Textbook section

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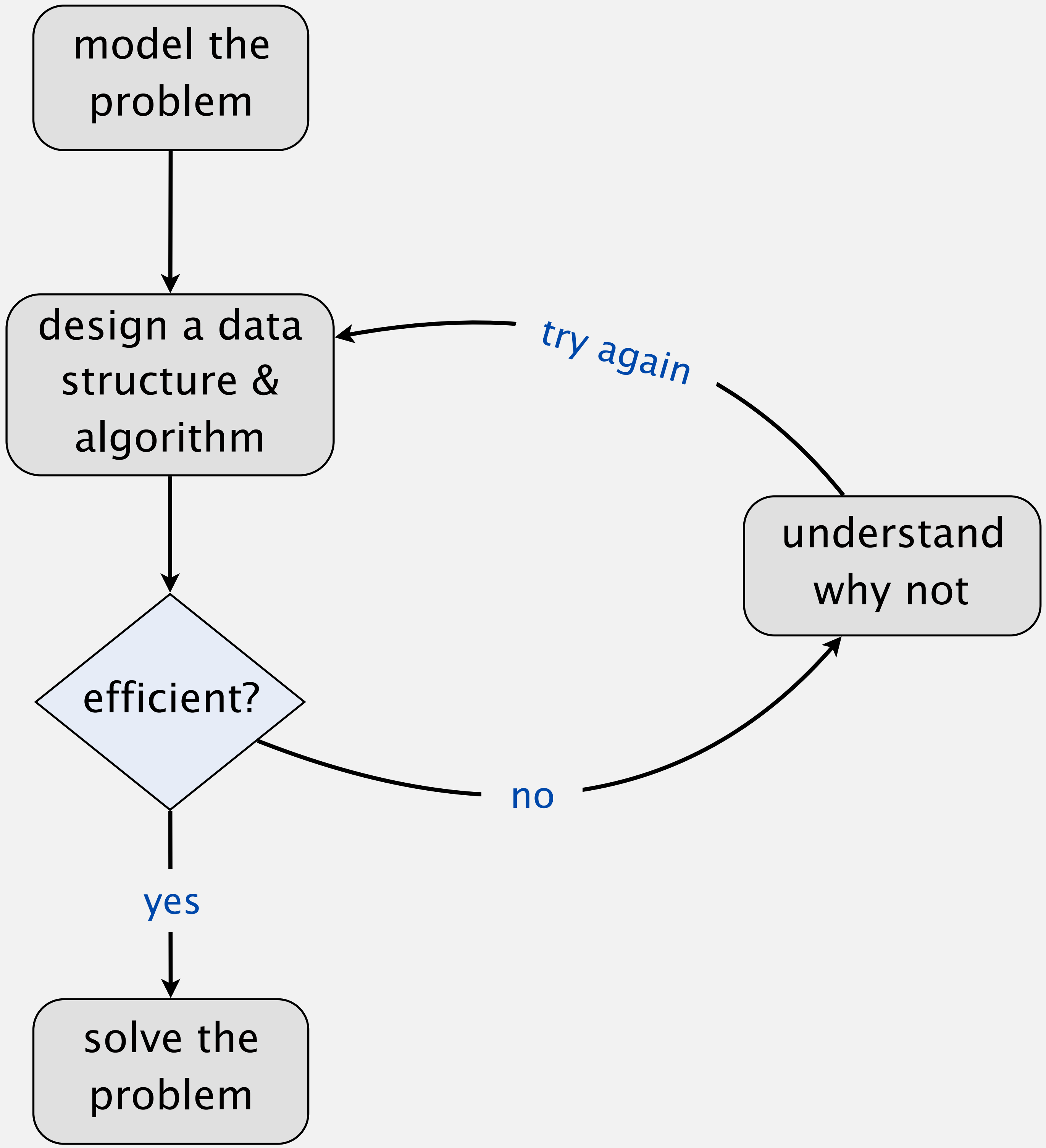
1.5 UNION–FIND

- *union–find data type*
- *quick-find*
- *quick-union*
- *improvements*
- *applications*

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm to solve a computational problem.



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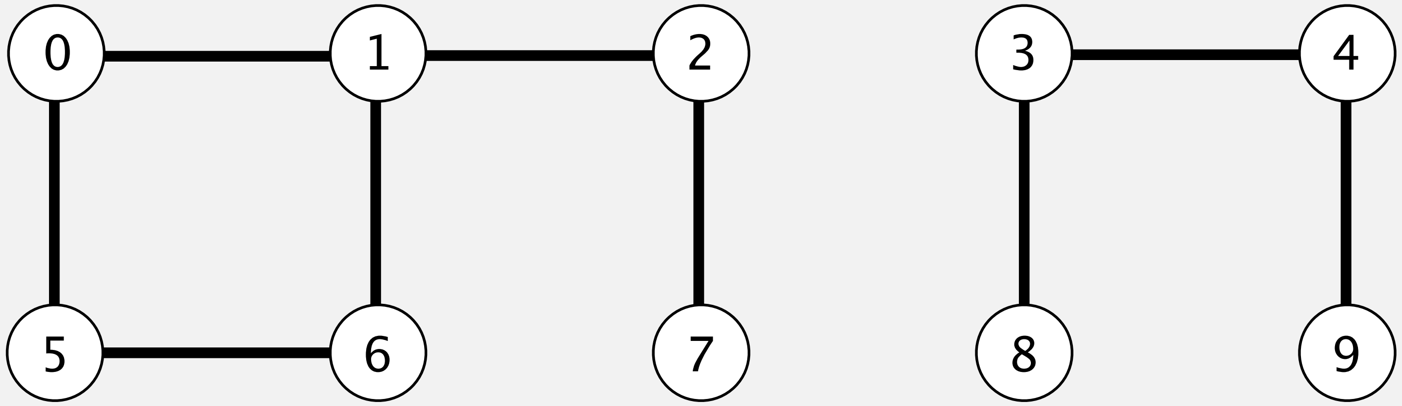
Problem: dynamic connectivity

Given n vertices, support two operations:

- Add edge: directly connect two vertices with an edge.
- Connection query: is there a path connecting two vertices?

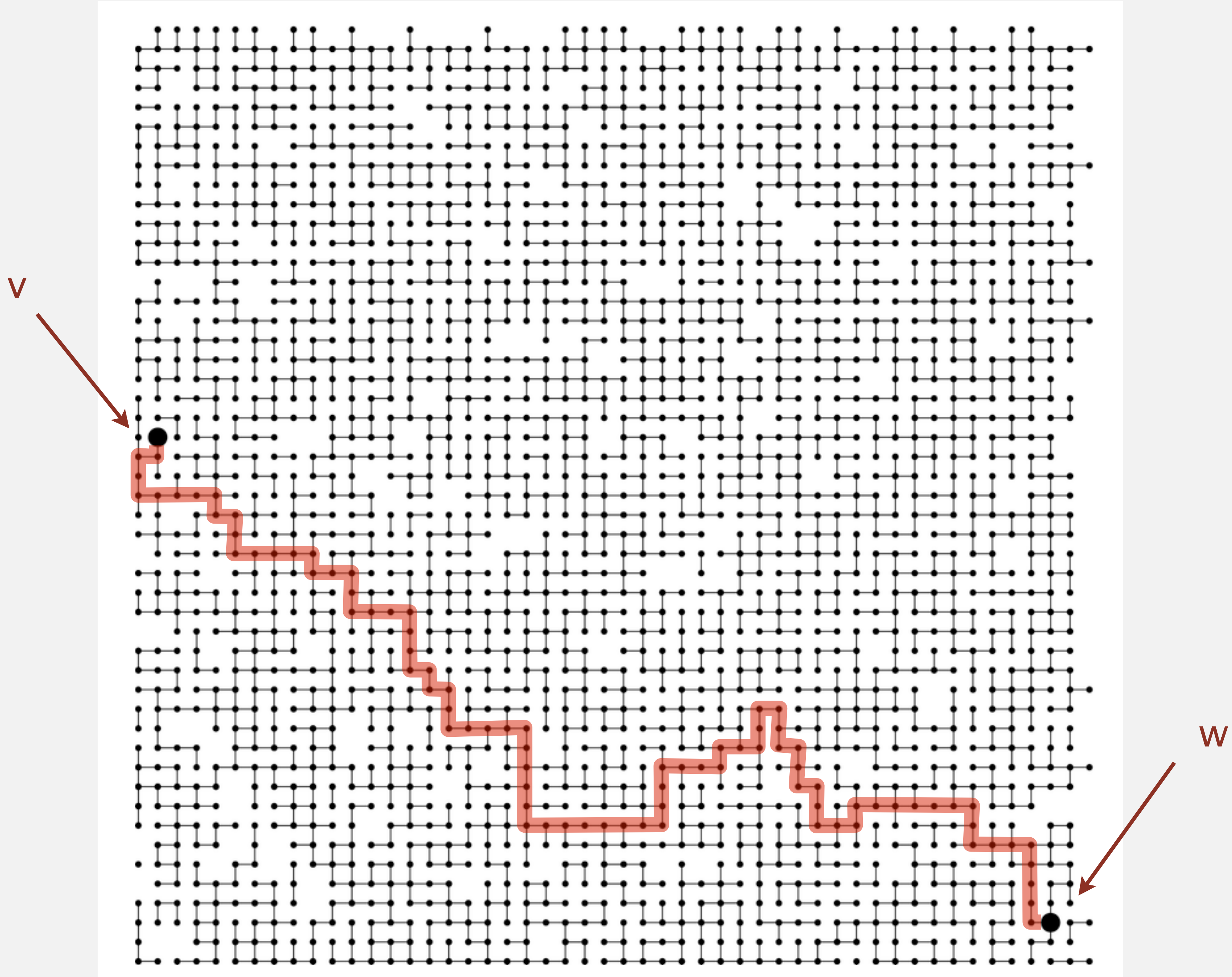
add edge 4–3
add edge 3–8
add edge 6–5
add edge 9–4
add edge 2–1
are 8 *and* 9 *connected?* ✔
are 5 *and* 7 *connected?* ✕
add edge 5–0
add edge 7–2
add edge 6–1
add edge 1–0
are 5 *and* 7 *connected?* ✔



A larger connectivity example

Q. Is there a path connecting vertices v and w ?

finding a path is a slightly harder problem
(stay tuned for graph algorithms in Chapter 4)



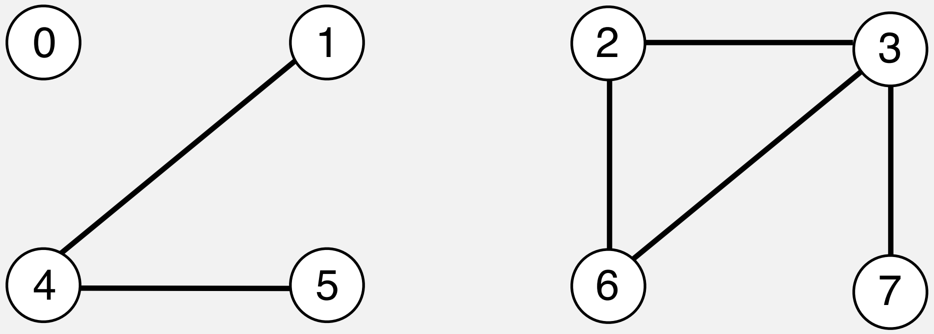
A. Yes.

Modeling the dynamic-connectivity problem

Note. Dynamic means not all edges given at once; interspersed with connection queries.

Key idea. Maintain disjoint sets that correspond to connected components.

Connected component. Maximal set of vertices that are mutually connected.



3 connected components

{ 0 } { 1, 4, 5 } { 2, 3, 6, 7 }

3 disjoint sets

Modeling the dynamic-connectivity problem

Key idea. Maintain disjoint sets that correspond to connected components.

- Add edge between vertices *v* and *w*.
- Are vertices *v* and *w* connected?

add edge 2–5

are vertices 5 and 6 connected?

3 connected components

2 connected components

union(2, 5)

find(5) == find(6) ✔

`{ 0 } { 1, 4, 5 } { 2, 3, 6, 7 }`

3 disjoint sets

`{ 0 } { 1, 2, 3, 4, 5, 6, 7 }`

2 disjoint sets

Connection queries are modeled with **two** calls to `find()`.

Union–find data type

Disjoint sets. A collection of sets; each element in exactly one set.

Find. Return a "canonical" element in the set containing the given vertex.
Union. Merge the set containing the first vertex with the set containing the second.

find(1) = find(4) = find(5) = 4

`{ 0 } { 1, 4, 5 } { 2, 3, 6, 7 }`

8 elements, 3 disjoint sets

union(2, 5)

`{ 0 } { 1, 2, 3, 4, 5, 6, 7 }`

2 disjoint sets

Simplifying assumption. The n elements are named 0, 1, …, $n - 1$.

Union–find data type (API*)

Goal. Design an efficient union–find data type.

- Number of elements n can be huge.
- Number of operations m can be huge.
- Union and find operations can be intermixed.

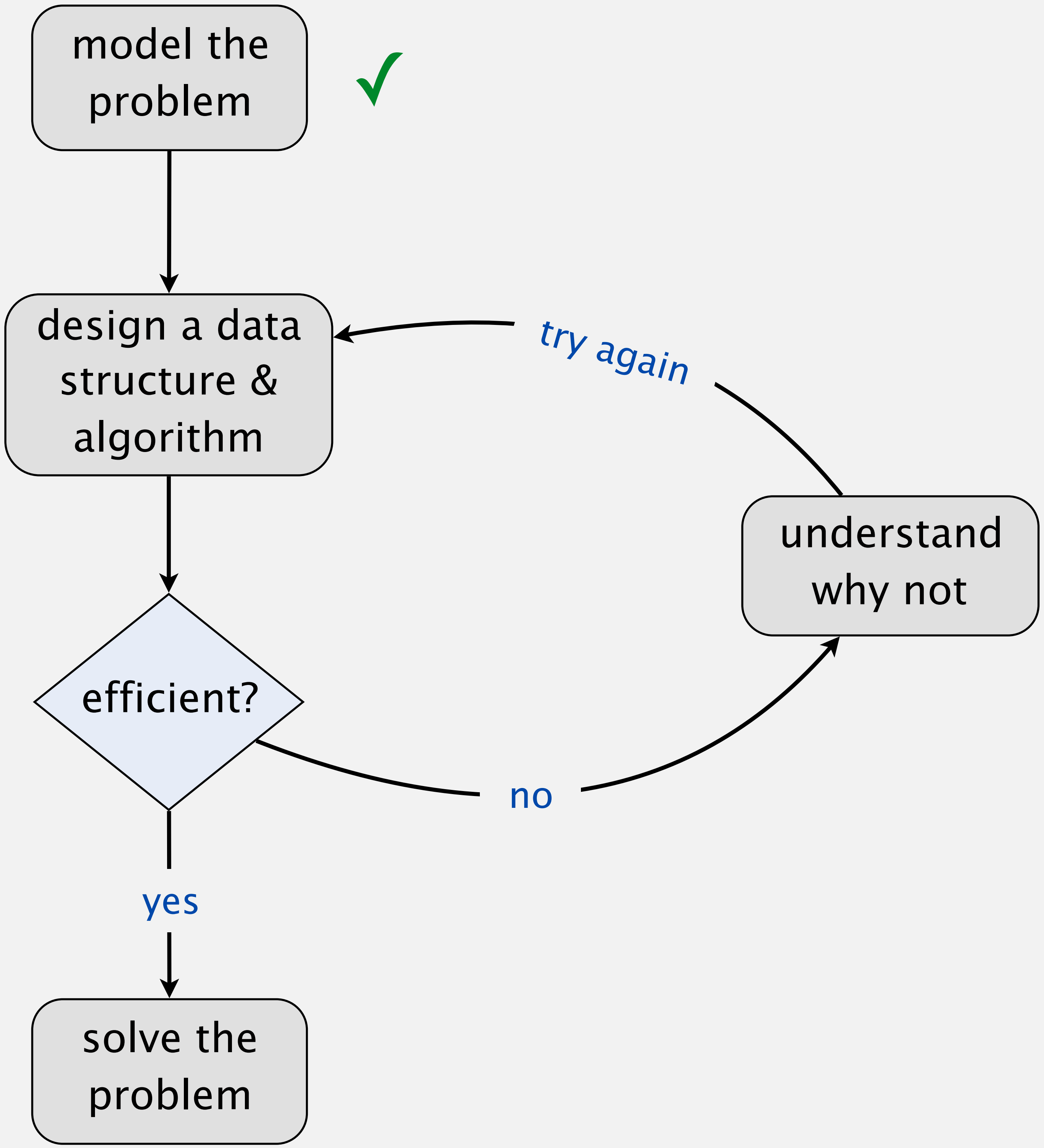
```
public class UF
```

	`UF(int n)`	*initialize union–find data structure with n singleton sets* (0 *to* n – 1)
`void`	`union(int p, int q)`	*merge sets containing elements p and q*
`int`	`find(int p)`	*canonical element in set containing p* (0 *to* n – 1)

*Application Programing Interface.

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm to solve a computational problem.



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1.5 Union–Find

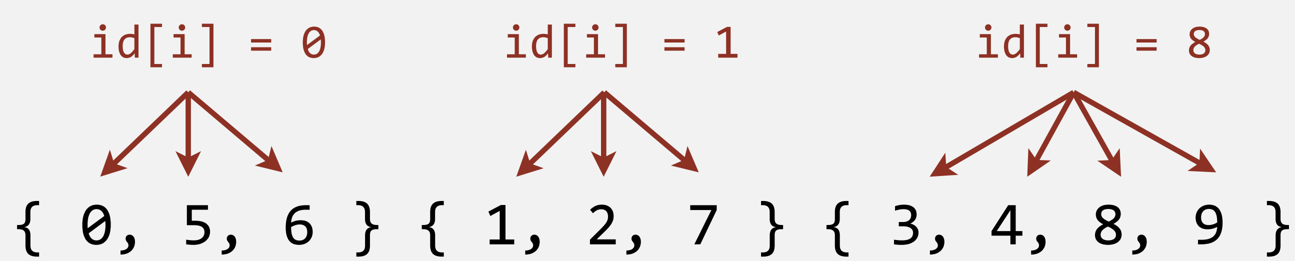
- union–find data type
- *quick-find*
- quick-union
- improvements
- applications

Quick-find [eager approach]

Data structure.

- Integer array `id[]` of length `n`.
- Interpretation: `id[p]` is canonical element in the set containing `p`.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8



3 disjoint sets

Q. How to implement `find(p)`?

A. Easy, just return `id[p]`.

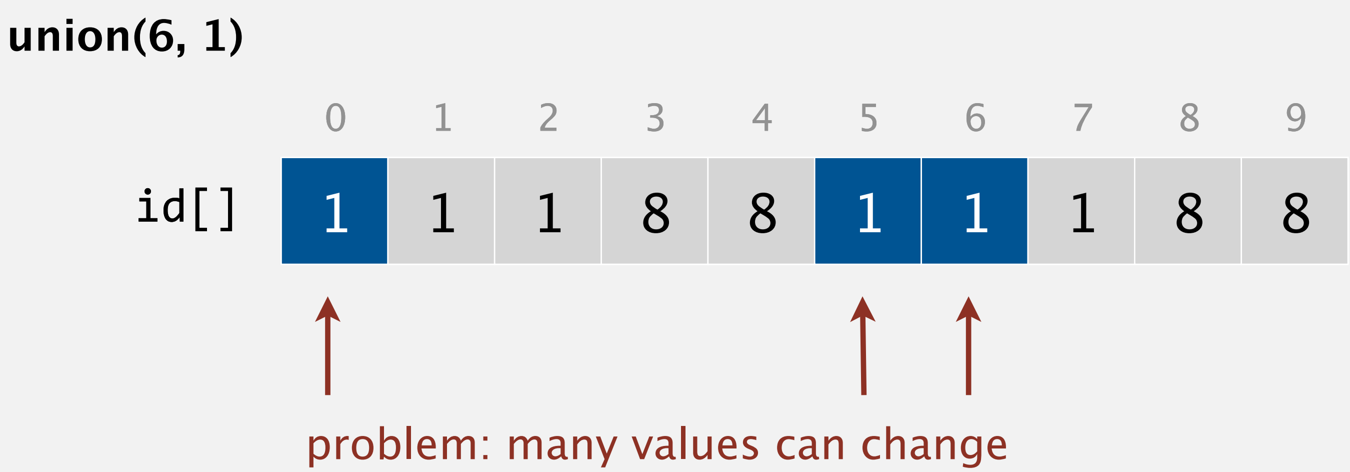
Q. How to implement `union(p, q)`?

(i.e. merge the sets containing p & q).

Quick-find [eager approach]

Data structure.

- Integer array `id[]` of length `n`.
- Interpretation: `id[p]` is canonical element in the set containing `p`.



Q. How to implement `union(p, q)`?

A. Change all entries whose identifier equals `id[p]` to `id[q]` (or vice versa).

Quick-find: Java implementation

```java
public class QuickFindUF
{
   private int[] id;

   public QuickFindUF(int n)
   {
      id = new int[n];
      for (int i = 0; i < n; i++)
         id[i] = i;
   }

   public int find(int p)
   {  return id[p];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each element to itself (n array accesses)

return the id of `p` (1 array access)

change all entries with `id[p]` to `id[q]` ($n + 2$ to $2n + 2$ array accesses)

https://algs4.cs.princeton.edu/15uf/QuickFindUF.java.html

Quick-find is too slow

Cost model. Number of array accesses (for read or write).

Rationale.

- Accessing memory is much slower than operations within CPU.
- If we had a more complex cost model (that included arithmetic ops), the constants might change, but not the order of growth.

algorithm	initialize	union	find
quick-find	n	n	1

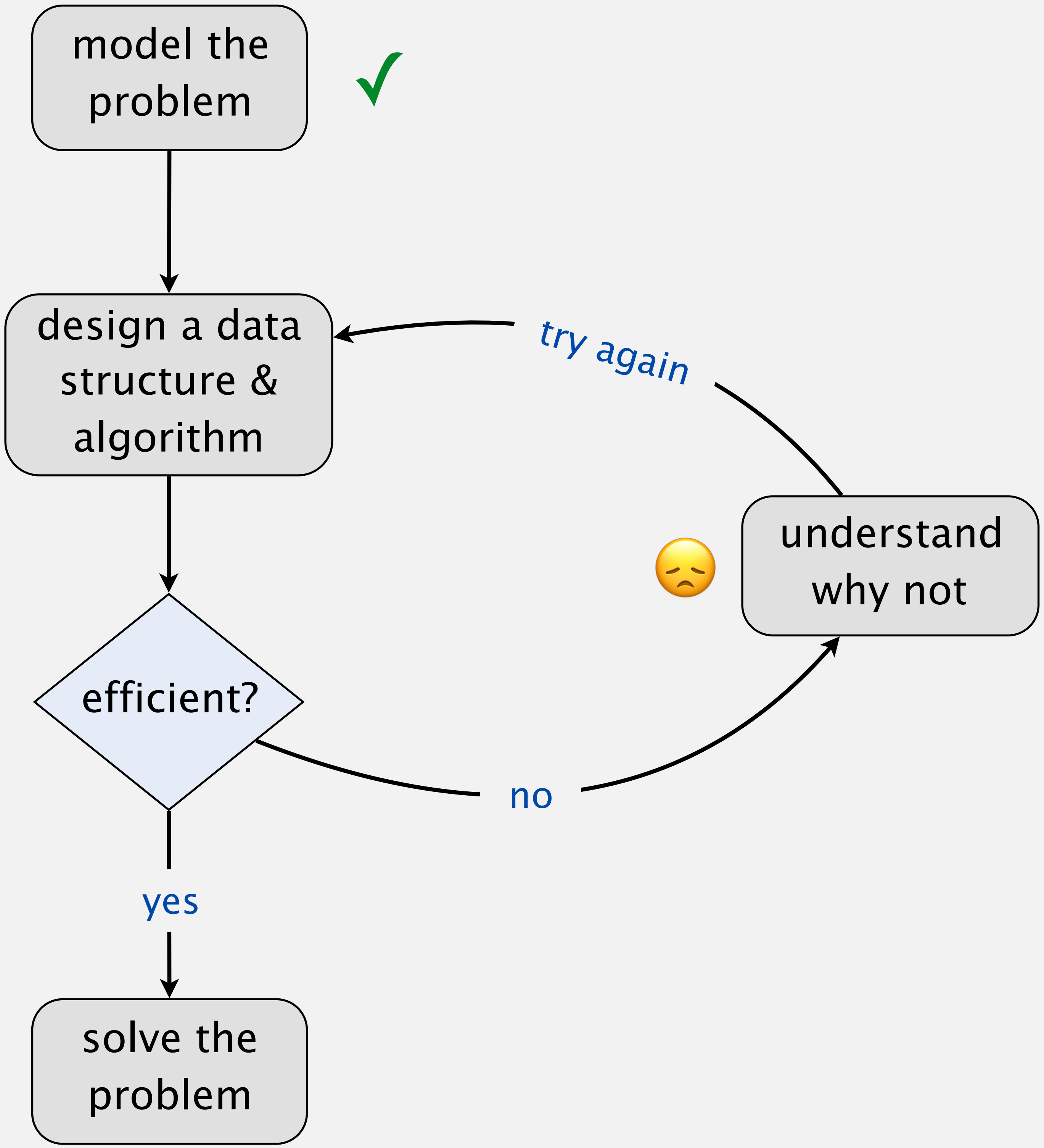
number of array accesses (ignoring leading constant)

Union is too expensive. Processing a sequence of n union operations on n elements takes more than n^2 array accesses.

quadratic

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm to solve a computational problem.



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1.5 UNION–FIND

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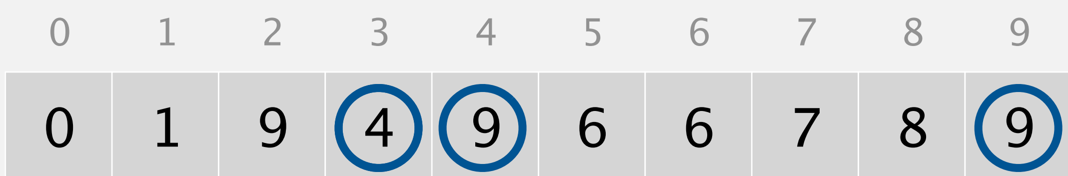
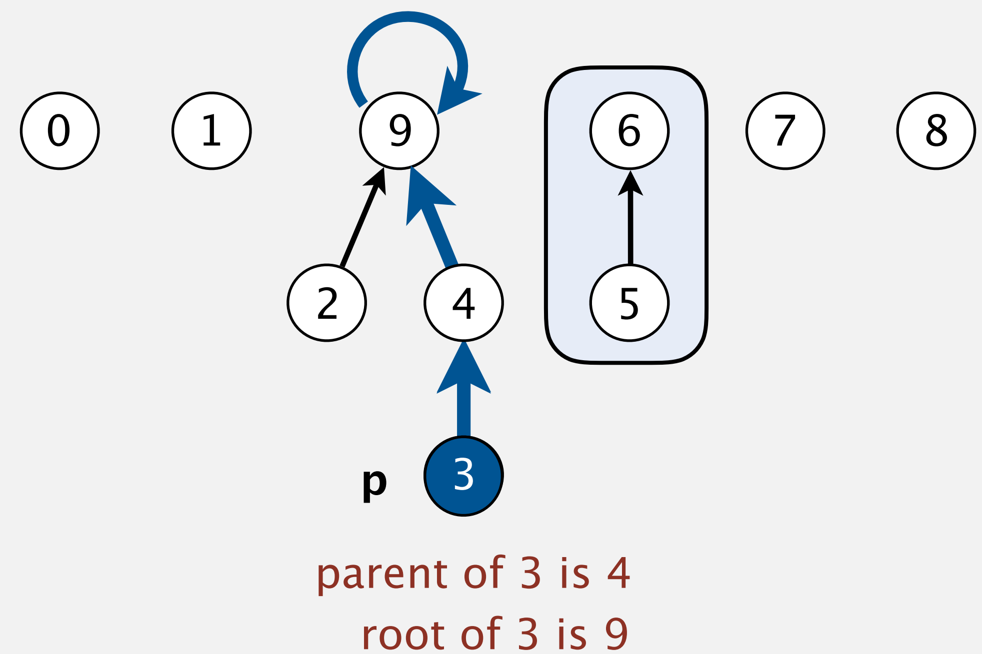
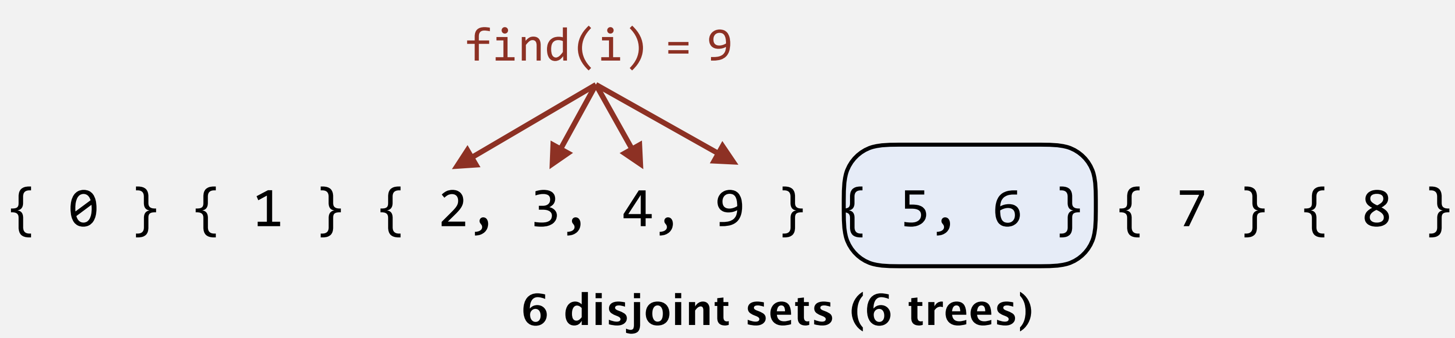
Quick-union [lazy approach]

Data structure.

- Integer array `parent[]` of length `n`, where `parent[i]` is parent of `i` in tree.
- Interpretation: elements in one tree correspond to one set.







Q. How to implement `find(p)` operation?

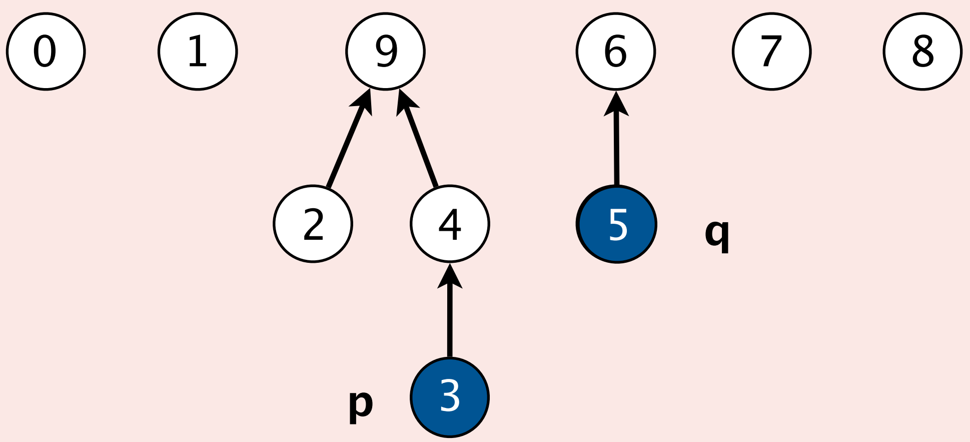
A. Return root of tree containing `p`.

Quick-union quiz

Data structure.

- Integer array `parent[]` of length `n`, where `parent[i]` is parent of `i` in tree.
- Interpretation: elements in one tree correspond to one set.

0	1	2	3	4	5	6	7	8	9
0	1	9	4	9	6	6	7	8	9



How to implement `union(3, 5)` **?**

A. Set `parent[3] = 5`.

B. Set `parent[9] = 5`.

C. Set `parent[9] = 6`.

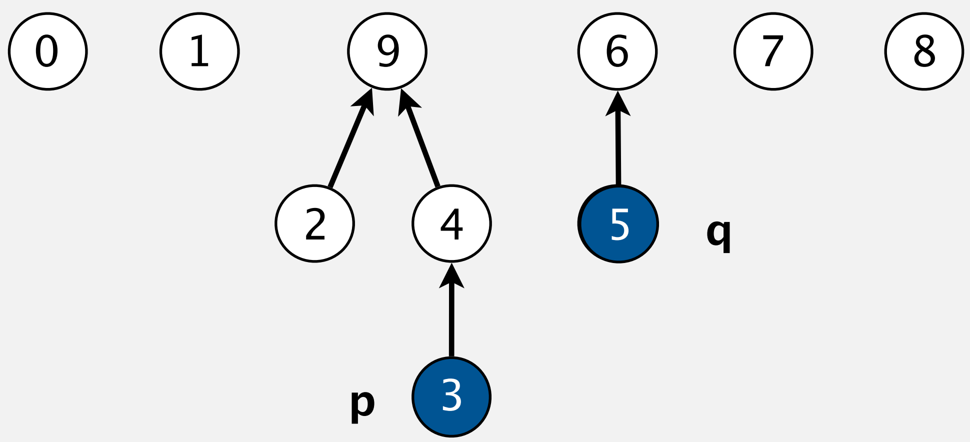
D. Set `parent[2] = parent[3] = parent[4] = parent[9] = 6`.

Quick-union [lazy approach]

Data structure.

- Integer array `parent[]` of length `n`, where `parent[i]` is parent of `i` in tree.
- Interpretation: elements in one tree correspond to one set.

	0	1	2	3	4	5	6	7	8	9
union(3, 5)	0	1	9	4	9	6	6	7	8	9



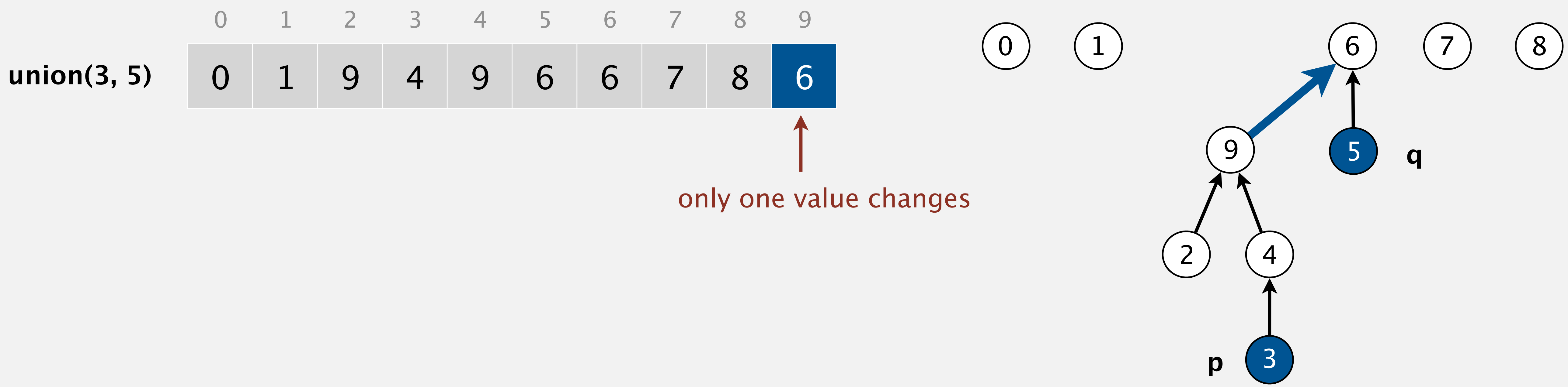
Q. How to implement `union(p, q)`?

A. Set parent of `p`'s root to parent of `q`'s root.

Quick-union [lazy approach]

Data structure.

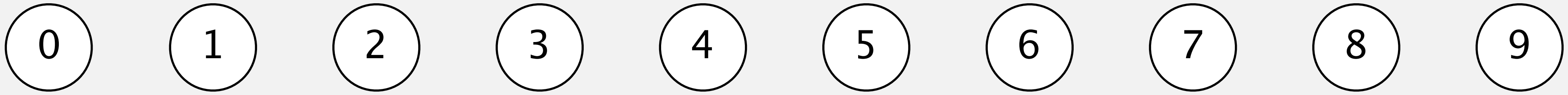
- Integer array `parent[]` of length `n`, where `parent[i]` is parent of `i` in tree.
- Interpretation: elements in one tree correspond to one set.



Q. How to implement `union(p, q)`?

A. Set parent of `p`'s root to parent of `q`'s root.

Quick-union demo



0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

Quick-union: Java implementation

```
public class QuickUnionUF
{
   private int[] parent;

   public QuickUnionUF(int n)
   {
      parent = new int[n];
      for (int i = 0; i < n; i++)
         parent[i] = i;
   }

   public int find(int p)
   {
      while (p != parent[p])
         p = parent[p];
      return p;
   }

   public void union(int p, int q)
   {
      int r1 = find(p);
      int r2 = find(q);
      parent[r1] = r2;
   }
}
```

set parent of each element to itself (n array accesses)

chase parent pointers until reach root (depth of p array accesses)

change root of p to point to root of q (depth of p and q array accesses)

https://algs4.cs.princeton.edu/15uf/QuickUnionUF.java.html

Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find
quick-find	n	n	1
quick-union	n	n	n

← worst case

number of array accesses (ignoring leading constant)

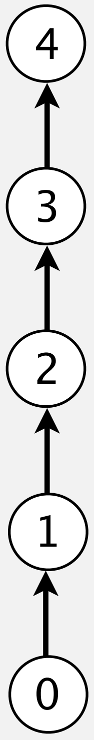
Quick-find defect:

Union too expensive (could be more than n array accesses).

Quick-union defect.

- Trees can get tall.
- Find too expensive (could be more than n array accesses).

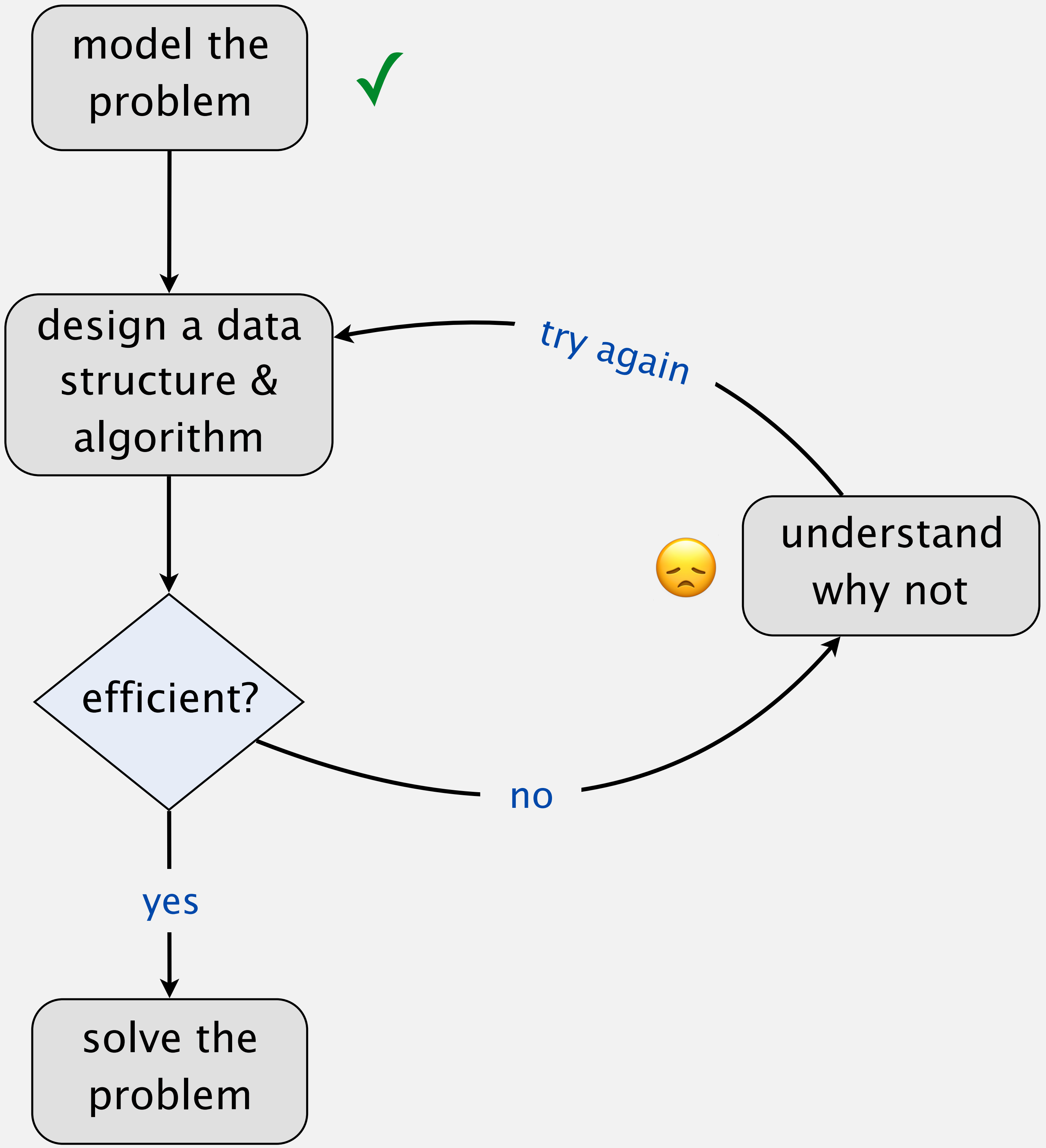
worst-case input



union(0, 1)
union(0, 2)
union(0, 3)
union(0, 4)

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm to solve a computational problem.



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1.5 Union–Find

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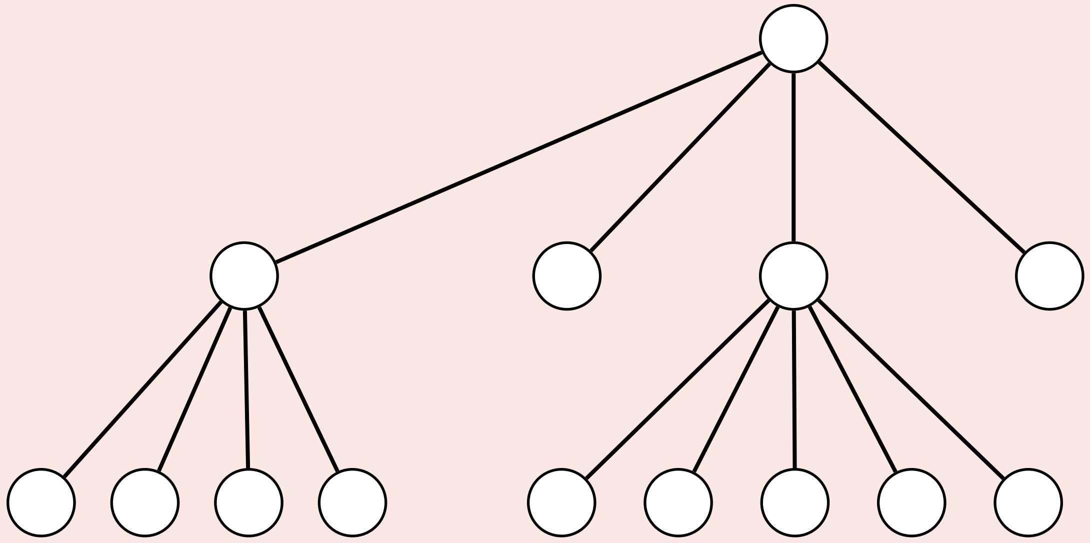
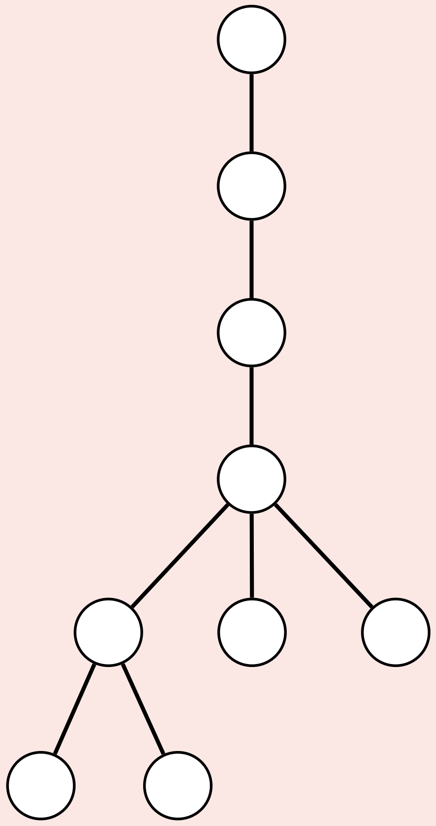
Weighted quick-union quiz

When merging two trees, which strategy is most effective?

- **A.** Link the root of the smaller tree to the root of the larger tree.
- **B.** Link the root of the larger tree to the root of the smaller tree.
- **C.** Link the root of the shorter tree to the root of the taller tree.
- **D.** Link the root of the taller tree to the root of the shorter tree.



shorter and larger tree
(height = 2, size = 14)

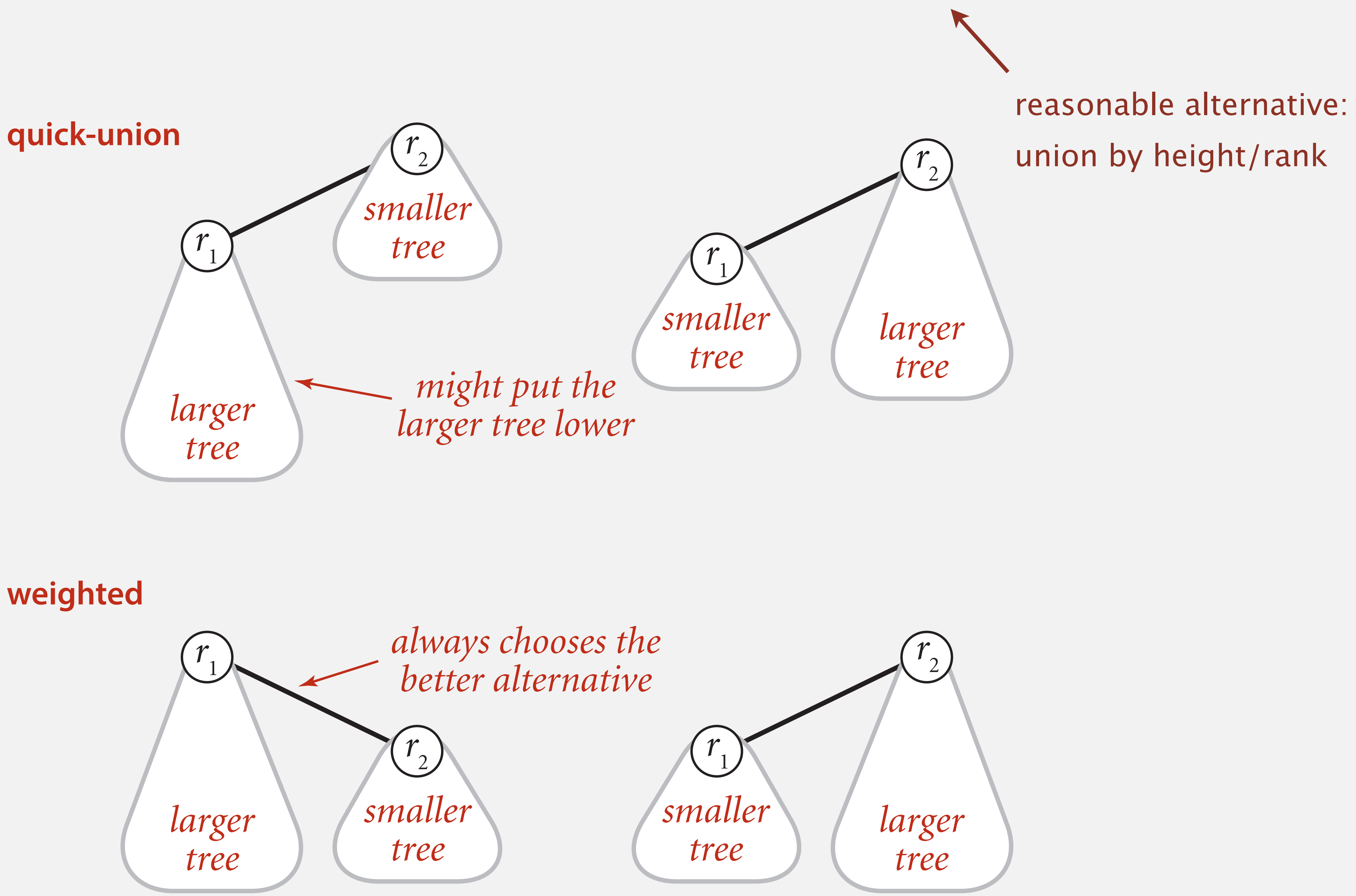


taller and smaller tree
(height = 5, size = 9)

Improvement 1: weighting

Weighted quick-union.

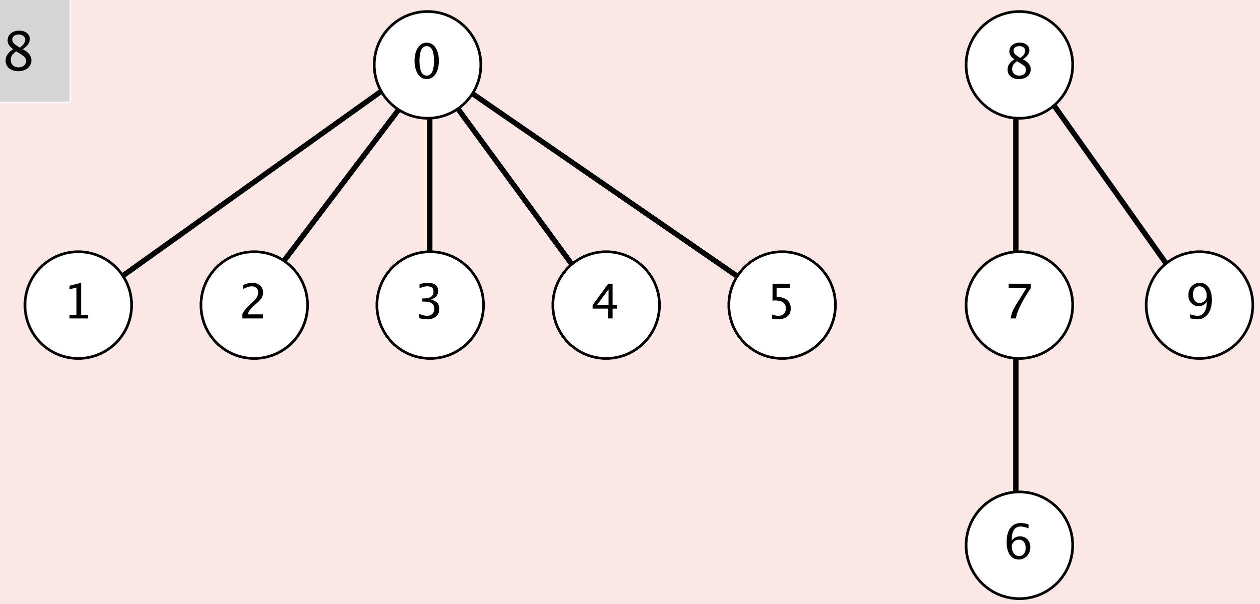
- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of elements).
- Always link root of smaller tree to root of larger tree.



Weighted quick-union quiz

Suppose that the `parent[]` array during weighted quick-union is:

	0	1	2	3	4	5	6	7	8	9
parent[]	0	0	0	0	0	0	7	8	8	8



Which `parent[]` entry changes during `union(2, 6)`?

A. `parent[0]`

B. `parent[2]`

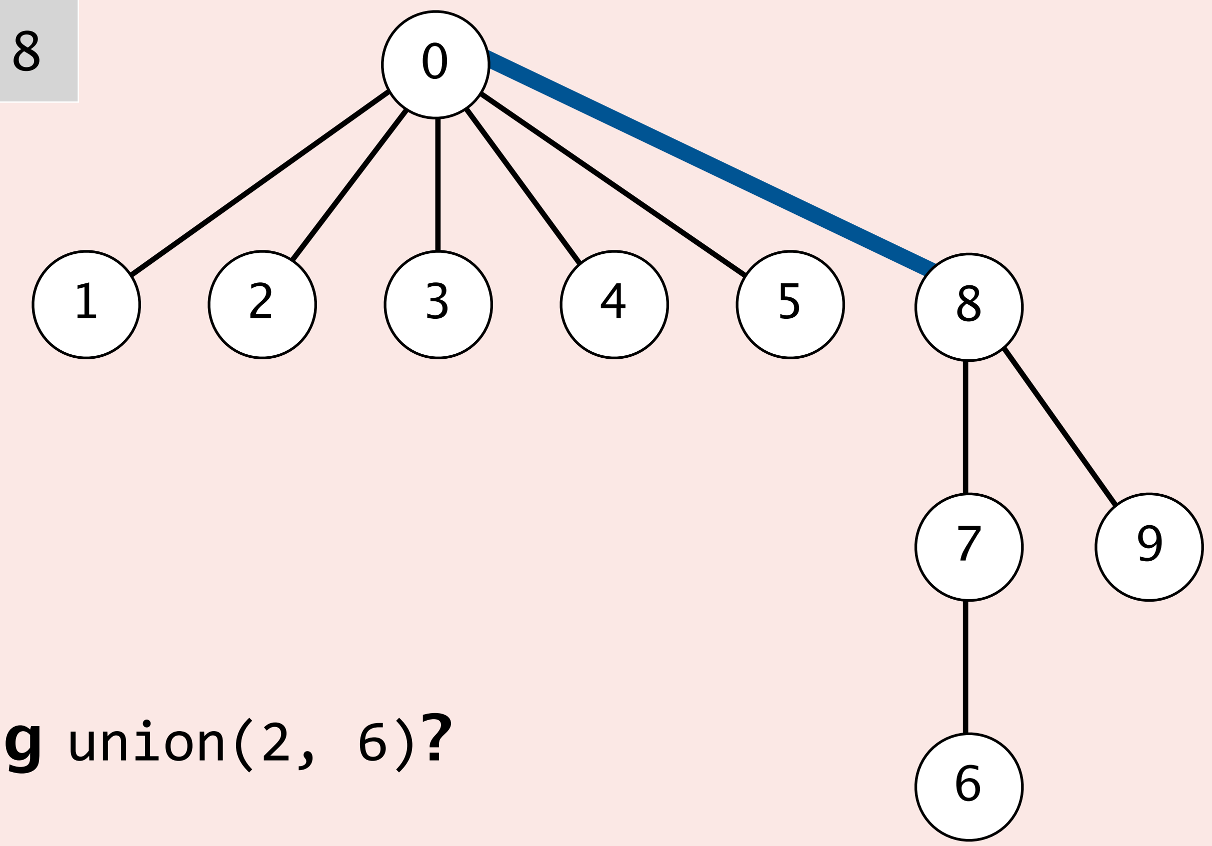
C. `parent[6]`

D. `parent[8]`

Weighted quick-union quiz

Suppose that the `parent[]` array during weighted quick-union is:

	0	1	2	3	4	5	6	7	8	9
parent[]	0	0	0	0	0	0	7	8	0	8



Which `parent[]` entry changes during `union(2, 6)`?

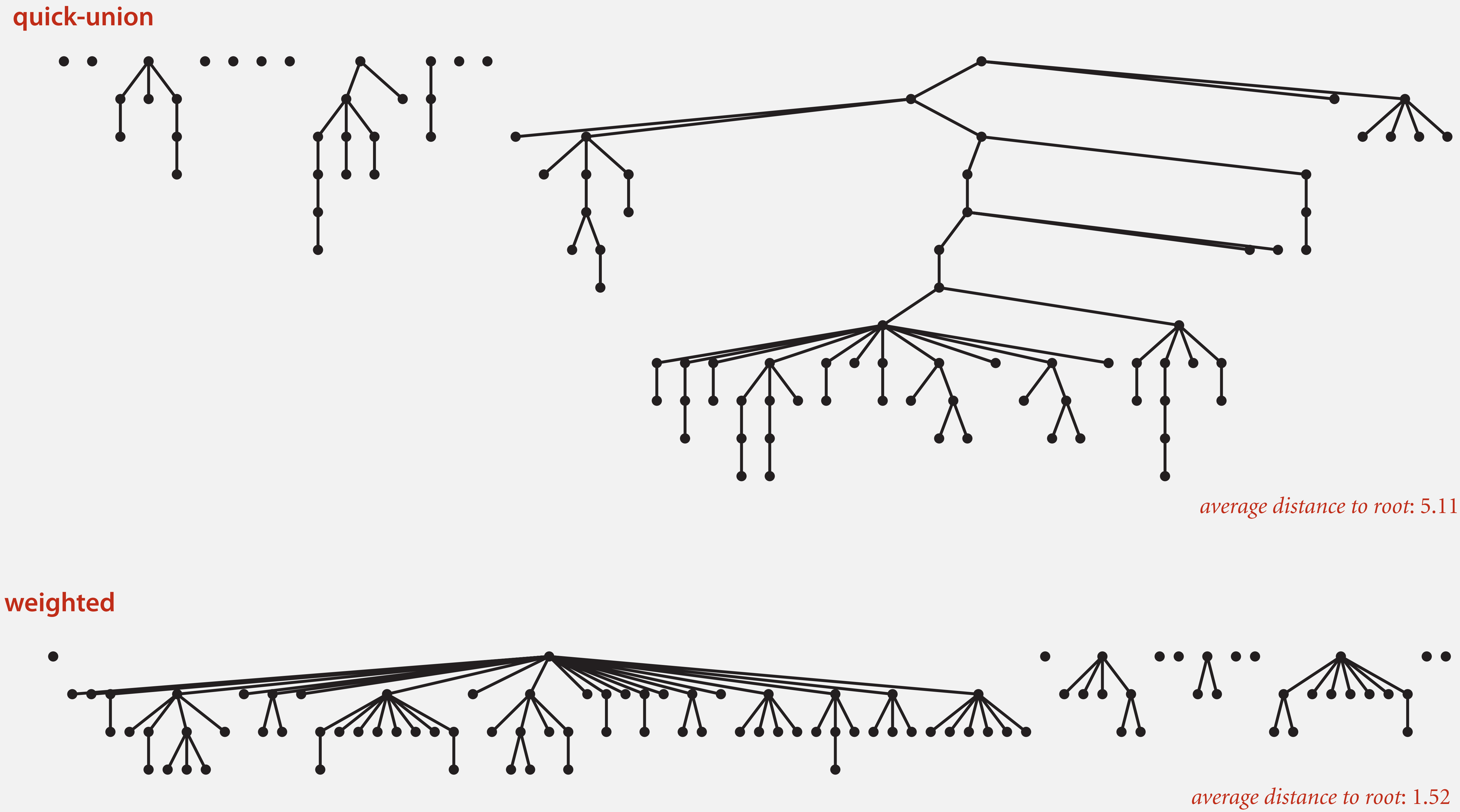
A. `parent[0]`

B. `parent[2]`

C. `parent[6]`

D. `parent[8]`

Quick-union vs. weighted quick-union: larger example



Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array `size[i]` to count number of elements in the tree rooted at `i`, initially `1`.

- Find: identical to quick-union.
- Union: link root of smaller tree to root of larger tree; update `size[]`.

Skipped in class

```
public void union(int p, int q)
{
   int r1 = find(p);
   int r2 = find(q);
   if (r1 == r2) return;

   if (size[r1] >= size[r2])
   { int temp = r1; r1 = r2; r2 = temp; }

   parent[r1] = r2;
   size[r2] += size[r1];
}
```

ensure `r1` is root of smaller tree

link root of smaller tree to root of larger tree

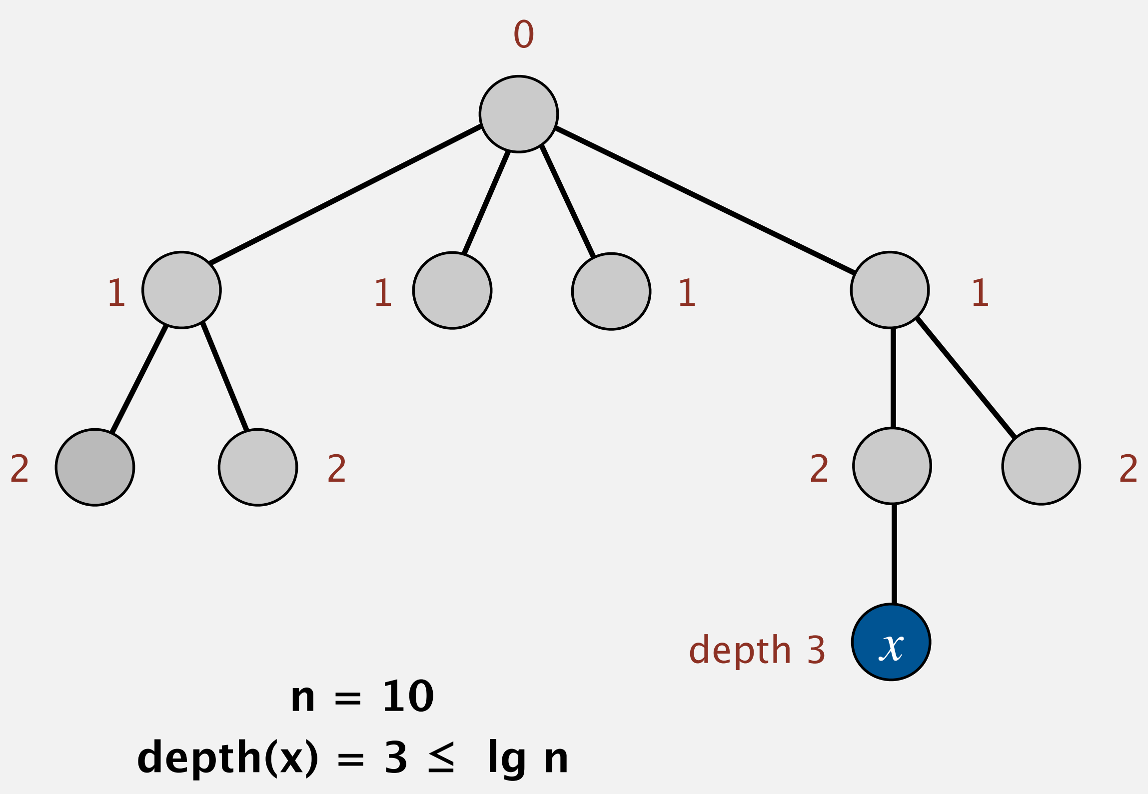
https://algs4.cs.princeton.edu/15uf/WeightedQuickUnionUF.java.html

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given two roots.

Proposition. Depth of any node x is at most $\lg n$. ← in computer science, lg means base-2 logarithm



Weighted quick-union analysis

Skipped in class

Running time.

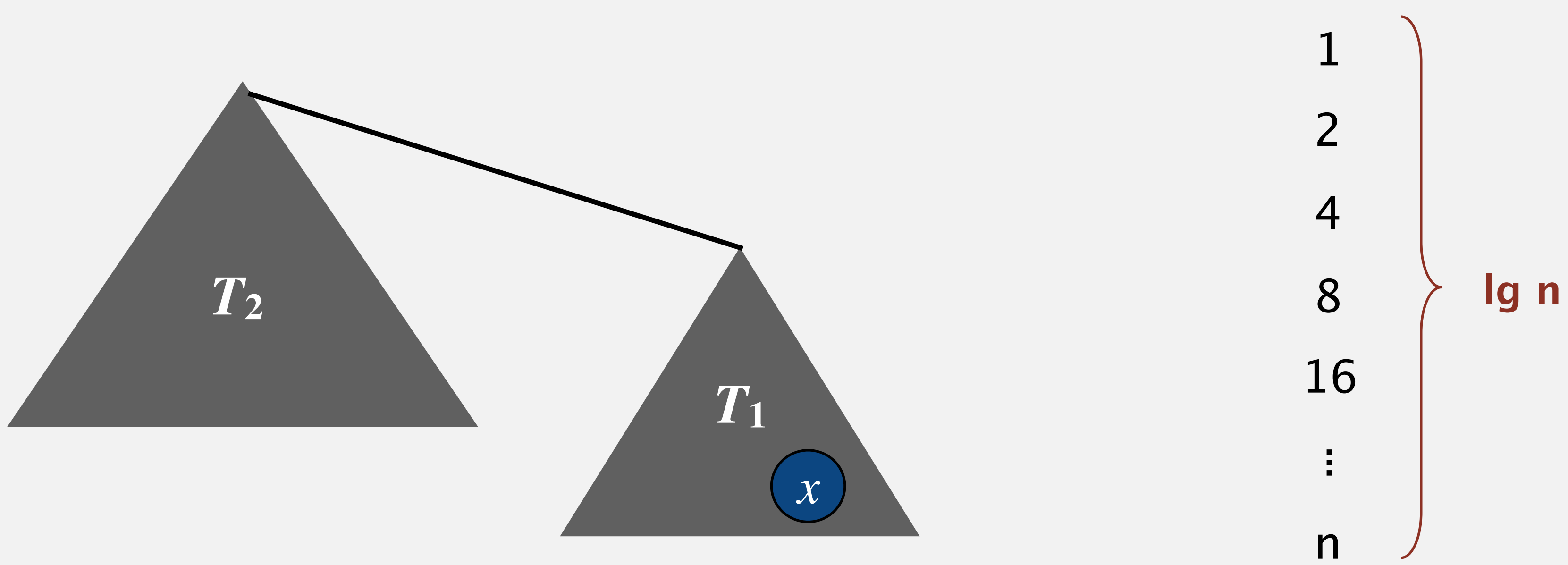
- Find: takes time proportional to depth of p.
- Union: takes constant time, given two roots.

Proposition. Depth of any node x is at most $\lg n$. ← in computer science, lg means base-2 logarithm

Pf. What causes the depth of element x to increase?

Increases by 1 when root of tree T_1 containing x is linked to root of tree T_2.

- The size of the tree containing x at least doubles since $|T_2| \geq |T_1|$.
- Size of tree containing x can double at most $\lg n$ times. Why?



Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given two roots.

Proposition. Depth of any node x is at most $\lg n$.

algorithm	initialize	union	find
quick-find	n	n	1
quick-union	n	n	n
weighted quick-union	n	$\log n$	$\log n$

← log mean logarithm, for some constant base

number of array accesses (ignoring leading constant)

Summary

Key point. Weighted quick-union makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
quick-find	$m\ n$
quick-union	$m\ n$
weighted quick-union	$n + m \log n$
QU + path compression	$n + m \log n$
weighted QU + path compression	$n + m \log^* n$

order of growth for m union-find operations on a set of n elements

Ex. [10^9 unions and finds with 10^9 elements]

- Weighted quick-union reduces run time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

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1.5 Union–Find

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- *applications*

Union-find applications

- Percolation. ⟵ first programming assignment
- Terrain analysis.
- Contiguous regions in images.
- Least common ancestors in trees.
- Games (Go, Hex, maze generation).
- Minimum spanning tree algorithms.
- Equivalence of finite state automata.
- Hoshen-Kopelman algorithm in physics.
- Hindley-Milner polymorphic type inference.
- Compiling equivalence statements in Fortran.
- Connectedness of nodes in a computer network.