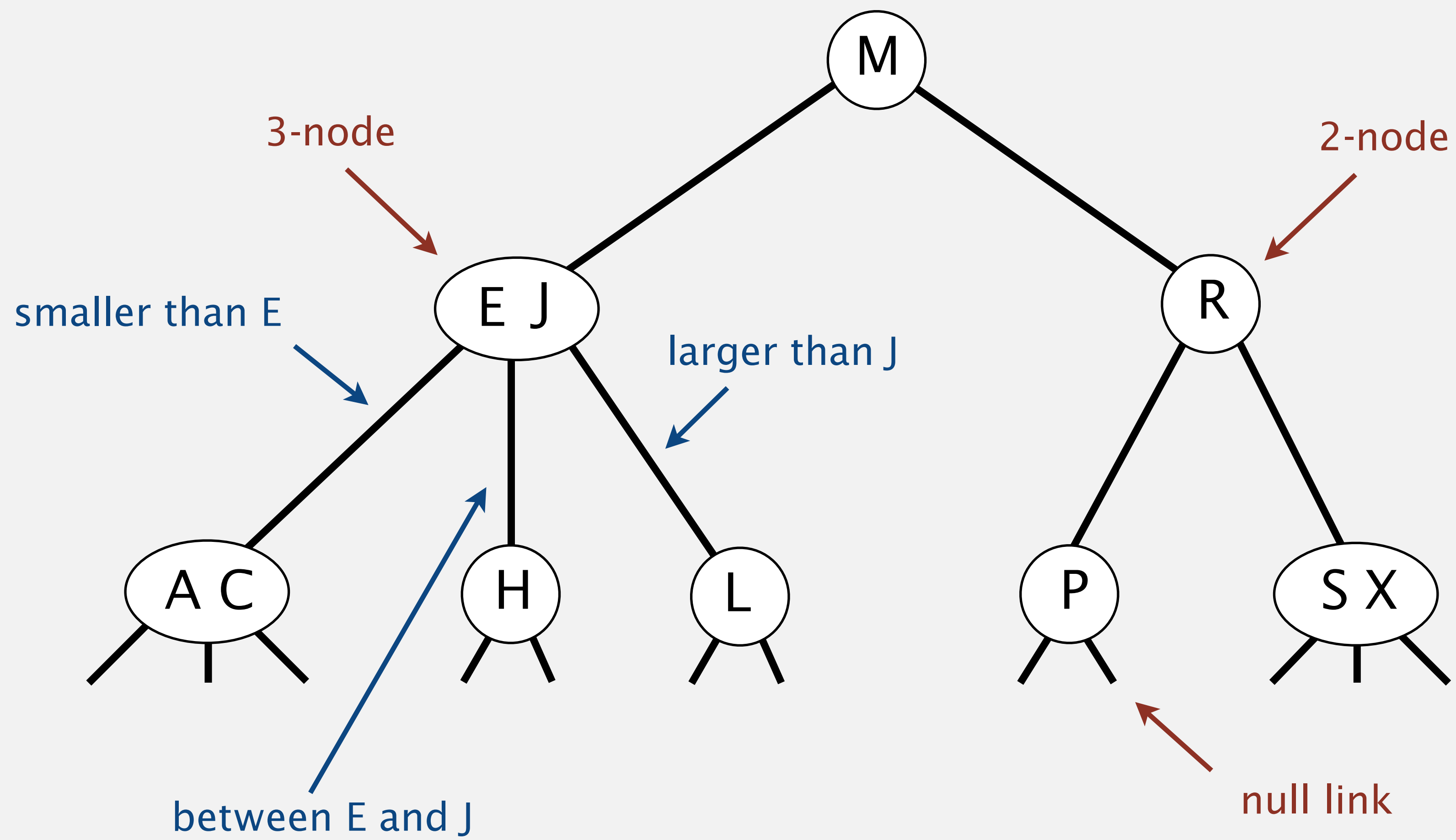
Allow 1 or 2 keys per node.

- 2-node: one key, two children.
- 3-node: two keys, three children.

Symmetric order. Inorder traversal yields keys in ascending order.

Perfect balance. Every path from root to null link has same length.

how to maintain?

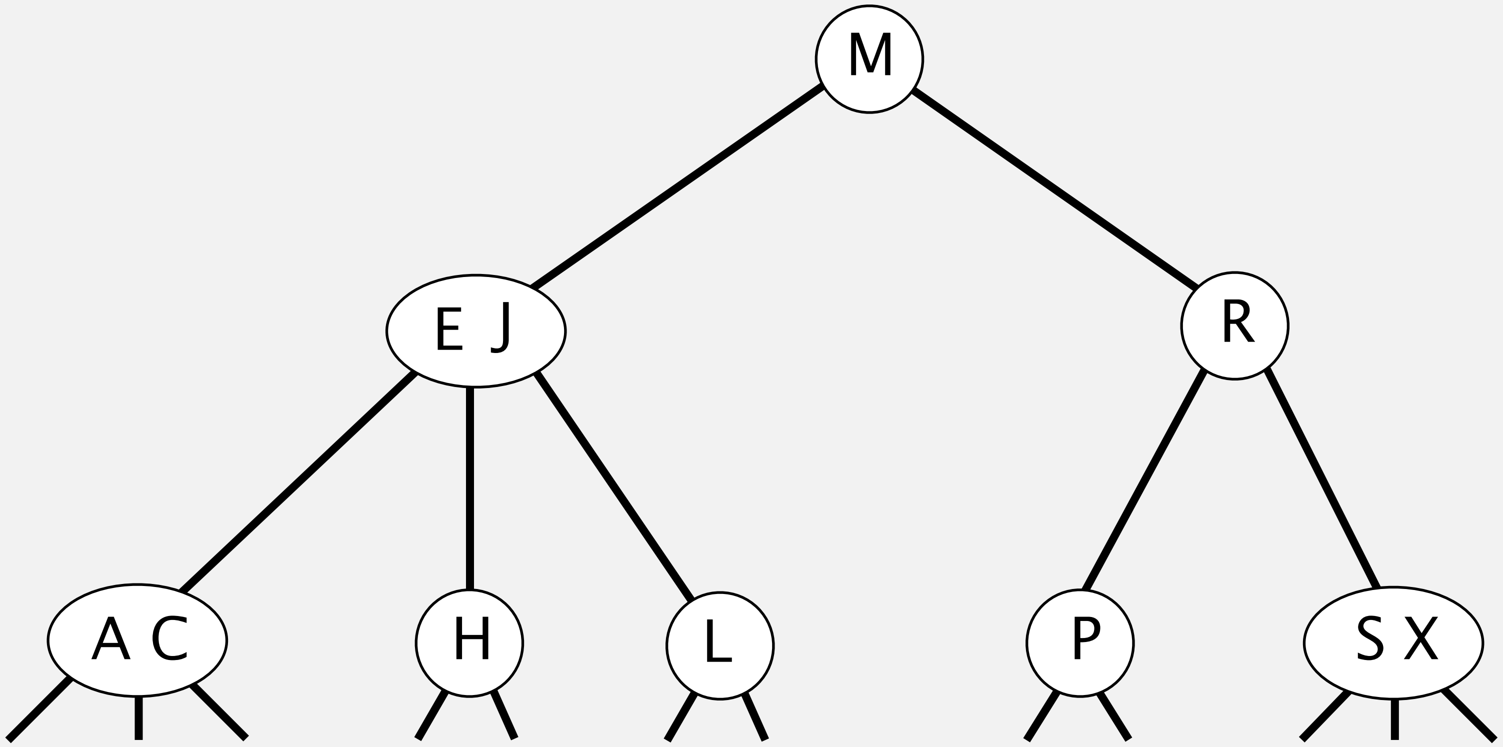


2-3 tree demo

Search.

- Compare search key against key(s) in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for H



2-3 tree demo: search

Search.

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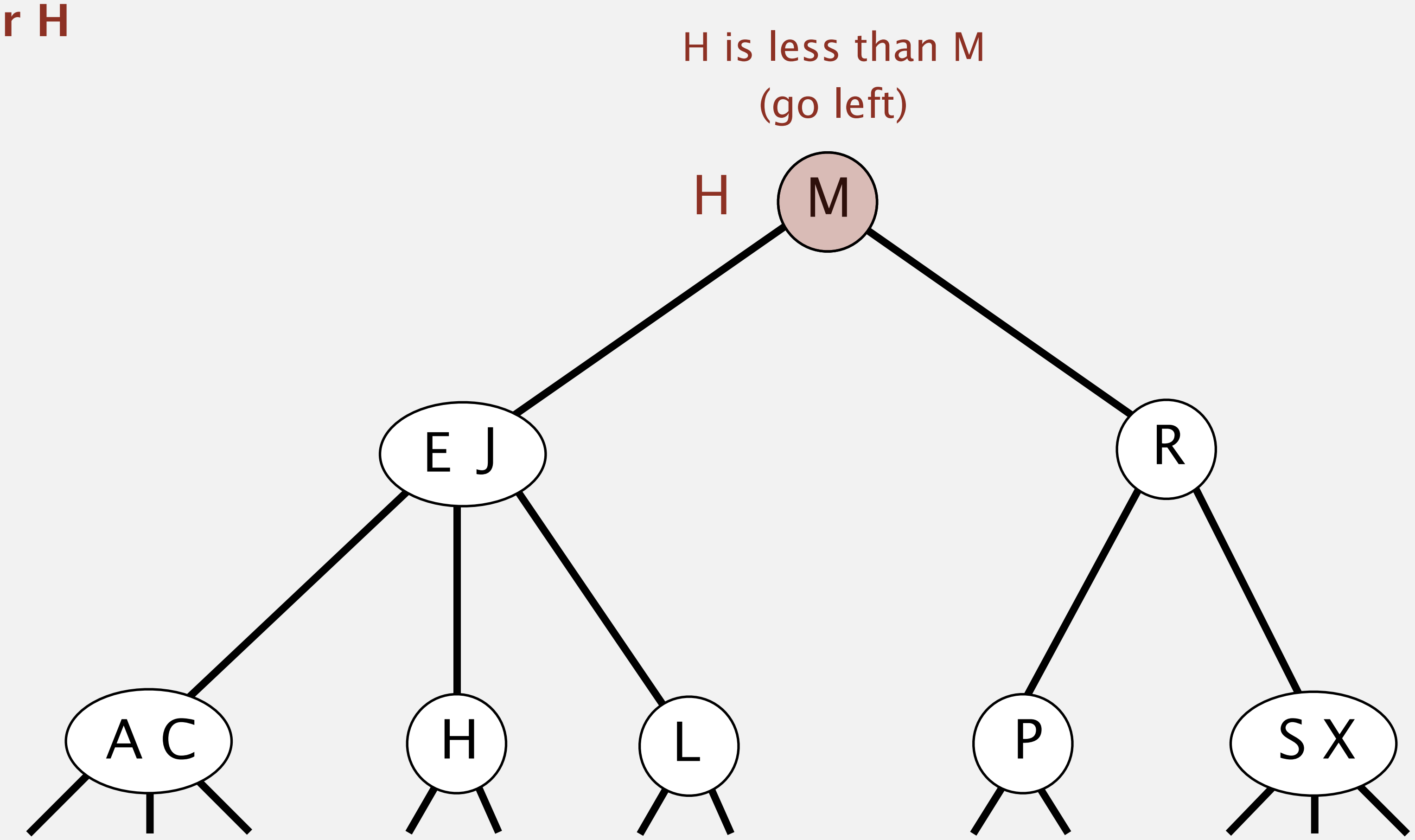
search for H



2-3 tree demo: search

Search.

- Compare search key against key(s) in node.
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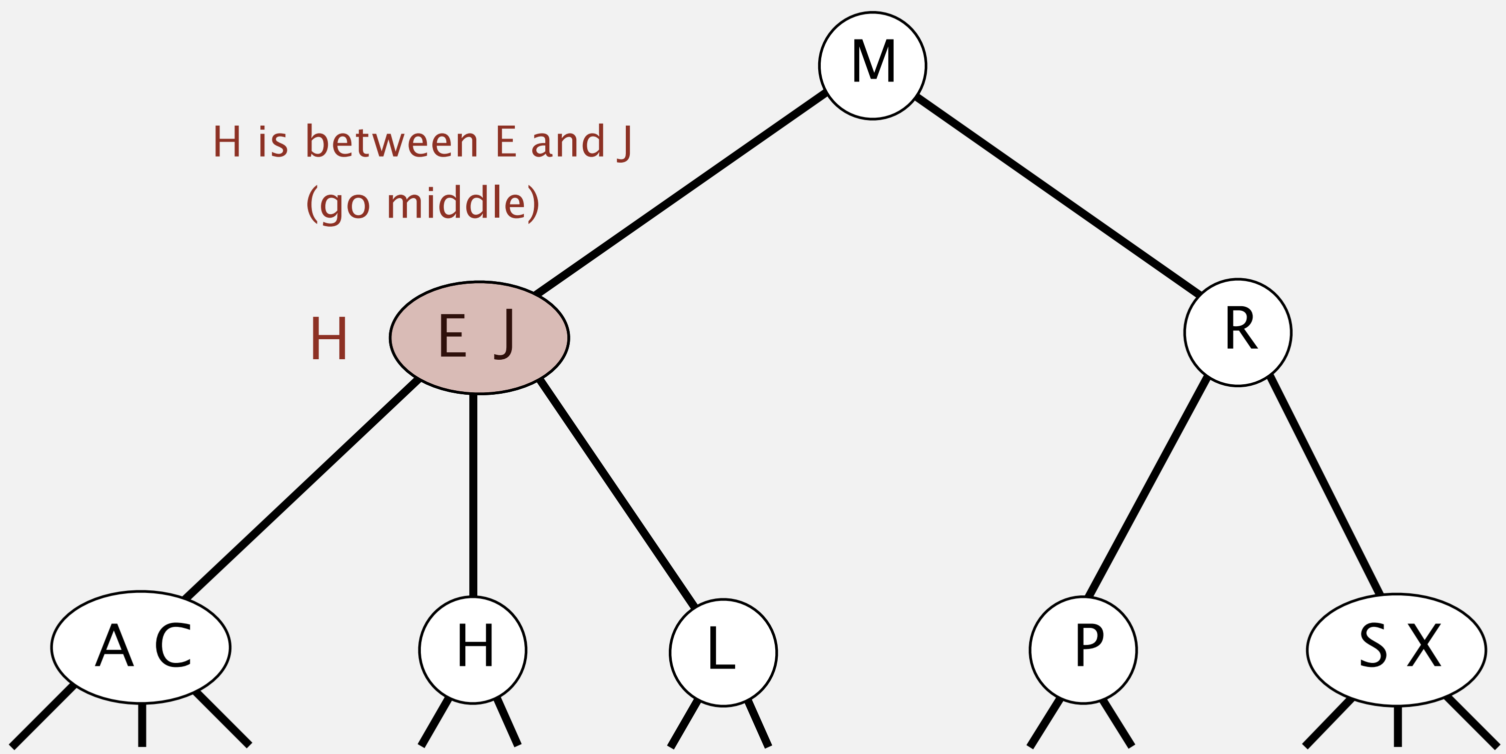
search for H

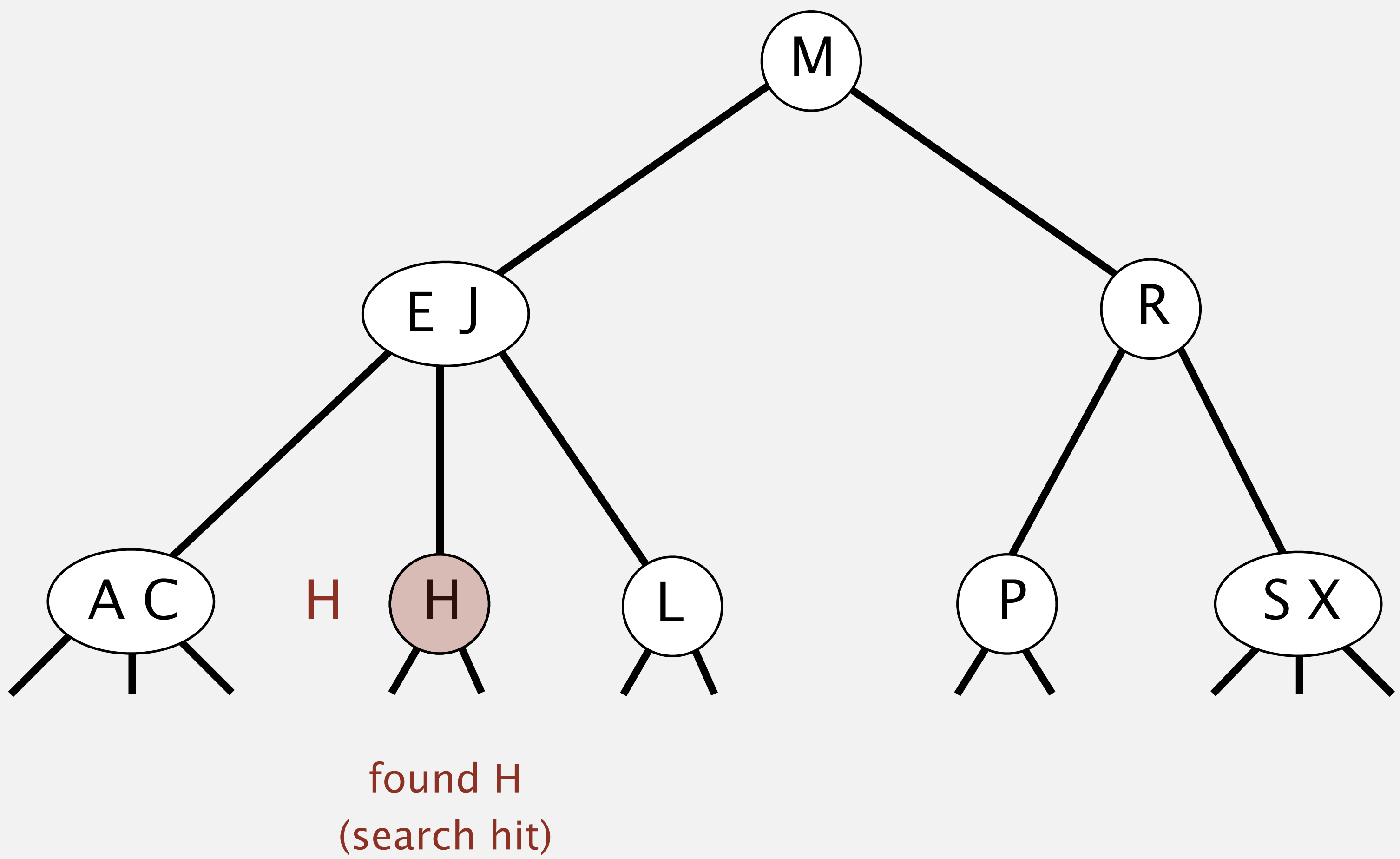


2-3 tree demo: search

Search.

- Compare search key against key(s) in node.
- Find interval containing search key.
- Follow associated link (recursively).

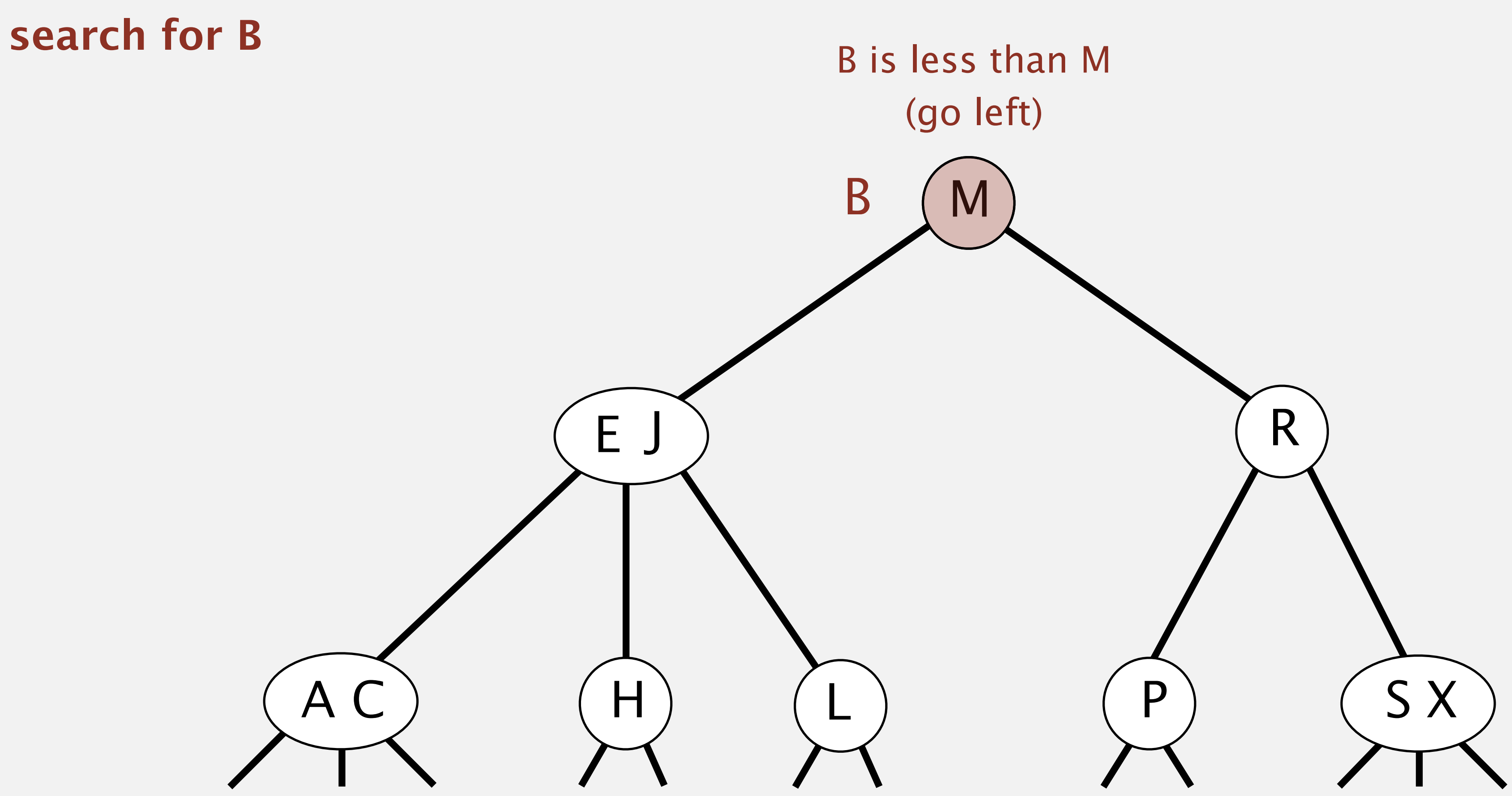
search for H



2-3 tree demo: search

Search.

- Compare search key against key(s) in node.
- Find interval containing search key.
- Follow associated link (recursively).

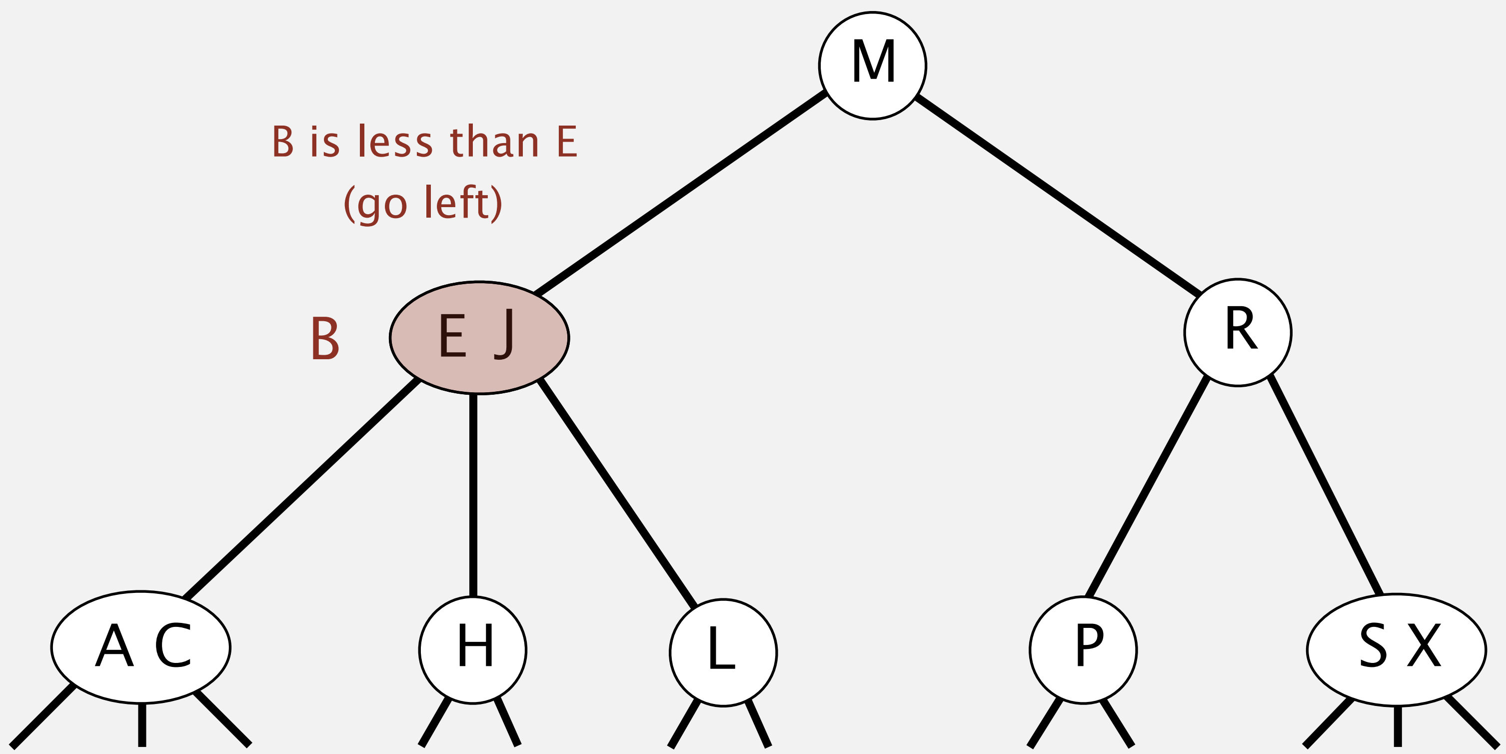


2-3 tree demo: search

Search.

- Compare search key against key(s) in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for B

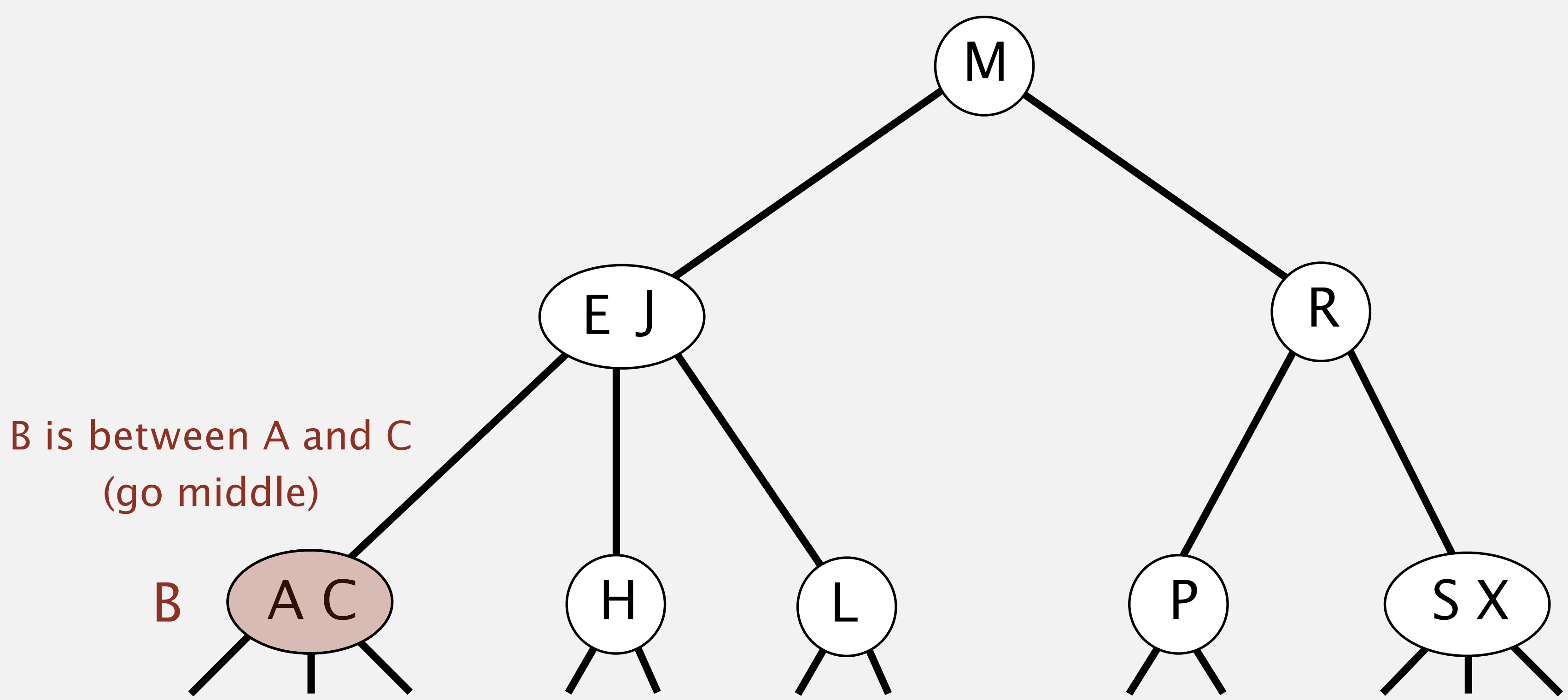


2–3 tree demo: search

Search.

- Compare search key against key(s) in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for B

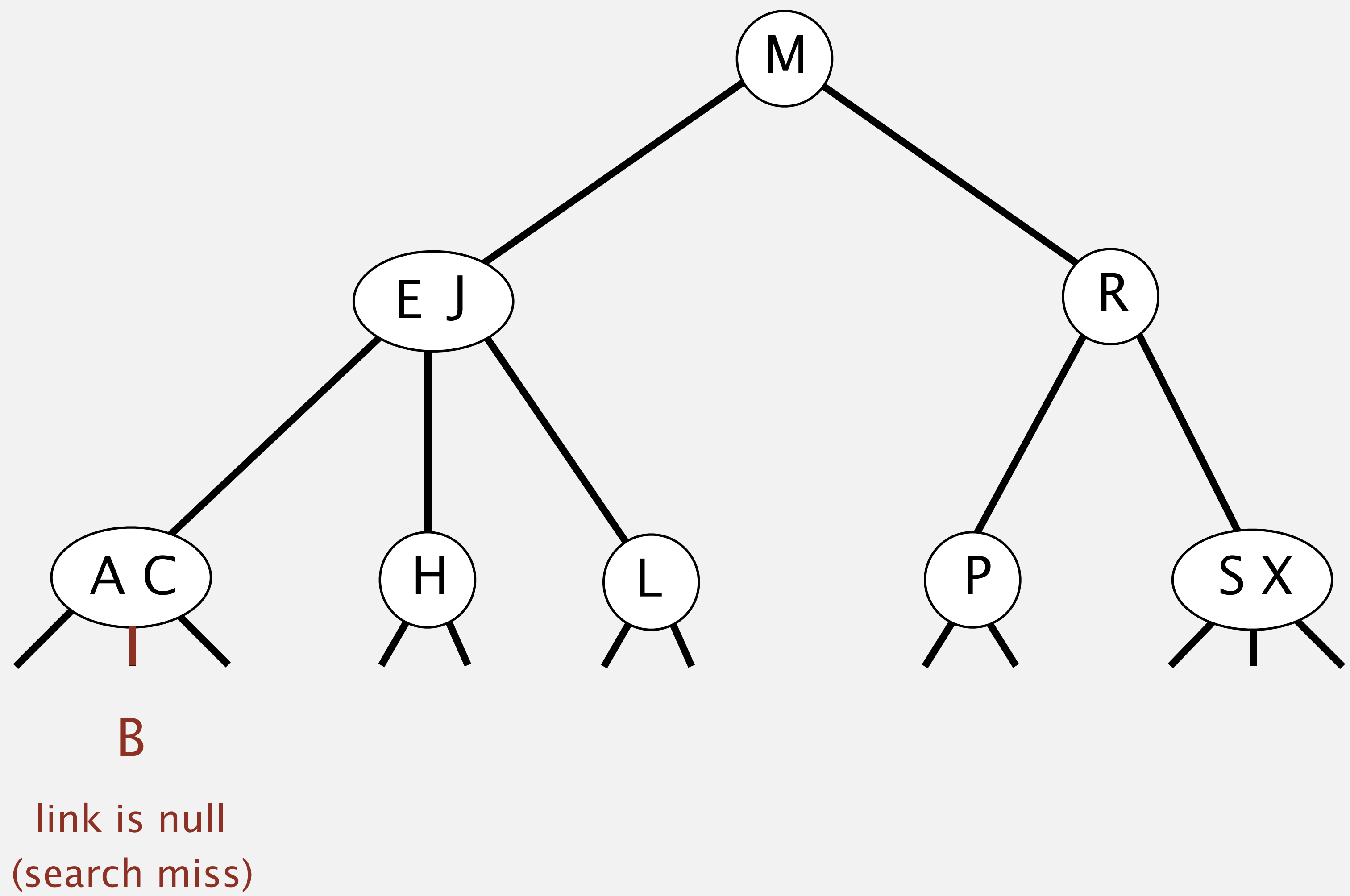


2-3 tree demo: search

Search.

- Compare search key against key(s) in node.
- Find interval containing search key.
- Follow associated link (recursively).

search for B

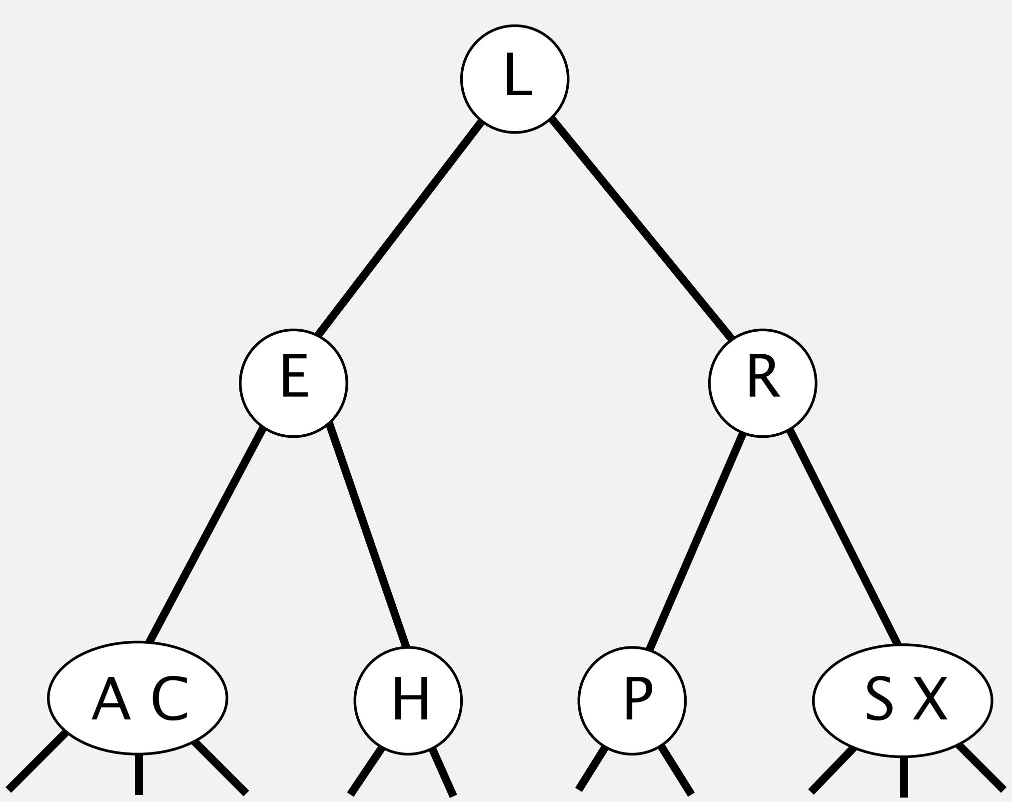


2-3 tree: insertion

Insertion into a 2-node at bottom.

- Add new key to 2-node to create a 3-node.

insert G

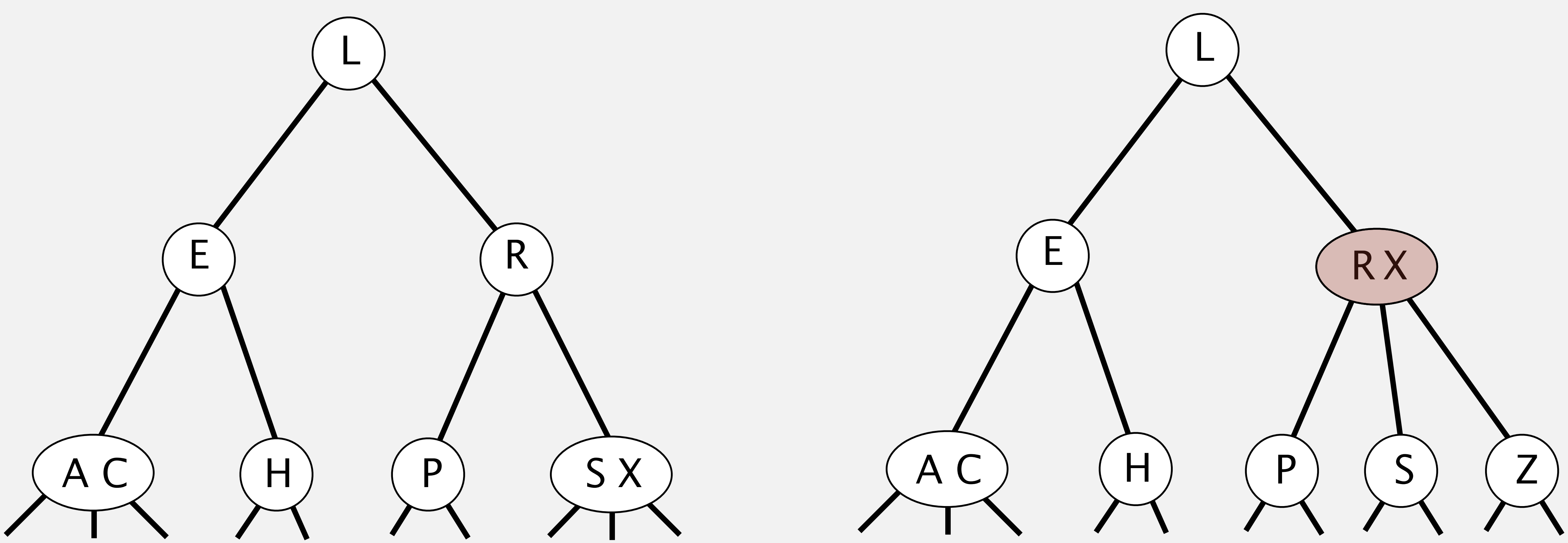


2–3 tree: insertion

Insertion into a 3-node at bottom.

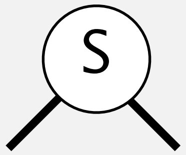
- Add new key to 3-node to create temporary 4-node.
- Move middle key in 4-node into parent.
- Repeat up the tree, as necessary.
- If you reach the root and it's a 4-node, split it into three 2-nodes.

insert Z



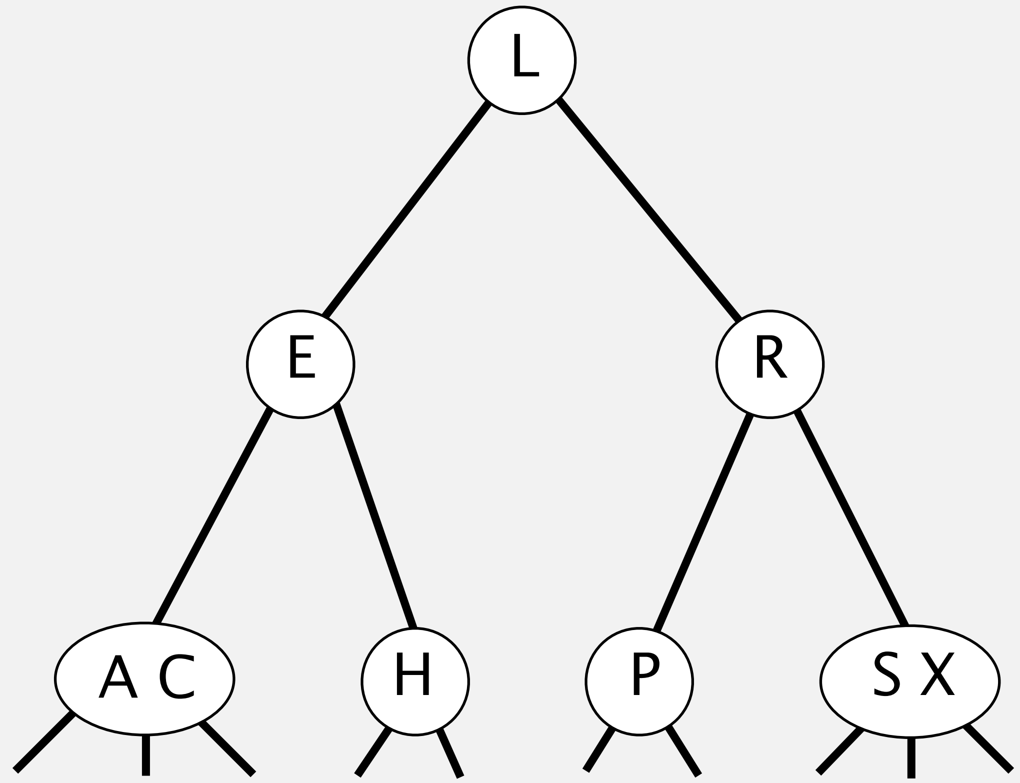
2–3 tree construction demo

insert S



2–3 tree construction demo

2–3 tree



Balanced search trees: quiz 1

What is the maximum height of a 2–3 tree with n keys?

A. $\sim \log_3 n$

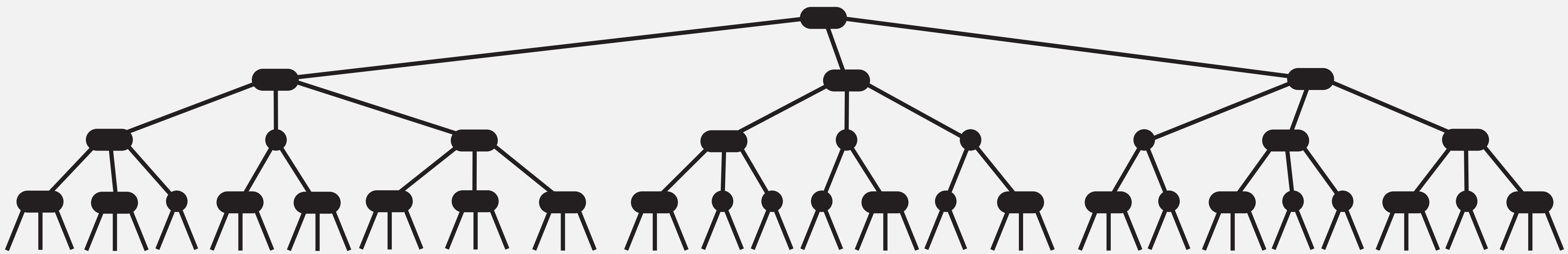
B. $\sim \log_2 n$

C. $\sim 2 \log_2 n$

D. $\sim n$

2-3 tree: performance

Perfect balance. Every path from root to null link has same length.



Tree height.

- Worst case: $\lg n$. [all 2-nodes]
- Best case: $\log_3 n \approx .631 \lg n$. [all 3-nodes]
- Between 12 and 20 for a million nodes.
- Between 18 and 30 for a billion nodes.

Bottom line. Guaranteed logarithmic performance for search and insert.

ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
2-3 tree	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`

but hidden constant c is large
(depends upon implementation)

2-3 tree: implementation?

Direct implementation is complicated, because:

- Maintaining multiple node types is cumbersome.
- Need multiple compares to move down tree.
- Need to move back up the tree to split 4-nodes.
- Large number of cases for splitting.

Bottom line. Could do it, but there's a better way.

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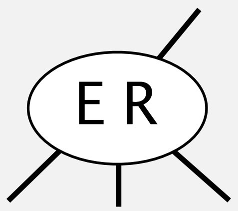
3.3 Balanced Search Trees

- 2–3 search trees
- *red–black BSTs*
- B-trees

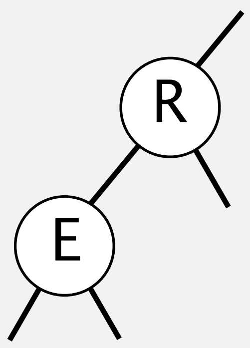
How to implement 2–3 trees with binary trees?

Challenge. How to represent a 3 node?



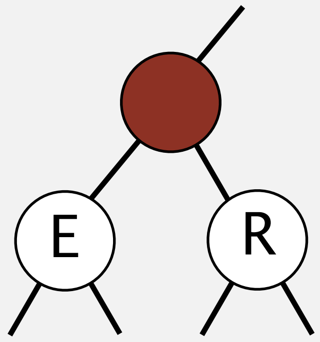
Approach 1. Regular BST.

- No way to tell a 3-node from two 2-nodes.
- Can't (uniquely) map from BST back to 2–3 tree.



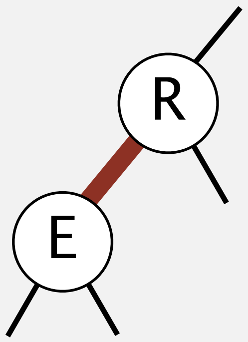
Approach 2. Regular BST with red "glue" nodes.

- Wastes space for extra node.
- Messy code.



Approach 3. Regular BST with red "glue" links.

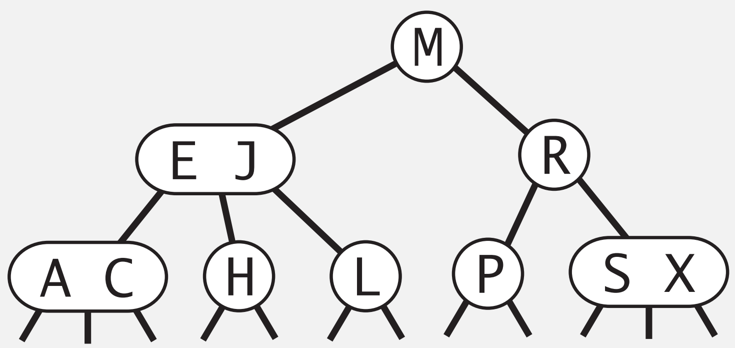
- Widely used in practice.
- Arbitrary restriction: red links lean left.



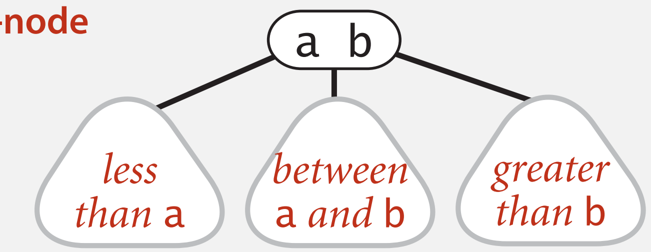
Left-leaning red–black BSTs: 1–1 correspondence with 2–3 trees

Key property. 1–1 correspondence between 2–3 and LLRB.

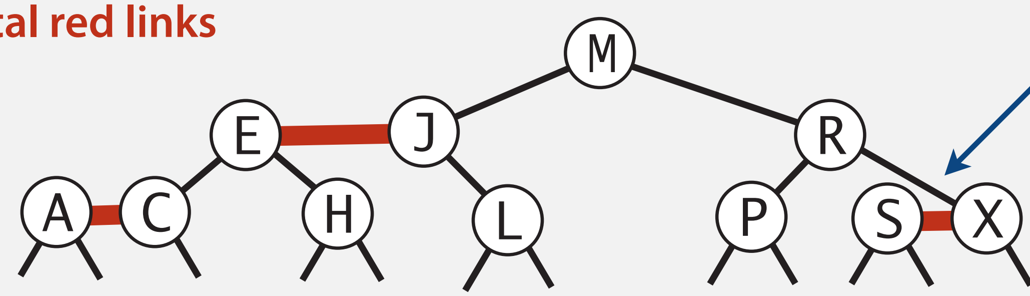
2-3 tree



3-node

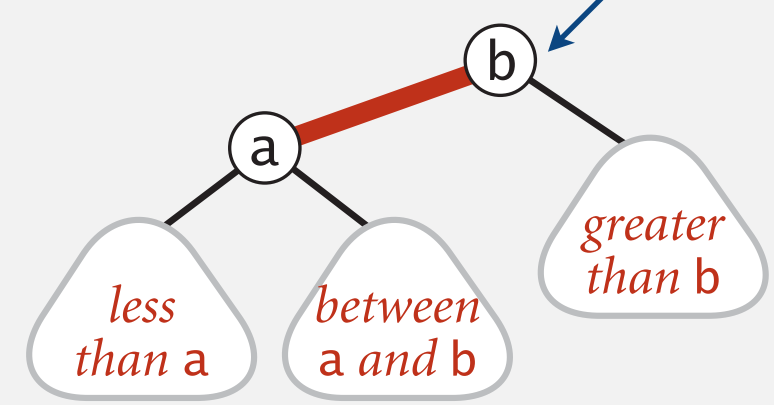


horizontal red links

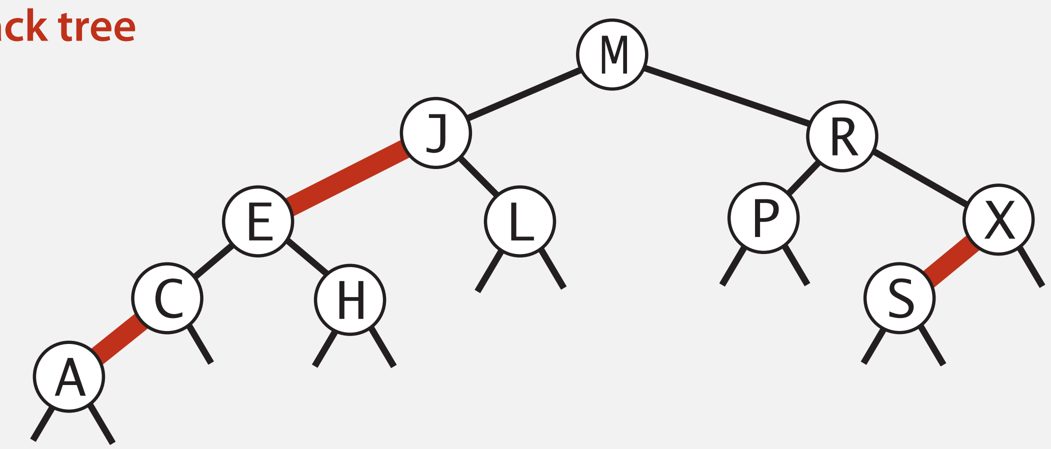


red links "glue" nodes within a 3-node

larger key is root

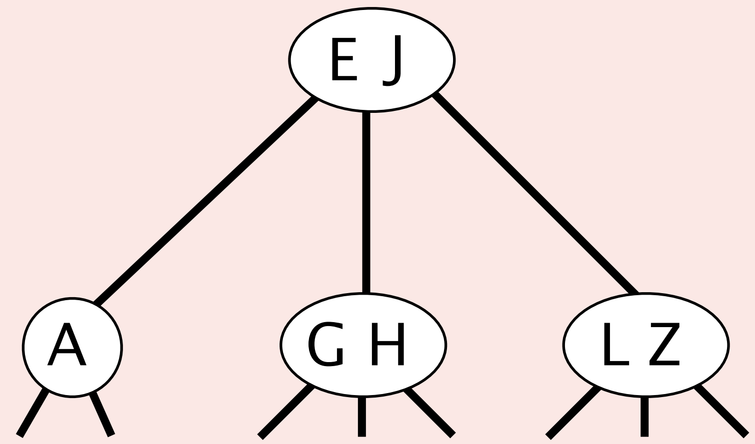


red–black tree

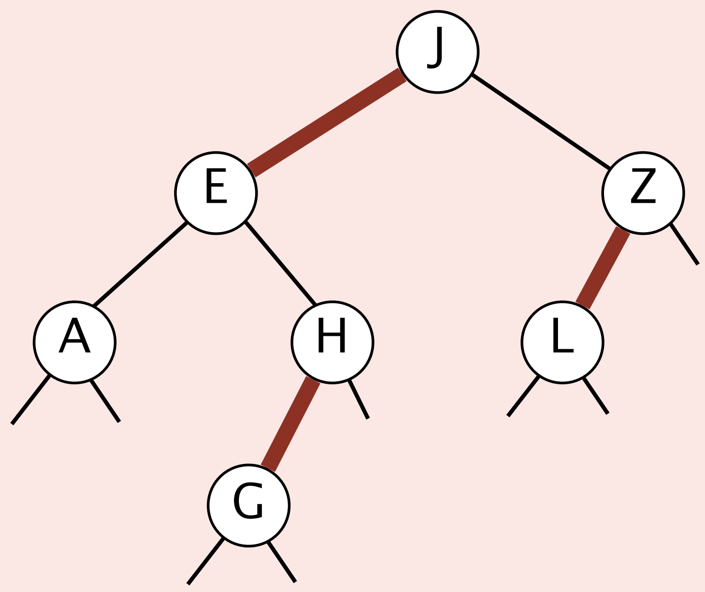


Balanced search trees: quiz 2

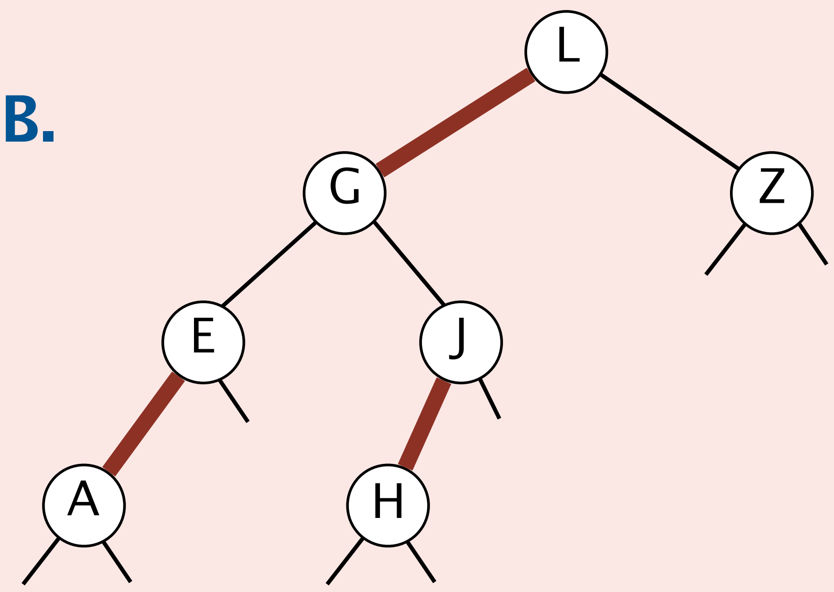
Which LLRB tree corresponds to the following 2–3 tree?



A.



B.



C. Both A and B.

D. Neither A nor B.

An equivalent definition of LLRB trees (without reference to 2-3 trees)

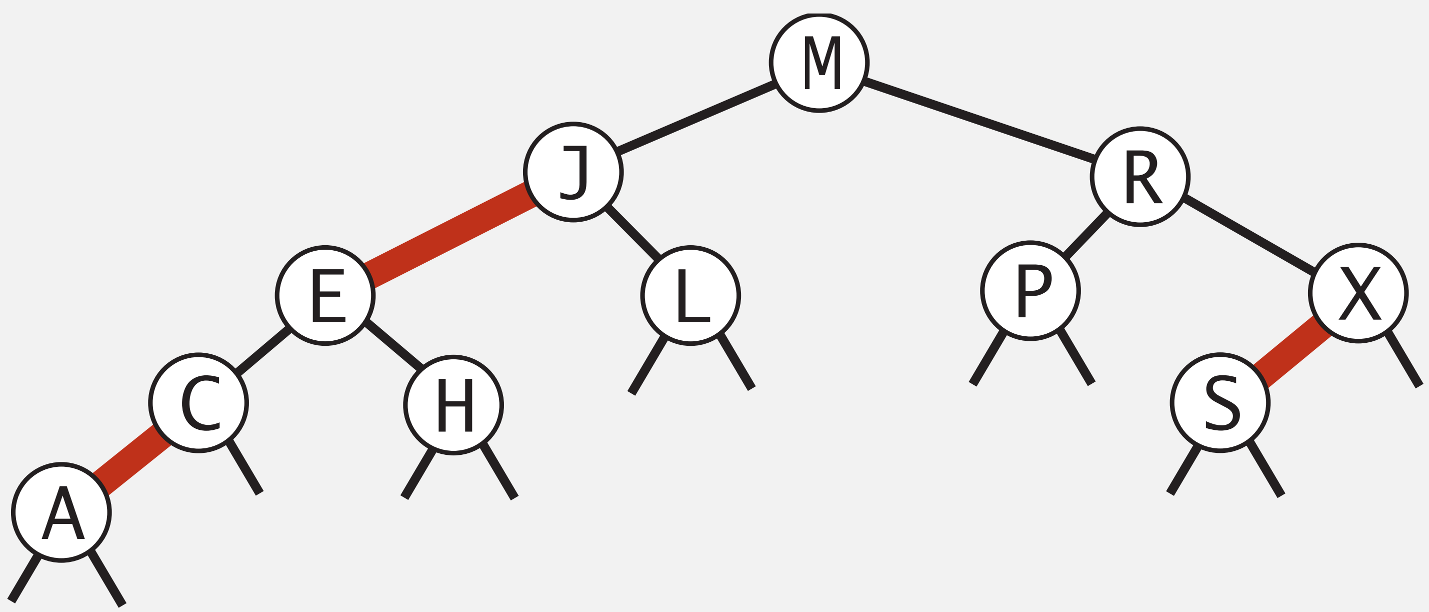
symmetric order

A BST such that:

- No node has two red links connected to it.
- Red links lean left.
- Every path from root to null link has the same number of black links.

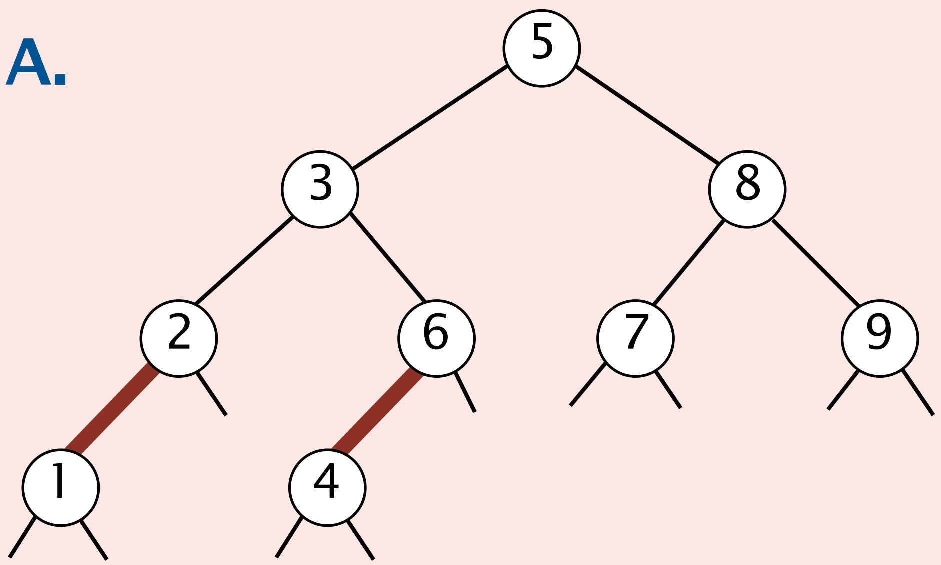
color invariants

"perfect black balance"

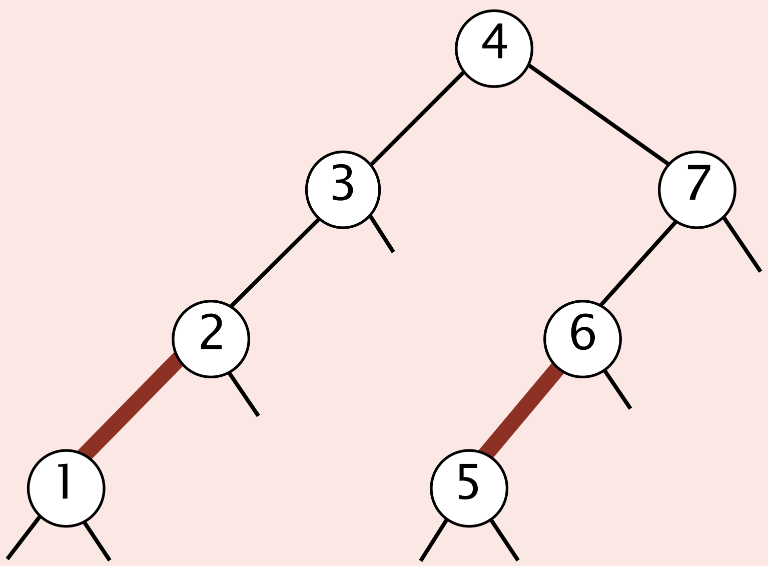


Which one of the following is a red–black BST?

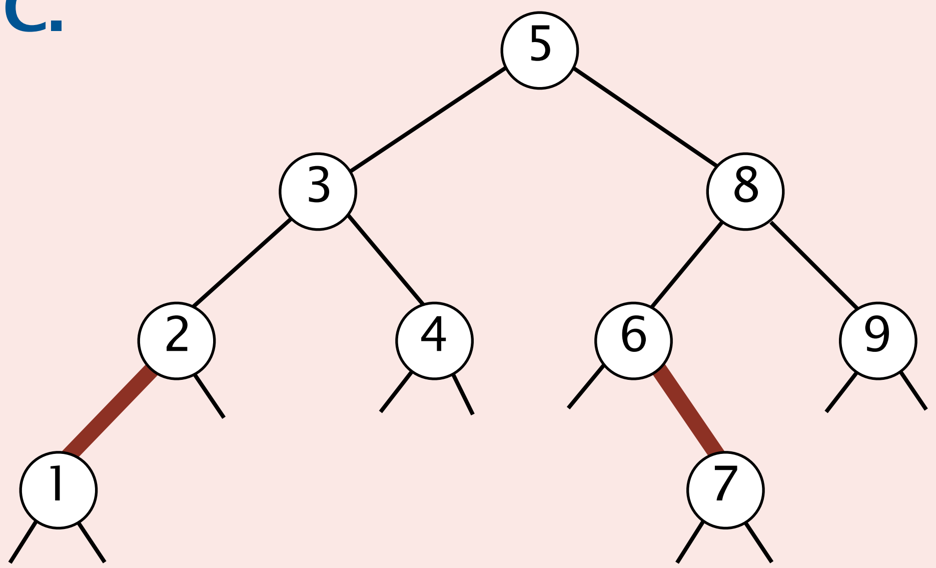
A.



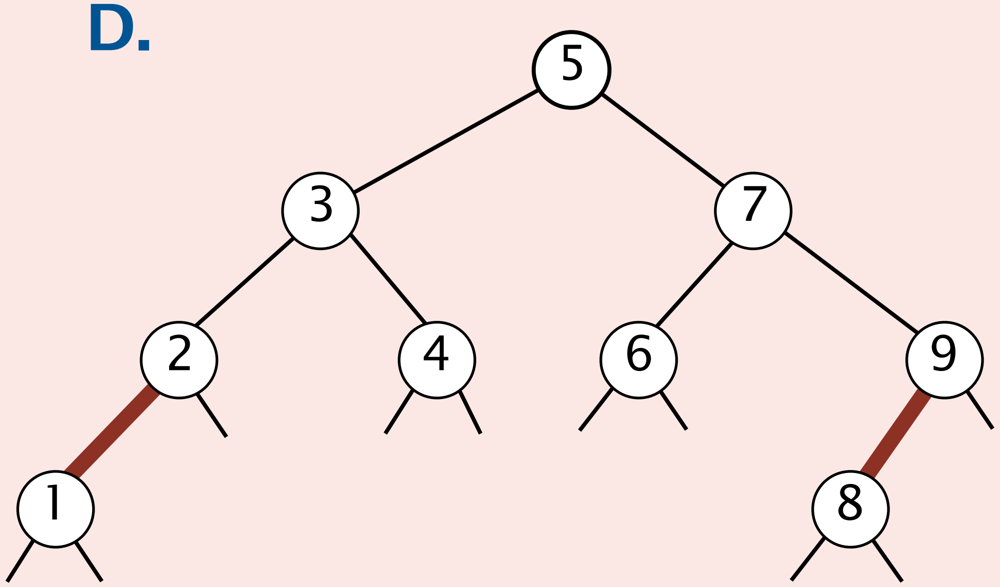
B.



C.



D.

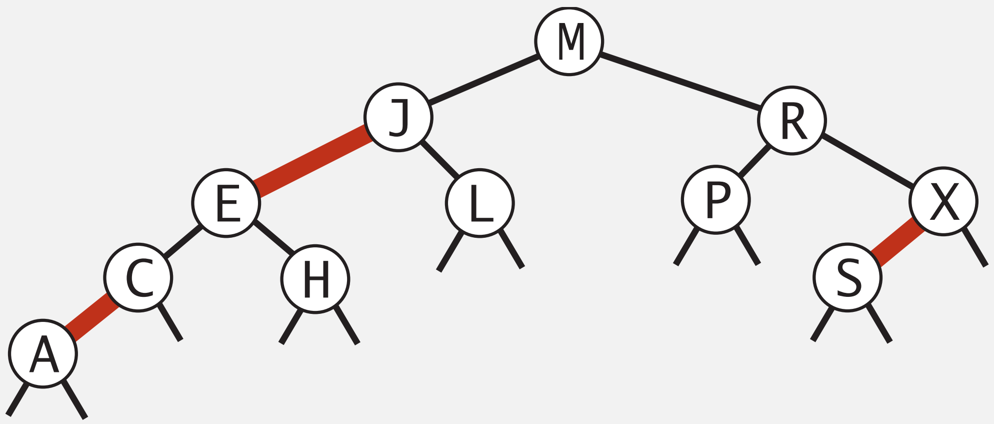


Search implementation for red-black BSTs

Observation. Search is the same as for elementary BST (ignore color).

but runs faster
(because of better balance)

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if       (cmp  < 0) x = x.left;
      else if (cmp  > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```



Remark. Many other ops (floor, iteration, rank, selection) are also identical.

Red-black BST representation

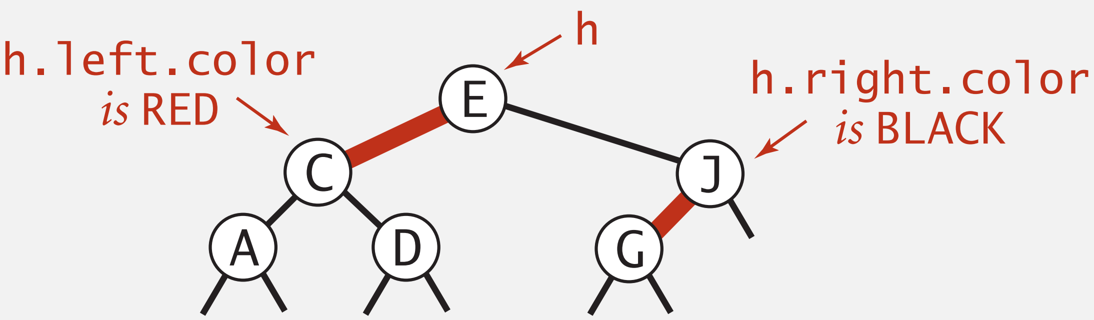
Each node is pointed to by precisely one link (from its parent) ⇒ can encode color of links in nodes.

```
private static final boolean RED   = true;
private static final boolean BLACK = false;

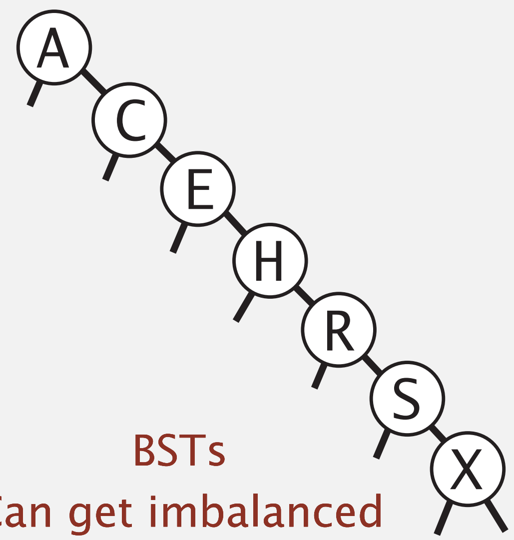
private class Node
{
   Key key;
   Value val;
   Node left, right;
   boolean color;    // color of parent link
}

private boolean isRed(Node x)
{
   if (x == null) return false;
   return x.color == RED;
}
```

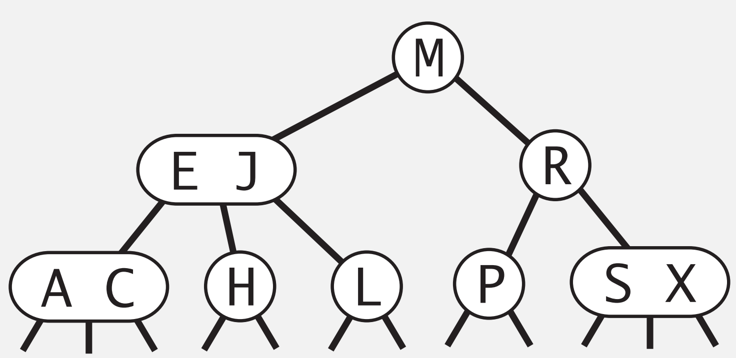
null links are black



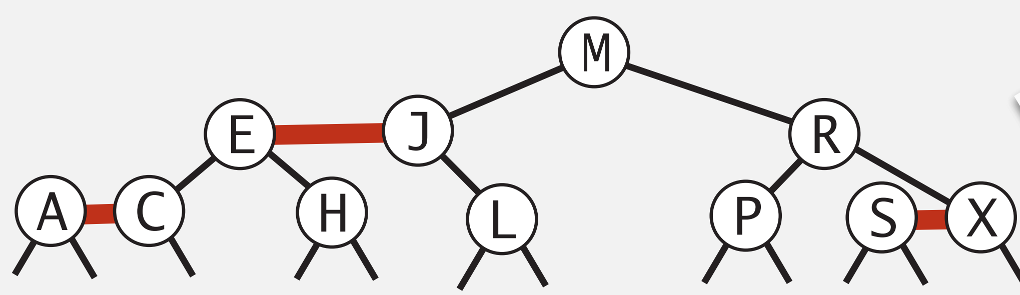
Review: the road to Left Leaning Red Black Trees



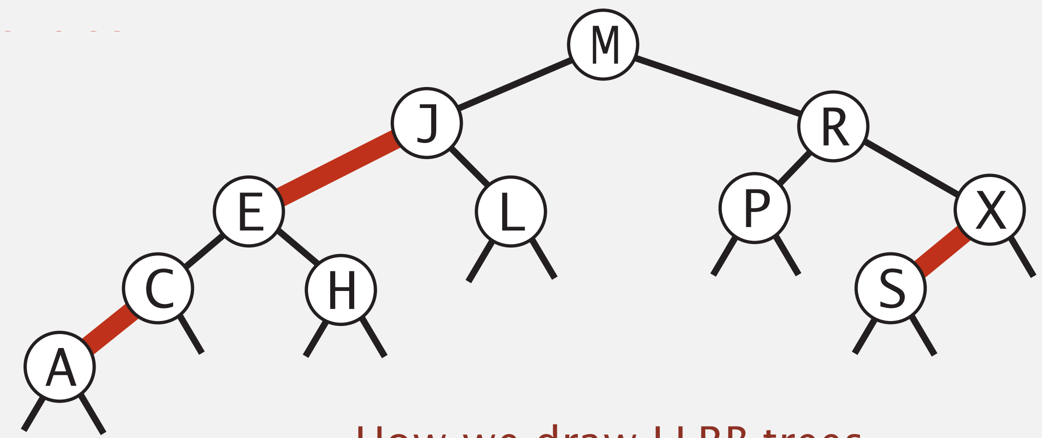
BSTs
Can get imbalanced



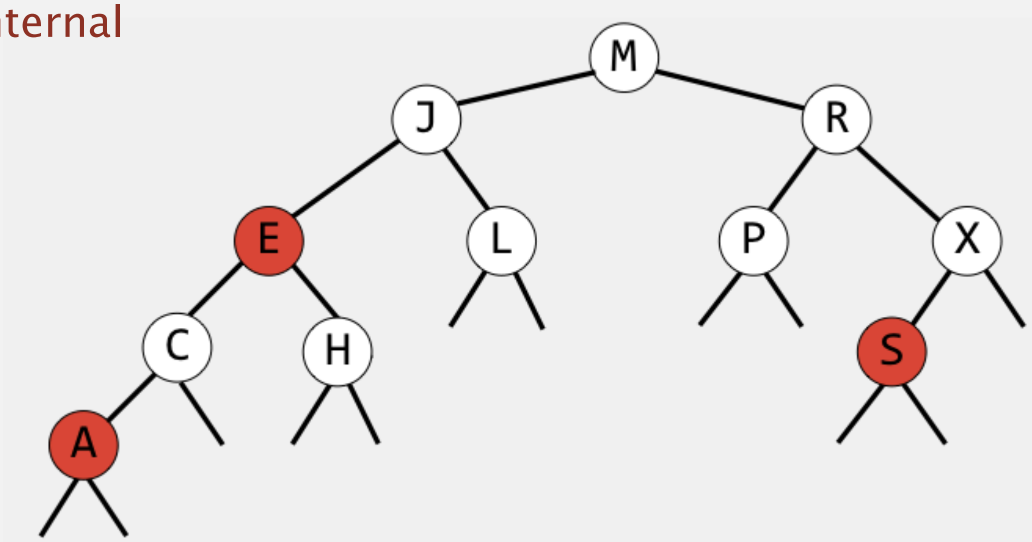
2-3 trees
Balanced but cumbersome



Imagine 3-nodes held together by internal glue links shown in red



How we draw LLRB trees



How we represent LLRB trees in code

Plan for rest of this lecture

LLRB search. Same as BST search; see above.

LLRB insert. Rest of this lecture.

LLRB delete. Tricky; see book.

LLRB operations.

- Insert requires operations called rotations and color flips.
- Derived via 1-1 correspondence with 2-3 tree operations
  (temporarily creating and splitting a 4-node)

Learning strategy.

We'll omit the correspondence to 2-3 trees in the rest of the lecture and learn the LLRB operations directly.

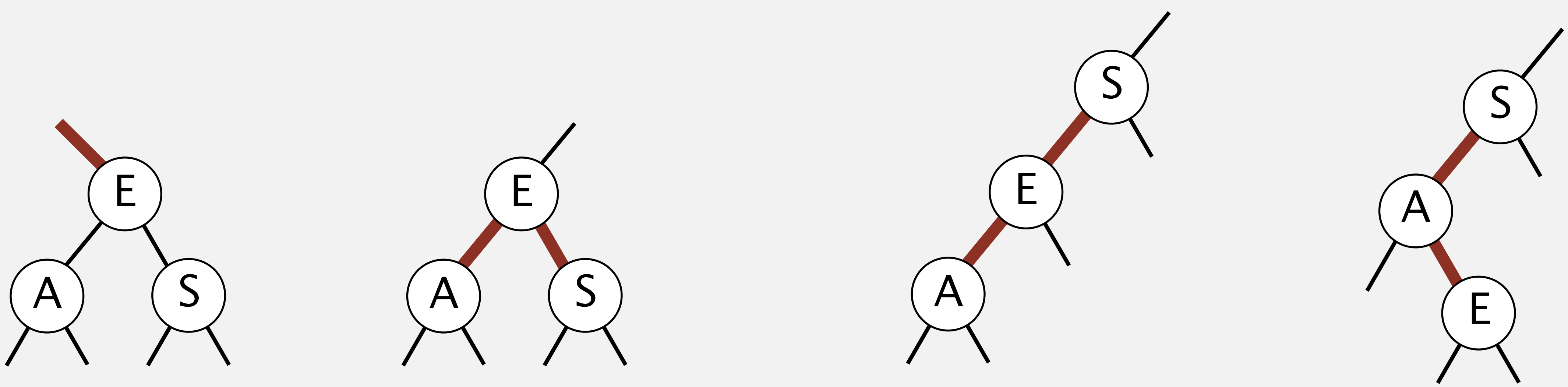
Insertion into a LLRB tree: overview

Basic strategy. Maintain 1–1 correspondence with 2–3 trees.

During internal operations, maintain:

- Symmetric order.
- Perfect black balance. [but not necessarily color invariants]

Examples of violations of color invariants:



right–leaning red link

two red children (a temporary 4–node)

left–left red (a temporary 4–node)

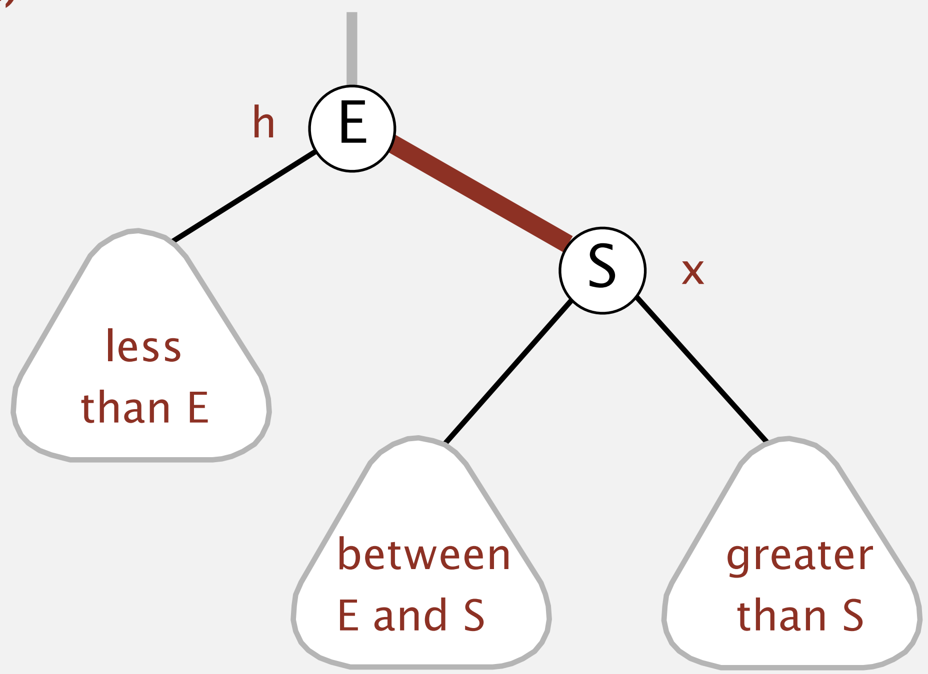
left–right red (a temporary 4–node)

To restore color invariant: apply rotations and color flips.

Elementary red-black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.

rotate E left
(before)

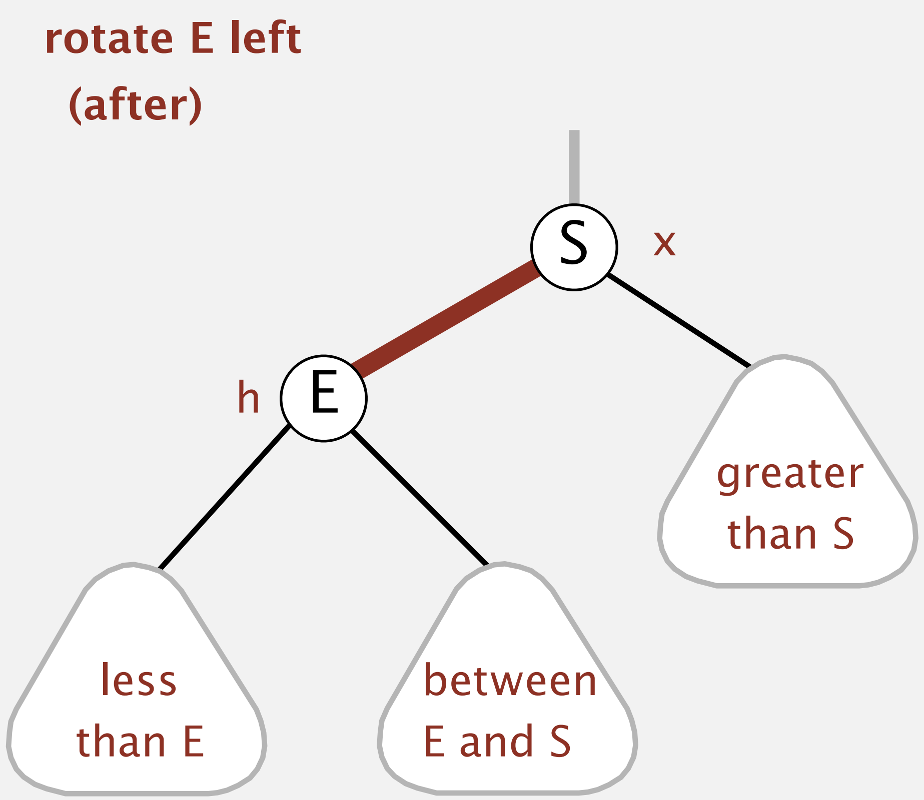



```
private Node rotateLeft(Node h)
{
   assert isRed(h.right);
   Node x = h.right;
   h.right = x.left;
   x.left = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red–black BST operations

Left rotation. Orient a (temporarily) right-leaning red link to lean left.




```
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   assert isRed(h.right);
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   h.right = x.left;
   x.left = h;
   x.color = h.color;
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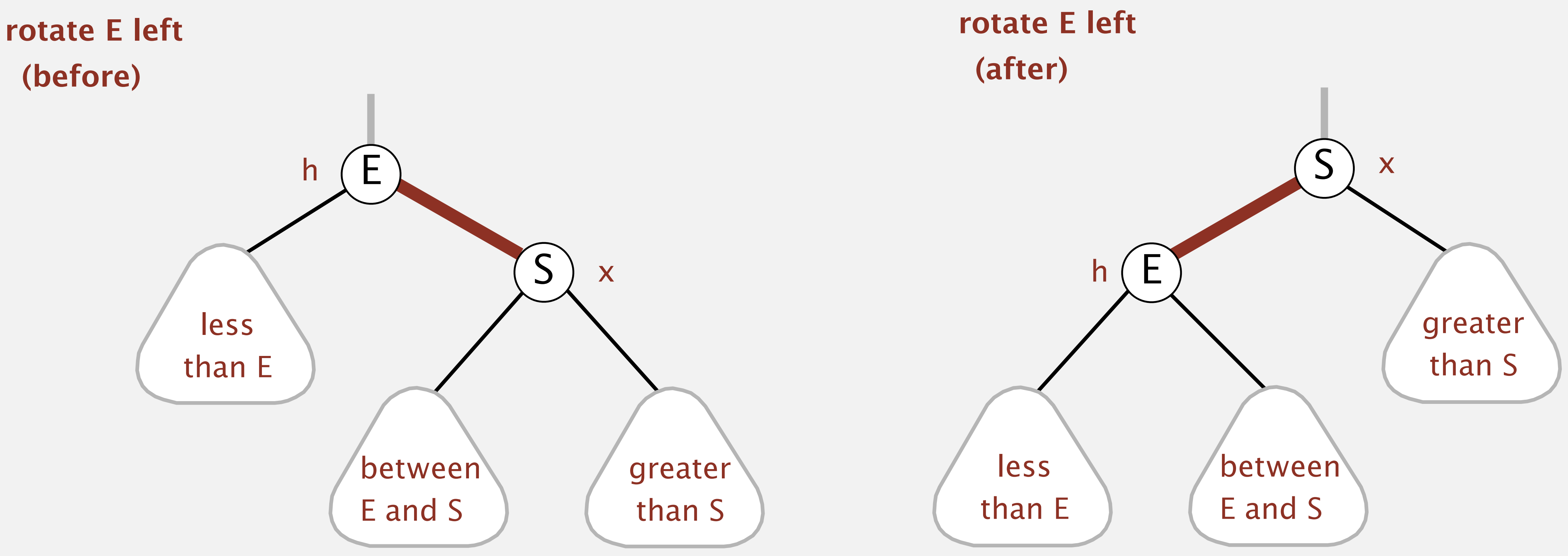
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

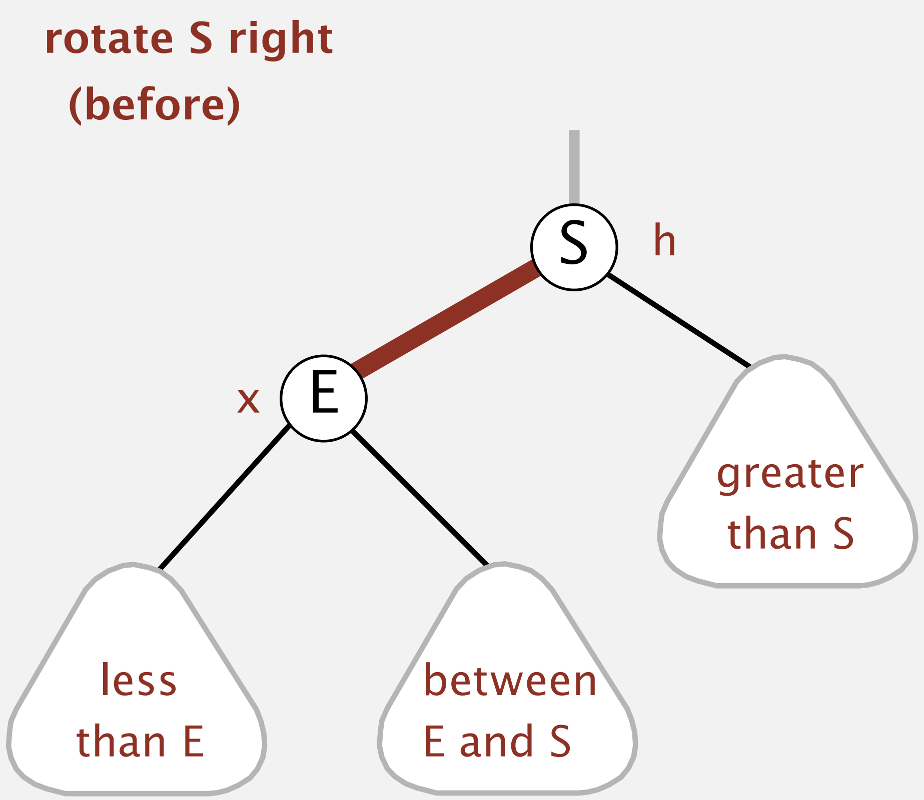
Left rotation. Orient a (temporarily) right-leaning red link to lean left.



Exercise. Verify that left rotation maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

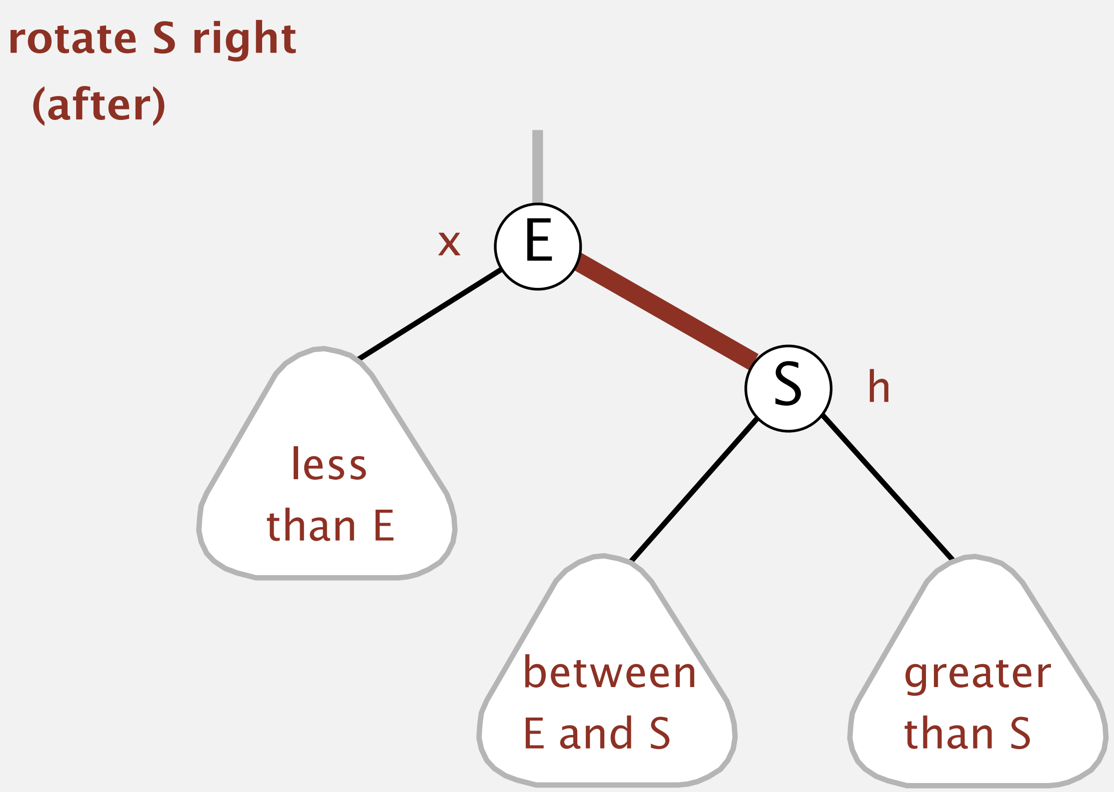



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Right rotation. Orient a left-leaning red link to (temporarily) lean right.

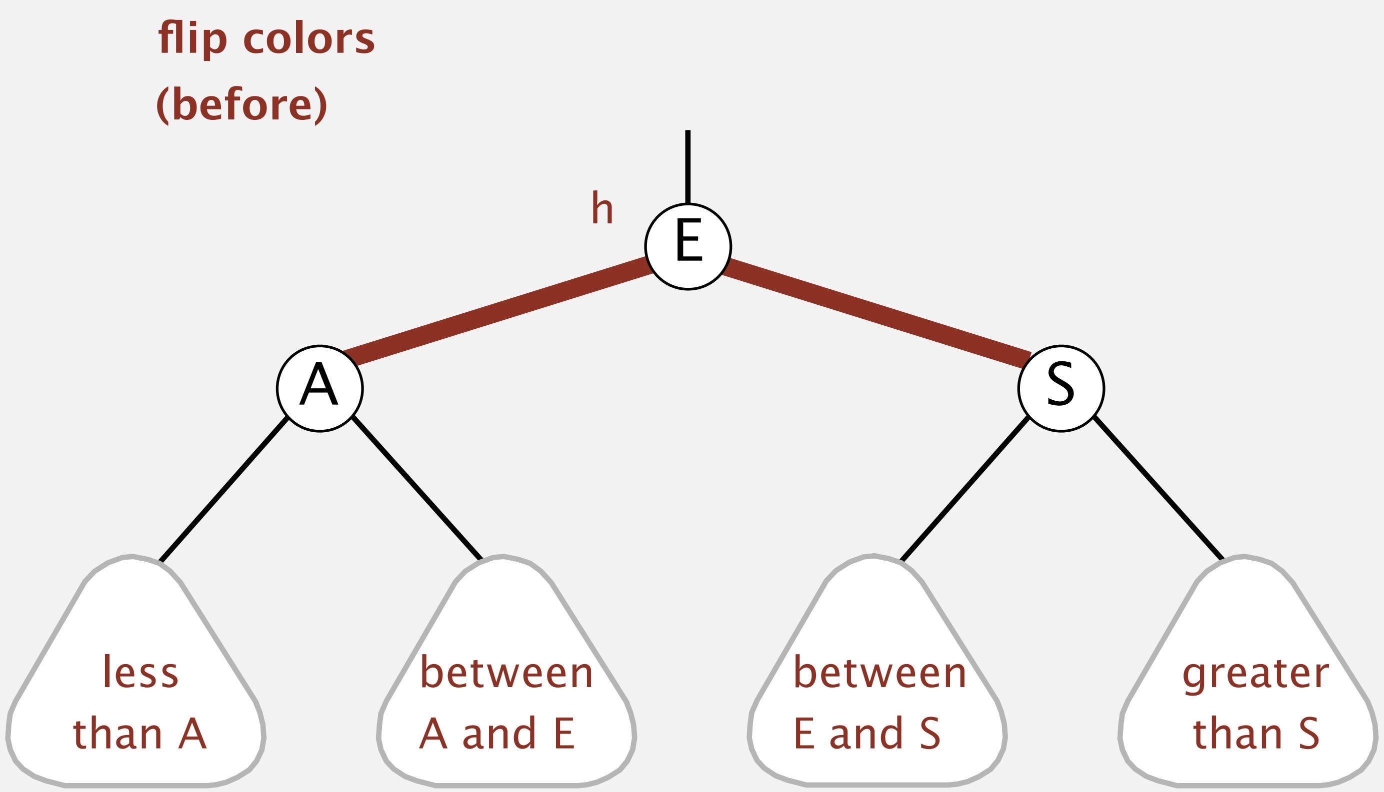



```
private Node rotateRight(Node h)
{
   assert isRed(h.left);
   Node x = h.left;
   h.left = x.right;
   x.right = h;
   x.color = h.color;
   h.color = RED;
   return x;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

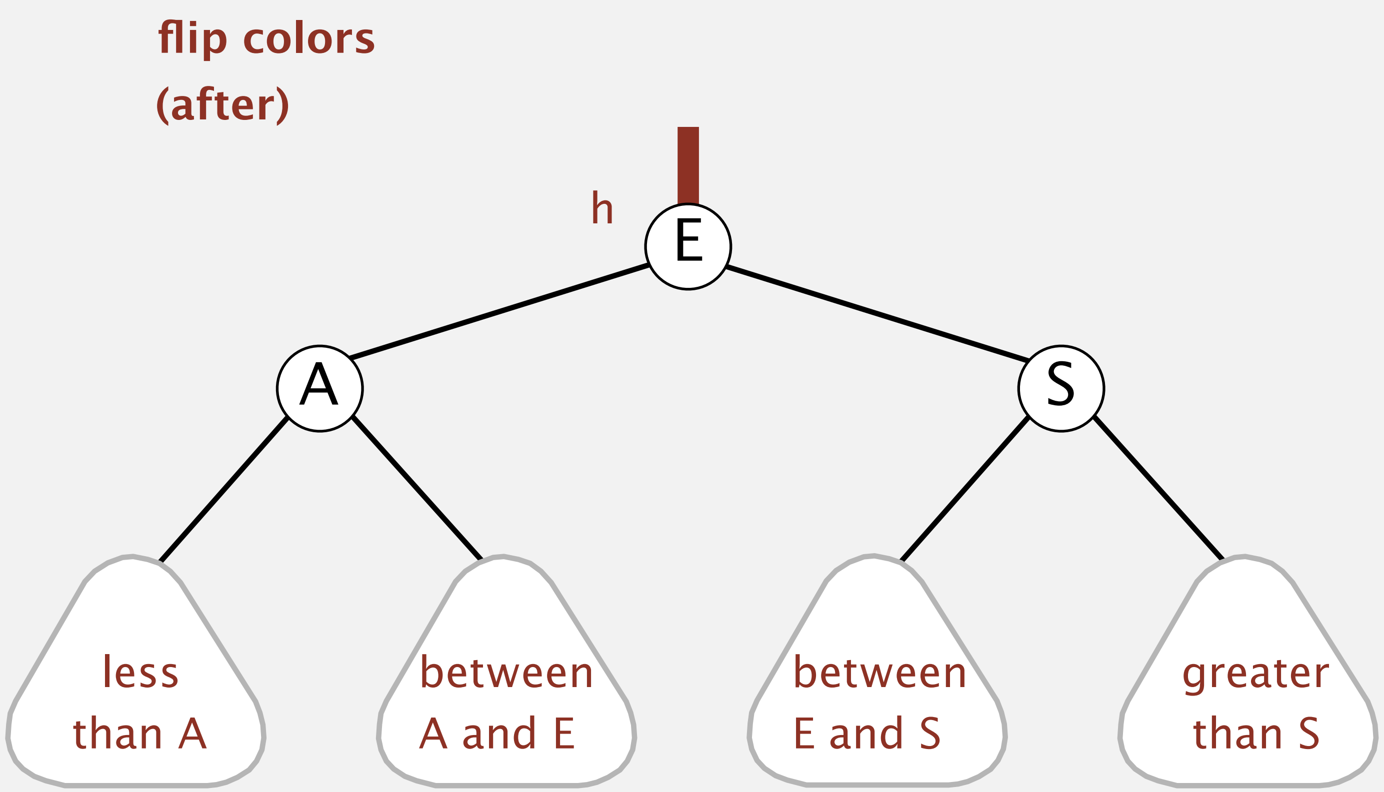



```
private void flipColors(Node h)
{
   assert !isRed(h);
   assert isRed(h.left);
   assert isRed(h.right);
   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

Invariants. Maintains symmetric order and perfect black balance.

Elementary red-black BST operations

Color flip. Recolor to split a (temporary) 4-node.

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   h.color = RED;
   h.left.color = BLACK;
   h.right.color = BLACK;
}
```

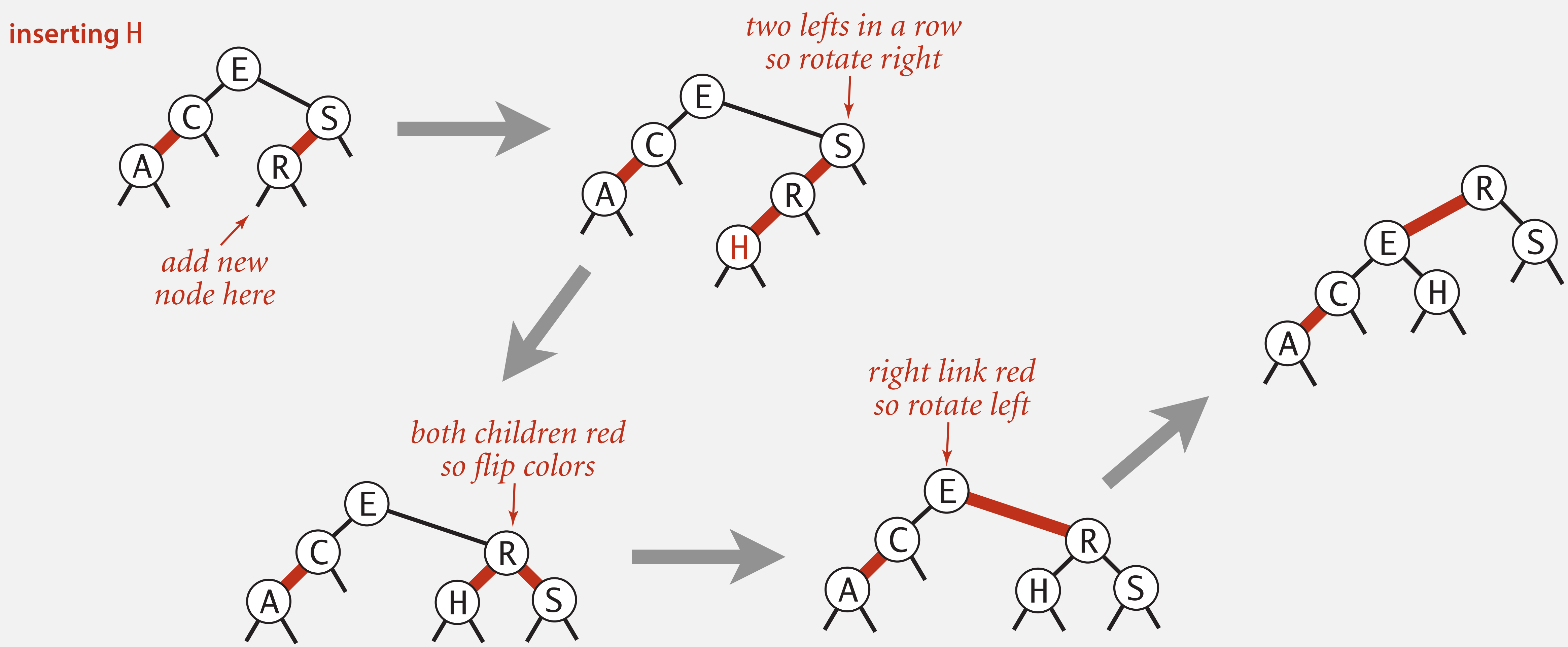
Invariants. Maintains symmetric order and perfect black balance.

Insertion into a LLRB tree

maintain symmetric order

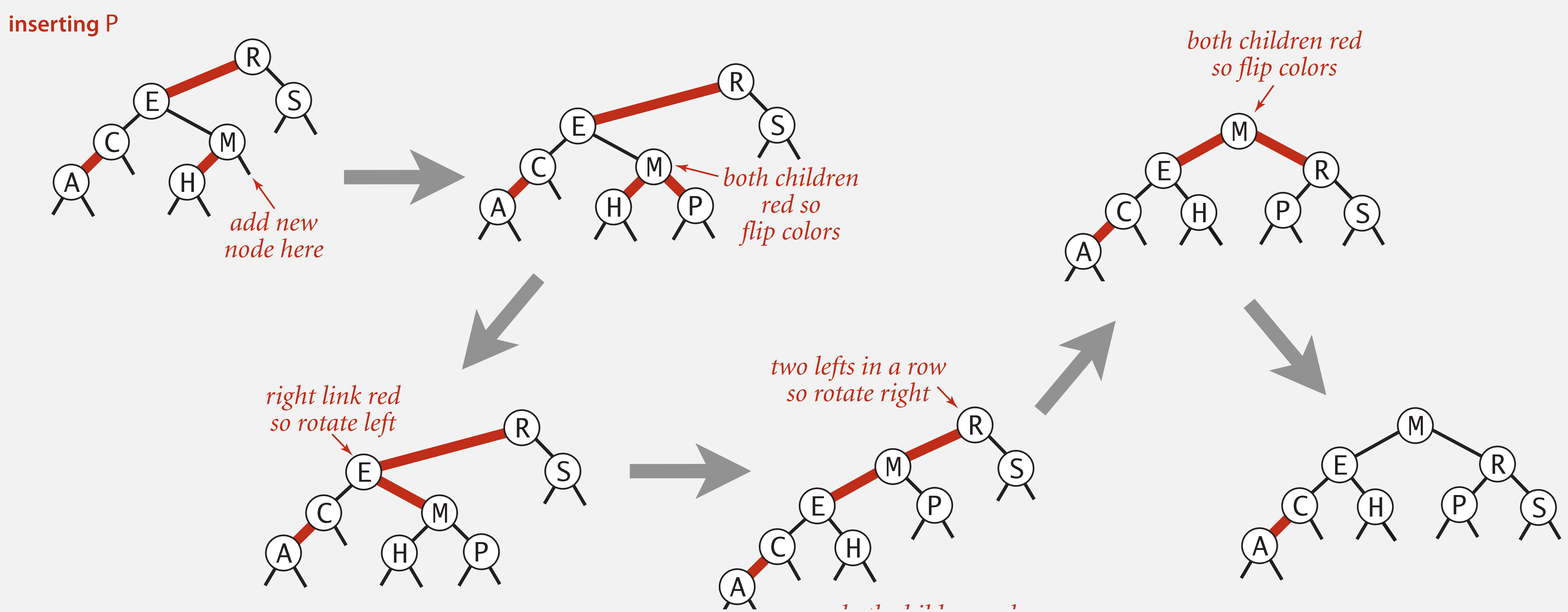
maintain perfect black balance

- Do standard BST insert; color new link red.
- Repeat until color invariants restored:
  - Both children red? Flip colors
  - Right link red? Rotate left
  - Two left reds in a row? Rotate right



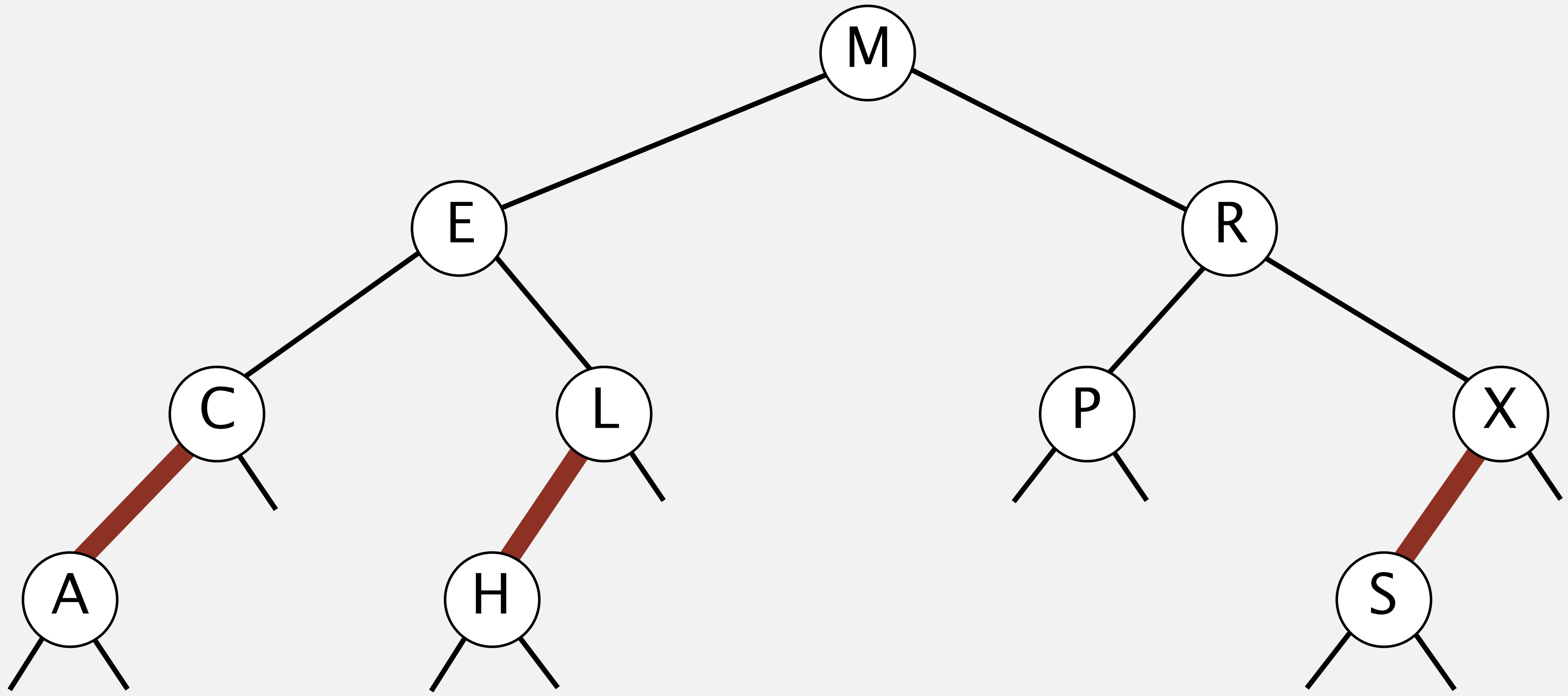
Insertion into a LLRB tree: passing red links up the tree

- Do standard BST insert; color new link red.
- Repeat until color invariants restored:
  - Both children red? Flip colors
  - Right link red? Rotate left
  - Two left reds in a row? Rotate right



Red-black BST construction demo

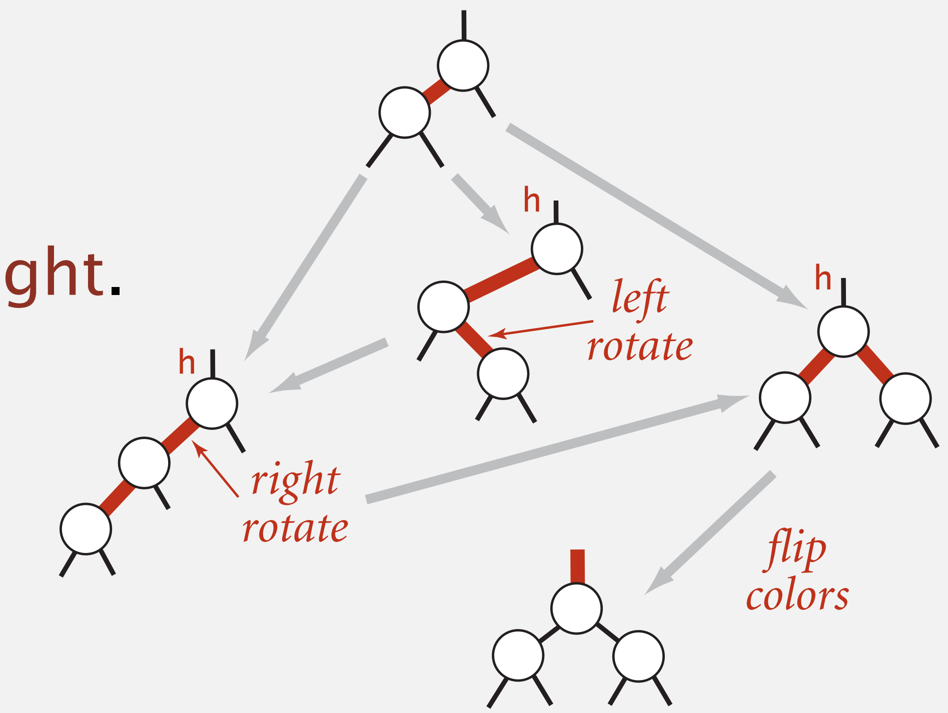
insert S E A R C H X M P L



Insertion into a LLRB tree: Java implementation

Can distill down to three cases!

- Right child red; left child black: rotate left.
- Left child red; left-left grandchild red: rotate right.
- Both children red: flip colors.




```
private Node put(Node h, Key key, Value val)
{
   if (h == null) return new Node(key, val, RED);
   int cmp = key.compareTo(h.key);
   if      (cmp  < 0) h.left  = put(h.left,  key, val);
   else if (cmp  > 0) h.right = put(h.right, key, val);
   else if (cmp == 0) h.val = val;

   if (isRed(h.right) && !isRed(h.left))     h = rotateLeft(h);
   if (isRed(h.left)  && isRed(h.left.left)) h = rotateRight(h);
   if (isRed(h.left)  && isRed(h.right))     flipColors(h);

   return h;
}
```

insert at bottom (and color it red)

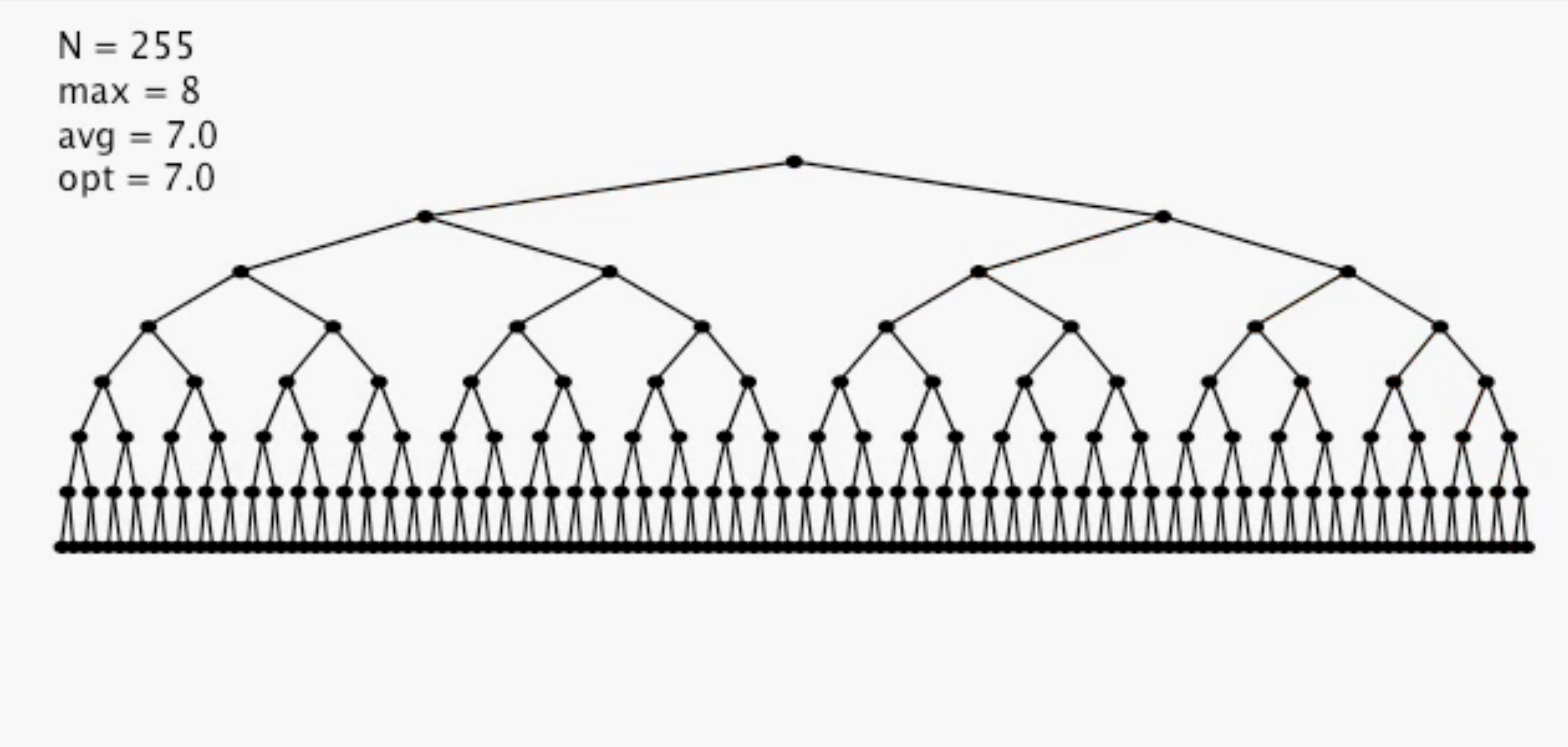
lean left

balance 4-node

split 4-node

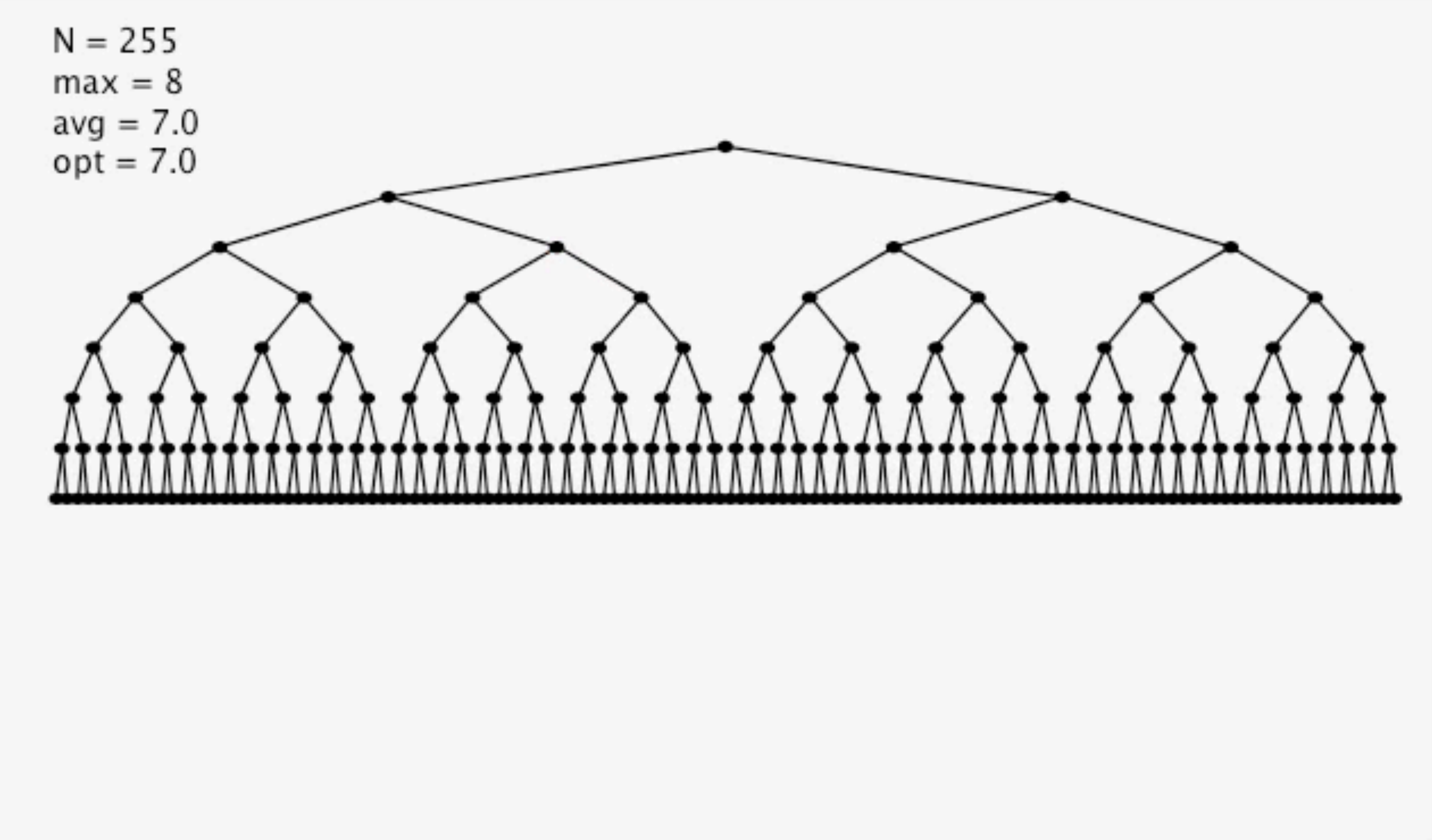
only a few extra lines of code provides near-perfect balance

Insertion into a LLRB tree: visualization



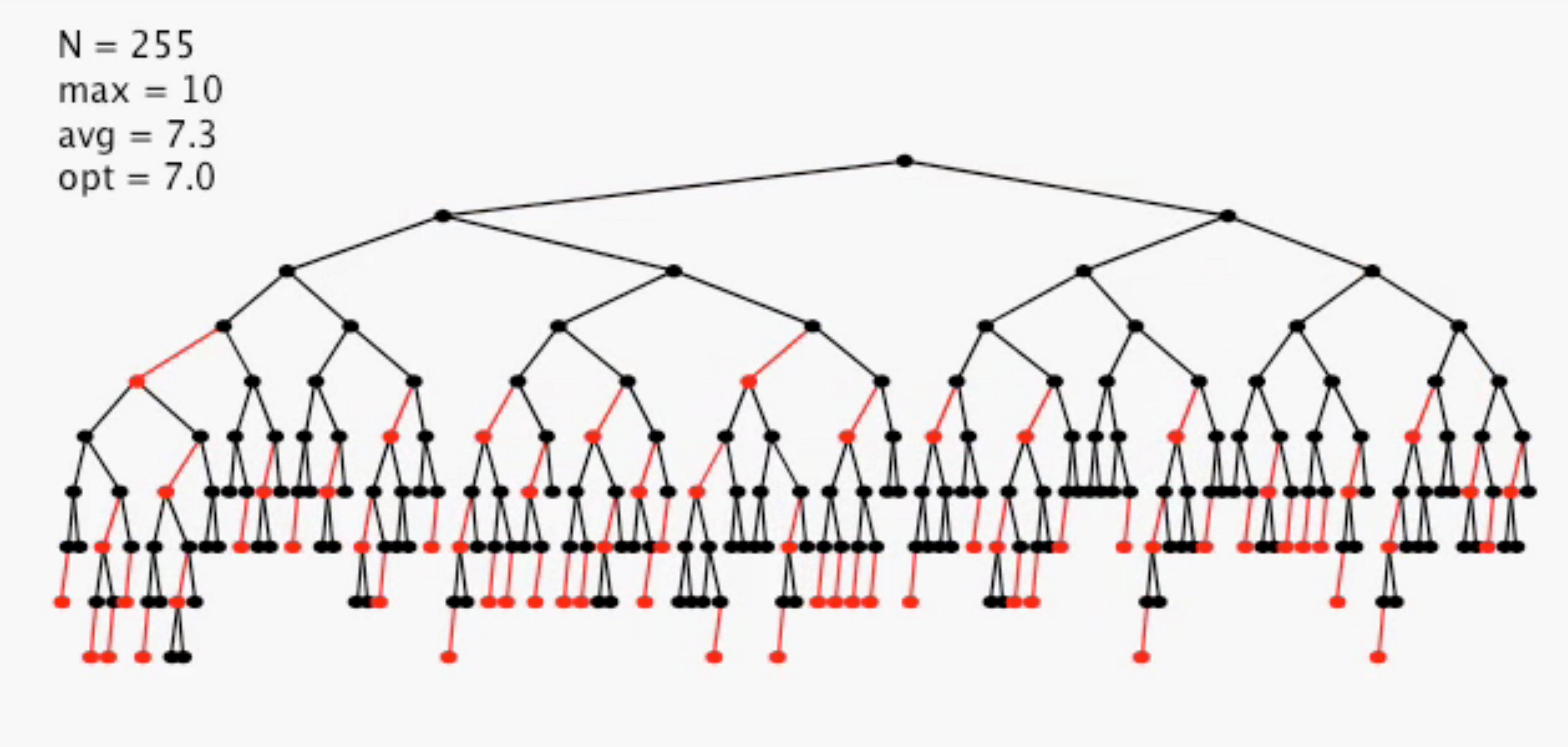
255 insertions in ascending order

Insertion into a LLRB tree: visualization



255 insertions in descending order

Insertion into a LLRB tree: visualization



255 random insertions

What is the maximum height of a LLRB tree with n keys?

A. $\sim \log_3 n$

B. $\sim \log_2 n$

C. $\sim 2 \log_2 n$

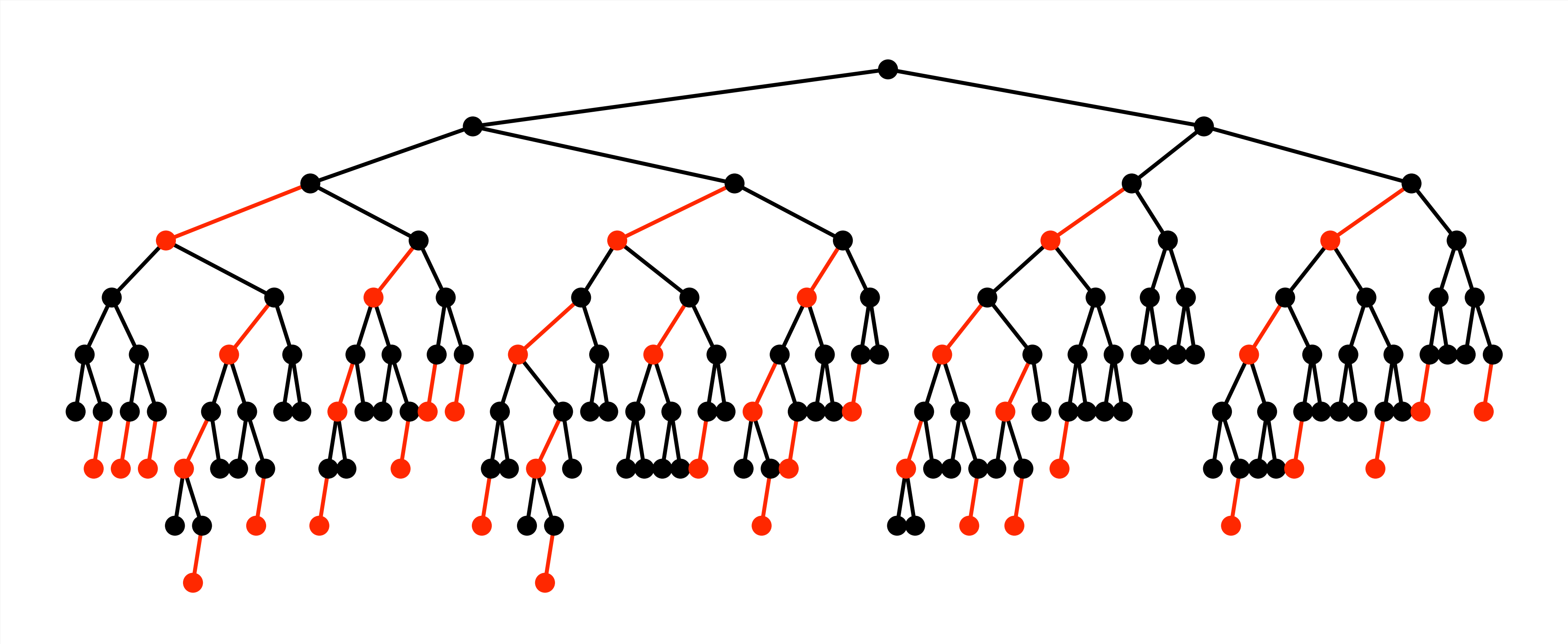
D. $\sim n$

Balance in LLRB trees

Proposition. Height of tree is $\leq 2 \lg n$ in the worst case.

Pf.

- Black height = height of corresponding 2–3 tree $\leq \lg n$.
- Never two red links in-a-row.



ST implementations: summary

implementation	guarantee			average case			ordered ops?	key interface
	search	insert	delete	search	insert	delete		
sequential search (unordered list)	n	n	n	n	n	n		`equals()`
binary search (ordered array)	$\log n$	n	n	$\log n$	n	n	✔	`compareTo()`
BST	n	n	n	$\log n$	$\log n$	$\sqrt{n}$	✔	`compareTo()`
2-3 tree	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`
red-black BST	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	$\log n$	✔	`compareTo()`

hidden constant c is small
(at most $2 \lg n$ compares)

Historical context: Guibas & Sedgewick 1978

A DICHROMATIC FRAMEWORK FOR BALANCED TREES

Leo J. Guibas
Xerox Palo Alto Research Center,
Palo Alto, California, and
Carnegie-Mellon University

and

Robert Sedgewick*
Program in Computer Science
Brown University
Providence, R. I.

ABSTRACT

In this paper we present a uniform framework for the implementation and study of balanced tree algorithms. We show how to imbed in this the way down towards a leaf. As we will see, this has a number of significant advantages over the older methods. We shall examine a number of variations on a common theme and exhibit full implementations which are notable for their brevity. One implementation is examined carefully, and some properties about its

Why "red-black"?

Xerox PARC innovations. [1970s]

- Alto.
- GUI.
- Ethernet.
- Smalltalk.
- Laser printing.
- Bitmapped display.
- WYSIWYG text editor.
- ...

XEROX PARC

Xerox Alto

Historical context: Sedgewick 2008

Left-leaning Red-Black Trees

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Princeton, NJ 08544

Abstract

The red-black tree model for implementing balanced search trees, introduced by Guibas and Sedgewick thirty years ago, is now found throughout our computational infrastructure. Red-black trees are described in standard textbooks and are the underlying data structure for symbol-table implementations within C++, Java, Python, BSD Unix, and many other modern systems. However, many of these implementations have sacrificed some of the original design goals (primarily in order to develop an effective implementation of the delete operation, which was incompletely specified in the original paper), so a new look is worthwhile. In this paper, we describe a new variant of red-black trees that meets many of the original design goals and leads to substantially simpler code for insert/delete, less than one-fourth as much code as in implementations in common use.

Balanced trees in the wild

Red–black trees are widely used as system symbol tables.

- Java: `java.util.TreeMap`, `java.util.TreeSet`.
- C++ STL: map, multimap, multiset.
- Linux kernel: completely fair scheduler, `linux/rbtree.h`.
- Emacs: conservative stack scanning.

Other balanced BSTs. AVL trees, splay trees, randomized BSTs, ….

B-trees (and cousins) are widely used for file systems and databases.

- Windows: NTFS.
- Mac: HFS, HFS+.
- Linux: ReiserFS, XFS, Ext3FS, JFS, BTRFS.
- Databases: ORACLE, DB2, INGRES, SQL, PostgreSQL.

War story: red–black BSTs

Telephone company contracted with database provider to build real-time database to store customer information.

Database implementation.

- Red–black BST.
- Exceeding height limit of 80 triggered error-recovery process.

should allow for $\leq 2^{40}$ keys

Extended telephone service outage.

- Main cause = height bound exceeded!
- Telephone company sues database provider.
- Legal testimony:

[redacted]

> " *If implemented properly, the height of a red–black BST with n keys is at most 2 lg n.* " — *expert witness*