Convolutional Codes

COS 463: Wireless Networks
Lecture 9
Kyle Jamieson

[Parts adapted from H. Balakrishnan]

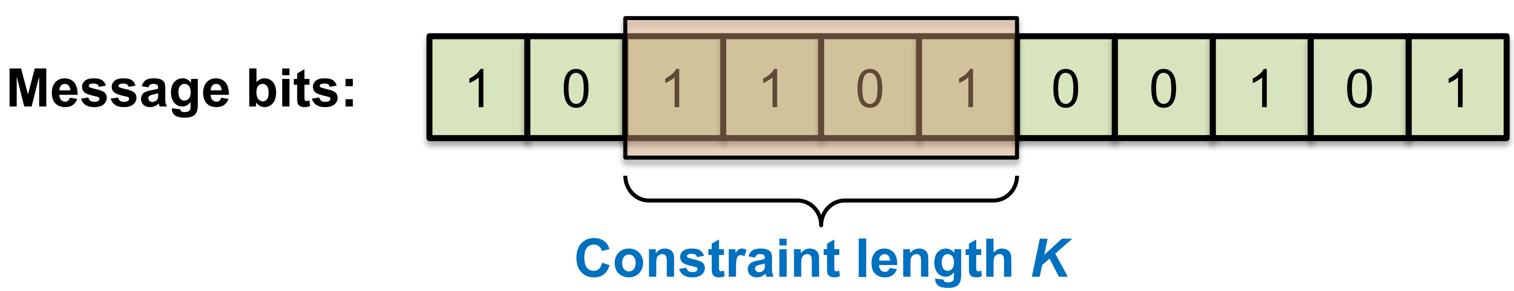
Today

1. **Encoding data using convolutional codes**
   – Encoder state
   – Changing code rate: Puncturing

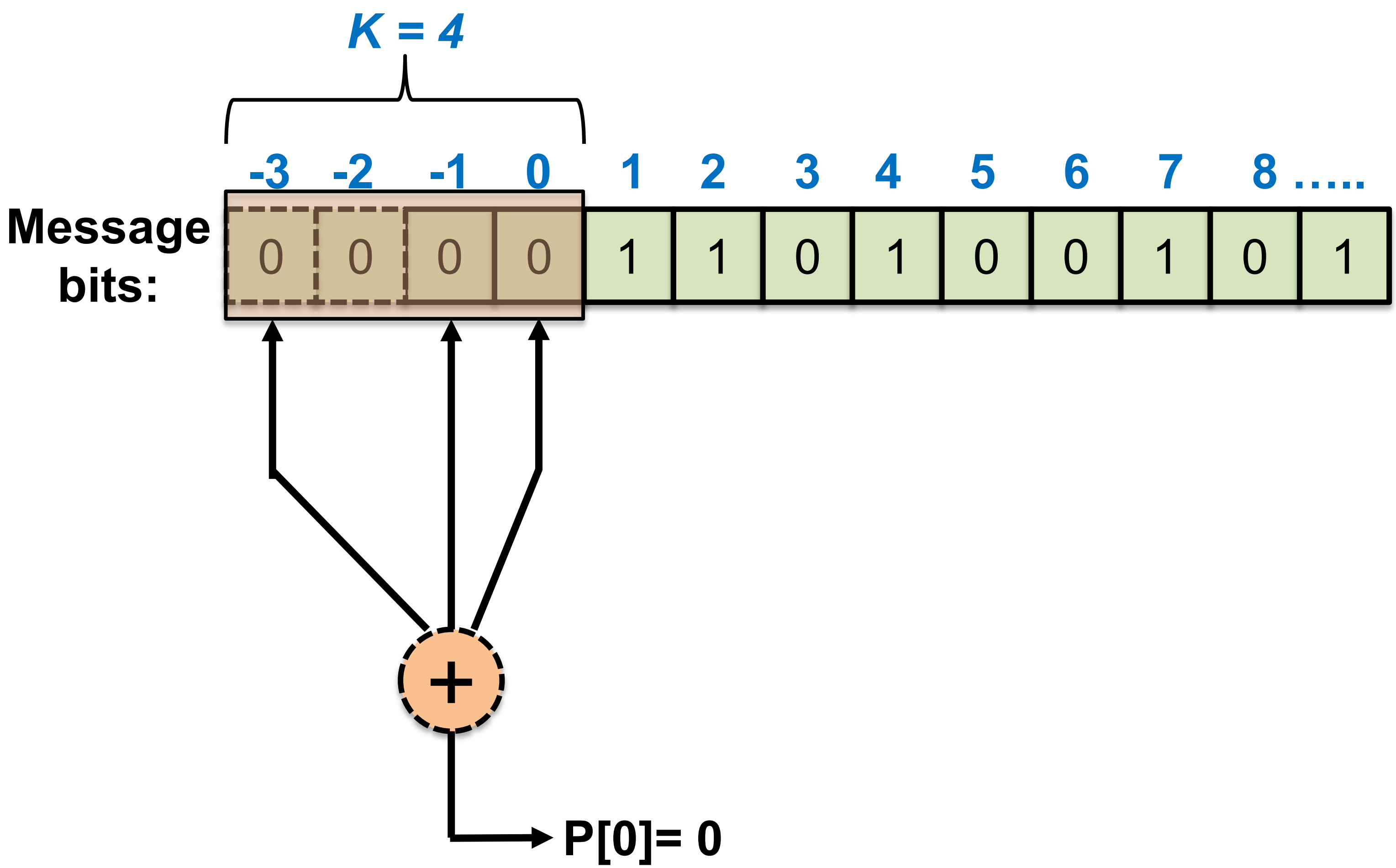
2. Decoding convolutional codes: Viterbi Algorithm

Convolutional Encoding

- Don't send message bits, send **only parity bits**
- Use a **sliding window** to select which message bits may participate in the parity calculations

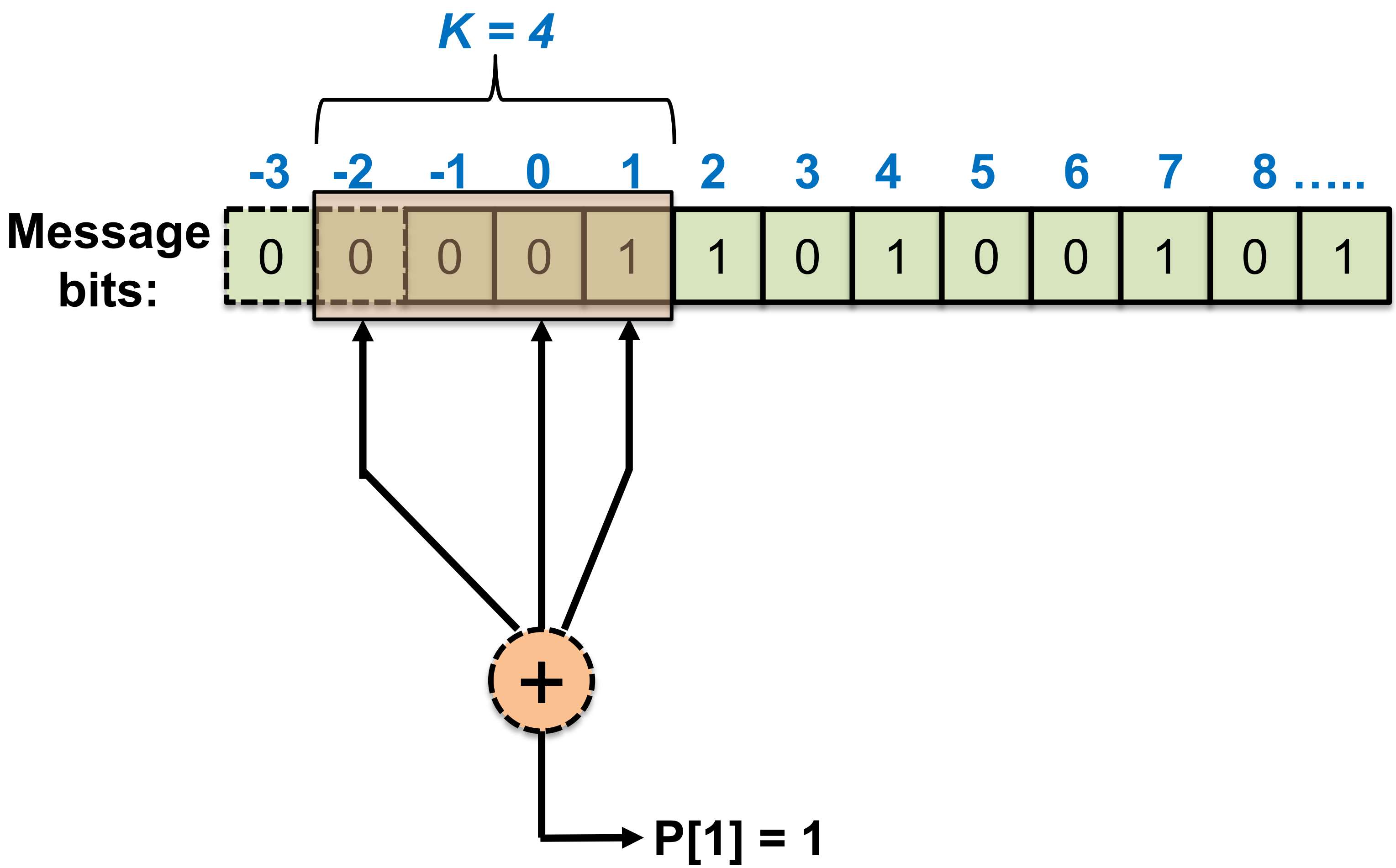


Sliding Parity Bit Calculation



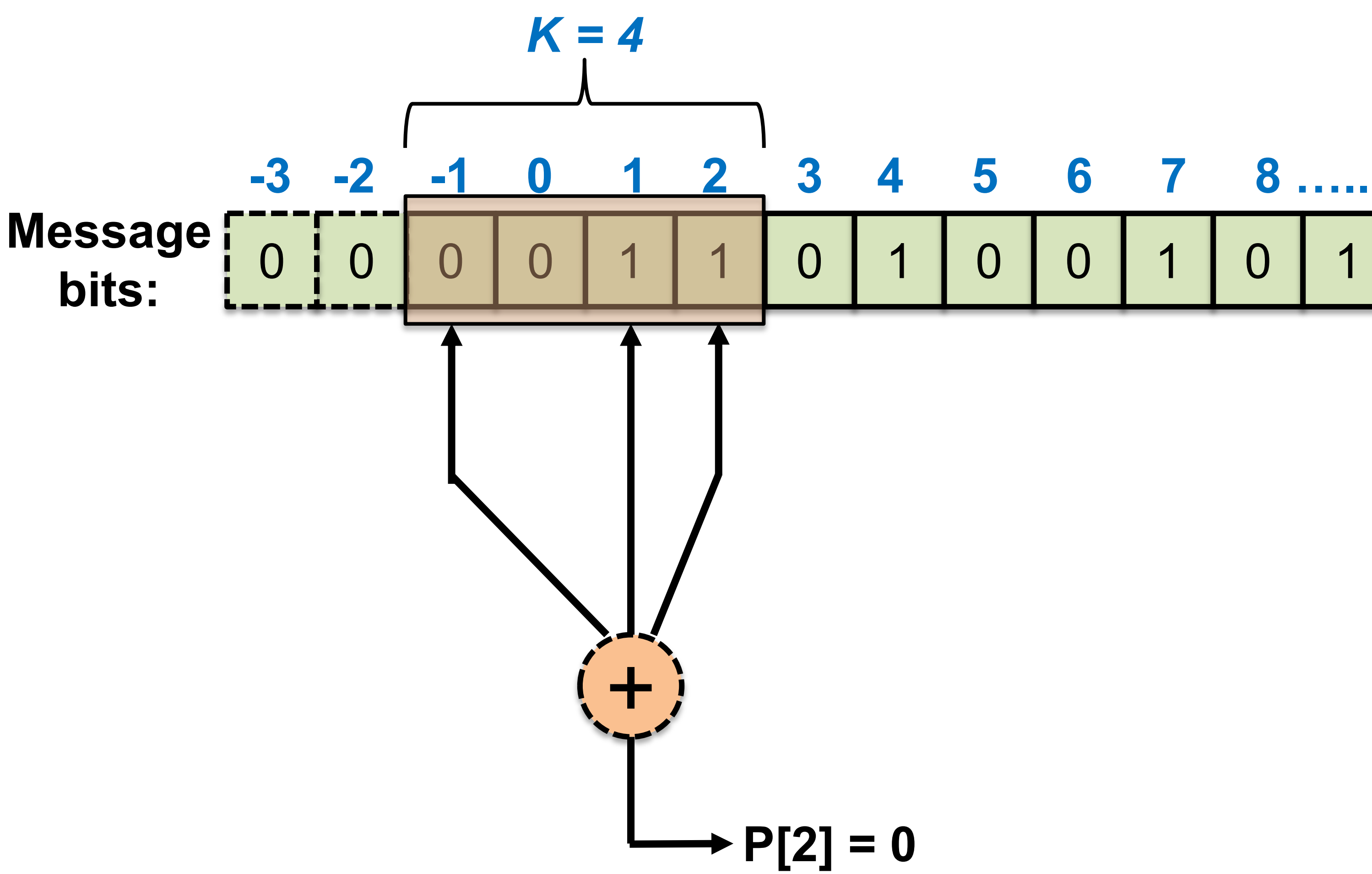
- **Output: 0**

Sliding Parity Bit Calculation



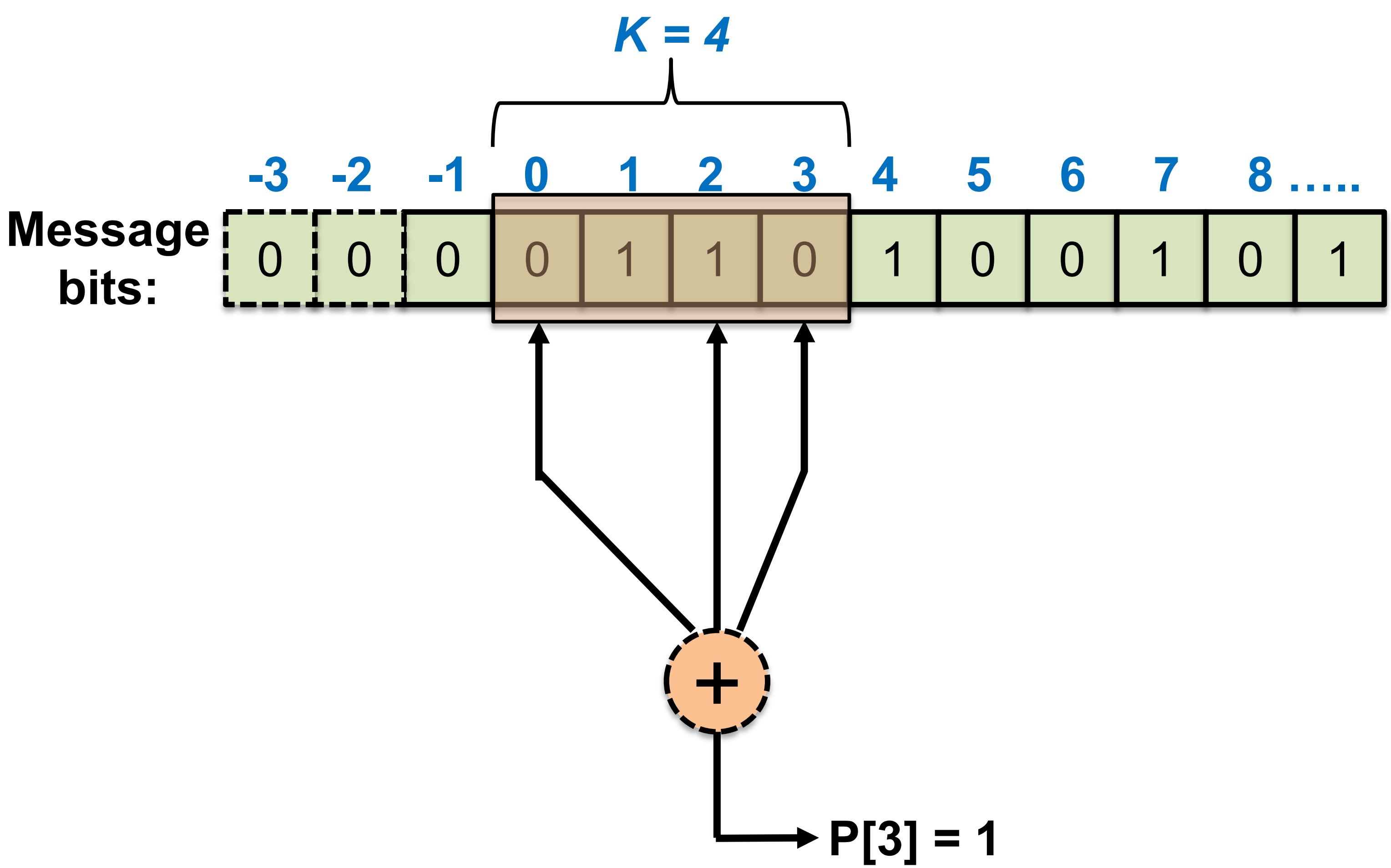
- **Output: 01**

Sliding Parity Bit Calculation



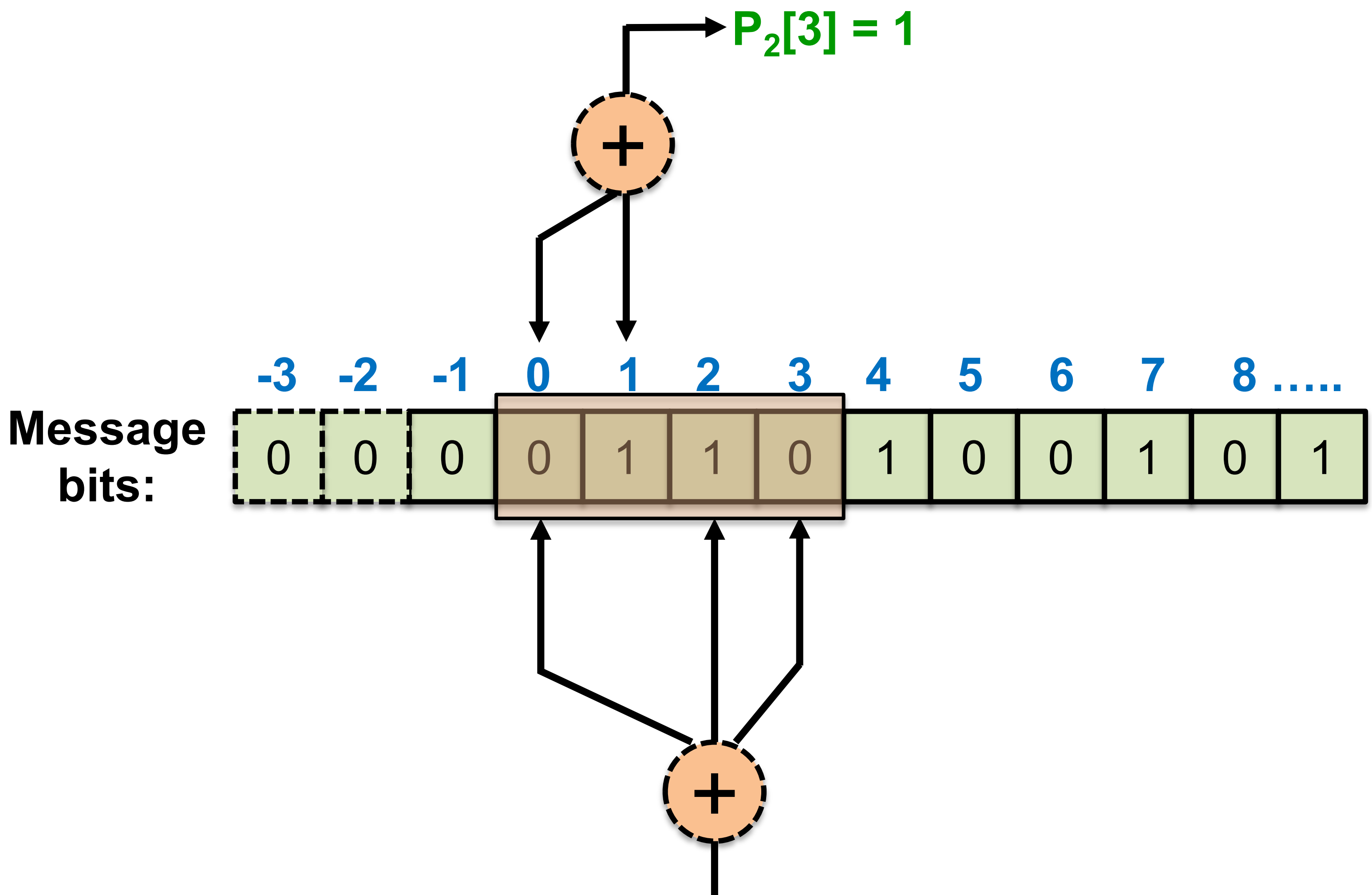
- **Output: 010**

Sliding Parity Bit Calculation



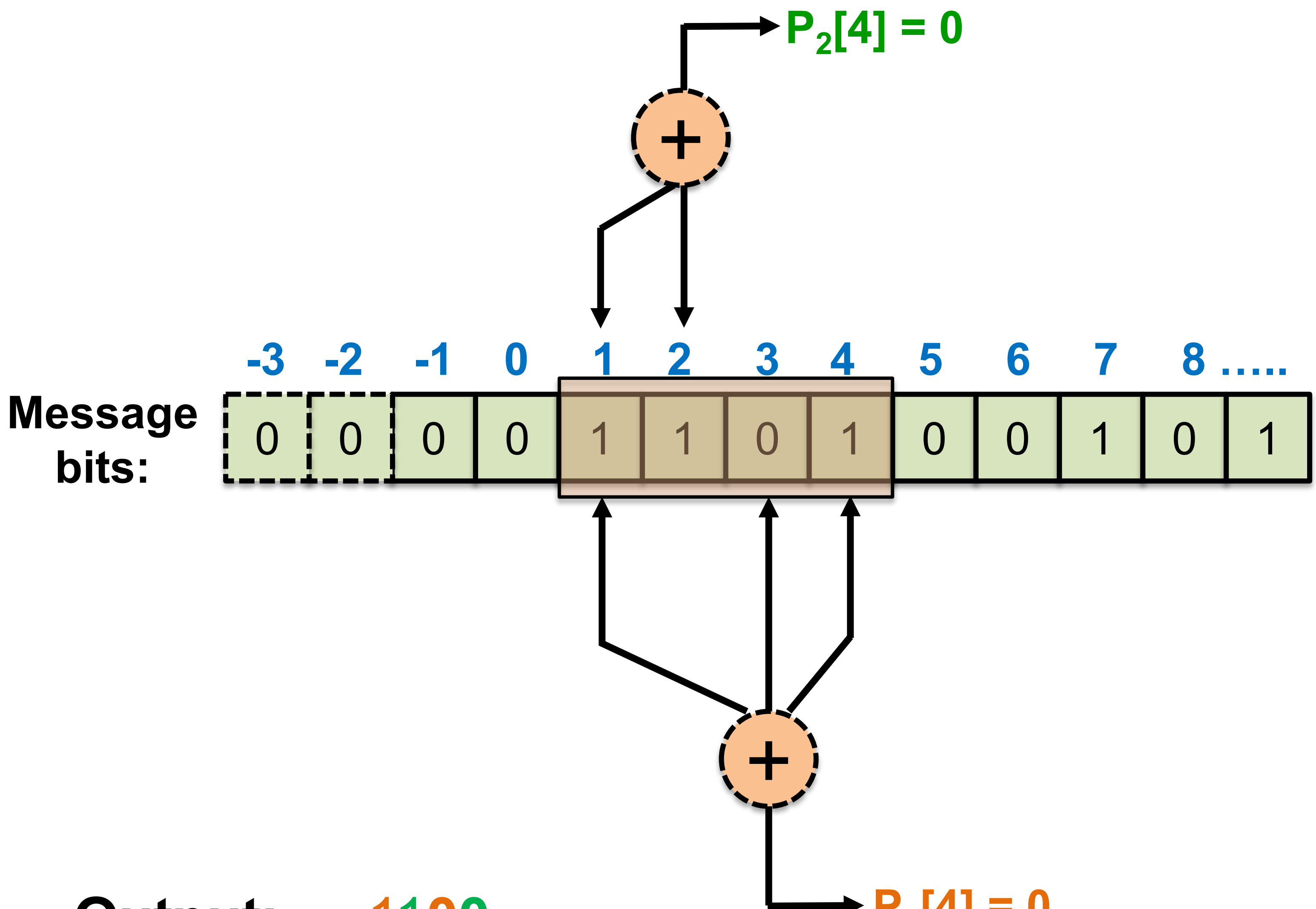
- **Output: 0100**

Multiple Parity Bits



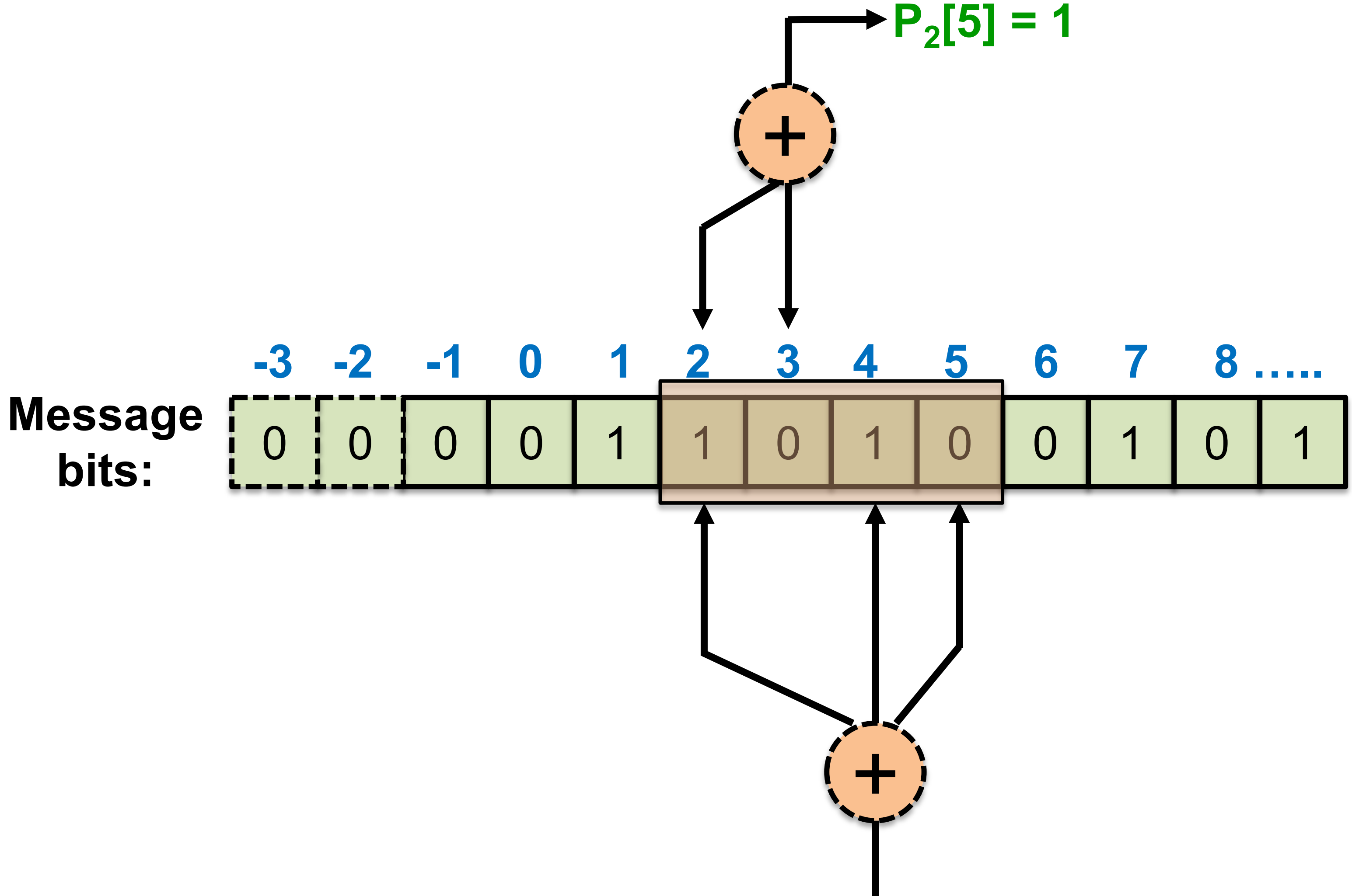
- **Output: ….11**

Multiple Parity Bits



- **Output: ….1100**

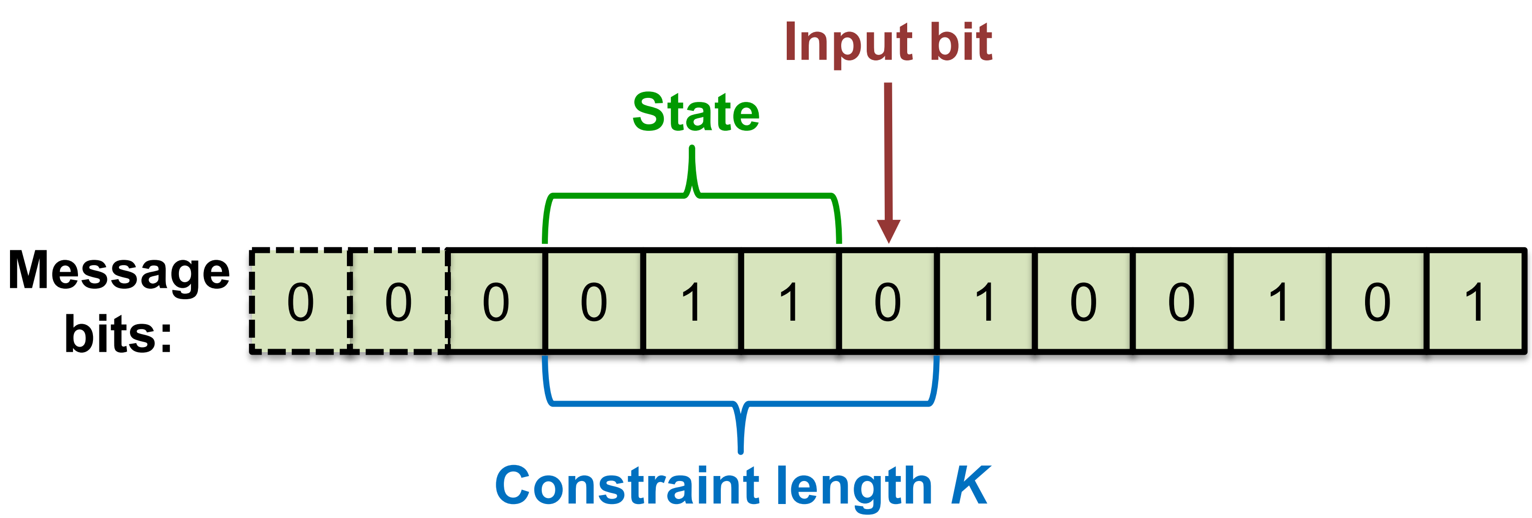
Multiple Parity Bits



- **Output: ….110001**

Encoder State

- **Input bit** and **K-1 bits of current state** determine state on next clock cycle
  - Number of states: 2^{K-1}



Constraint Length

- *K* is the **constraint length of the code**

- **Larger K:**
  - **Greater redundancy**
  - **Better error correction possibilities** (usually, not always)

Transmitting Parity Bits

- **Transmit the parity sequences, not the message itself**
  - Each message bit is "**spread across**" K bits of the output parity bit sequence
  - If using **multiple generators**, **interleave** the bits of each generator
    - *e.g.* (two generators):

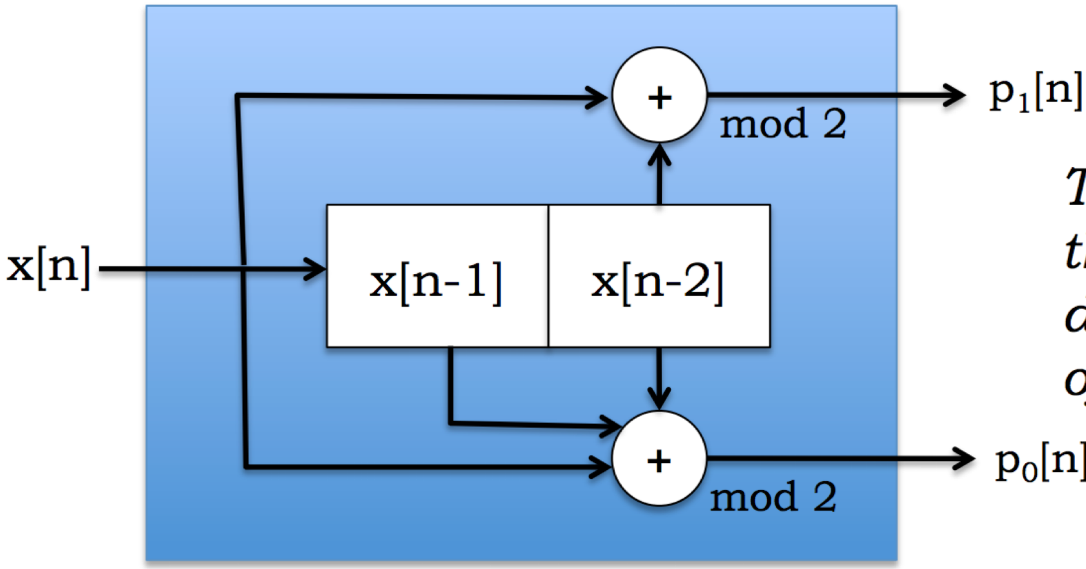
$$p_0[0], p_1[0], p_0[1], p_1[1], p_0[2], p_1[2]$$

Transmitting Parity Bits

- **Code rate** is 1 / #_of_generators
  - *e.g.*, 2 generators → rate = ½

- **Engineering tradeoff**:
  - More generators **improves bit-error correction**
    - But **decreases rate of the code** (the number of message bits/s that can be transmitted)

Shift Register View



The values in the registers define the ***state*** *of the encoder*

- **One message bit x[n] in, two parity bits out**
  - **Each timestep:** message bits **shifted right** by one, the **incoming bit** moves into the **left-most register**

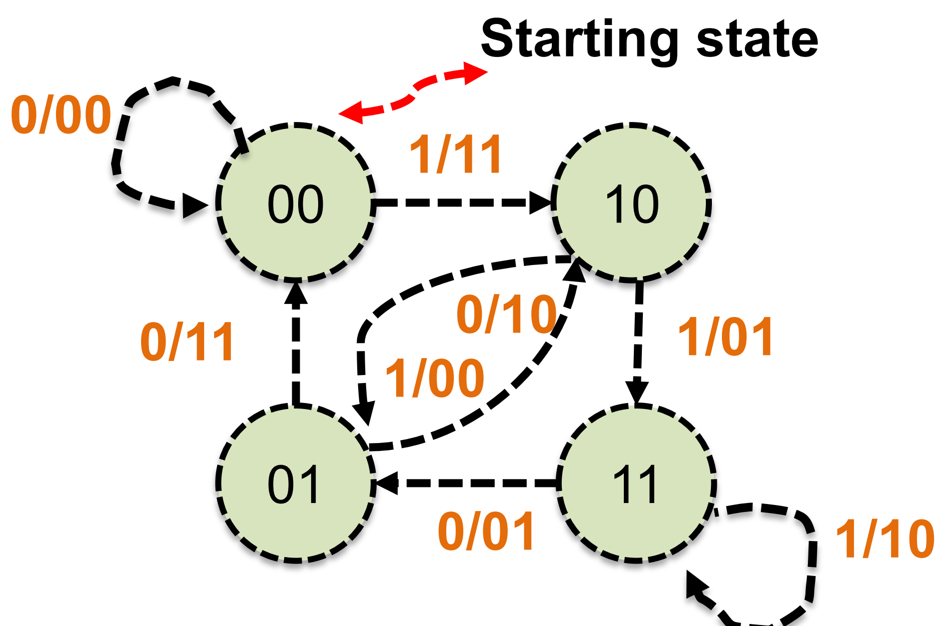
Today

1. **Encoding data using convolutional codes**
   - **Encoder state machine**
   - Changing code rate: Puncturing

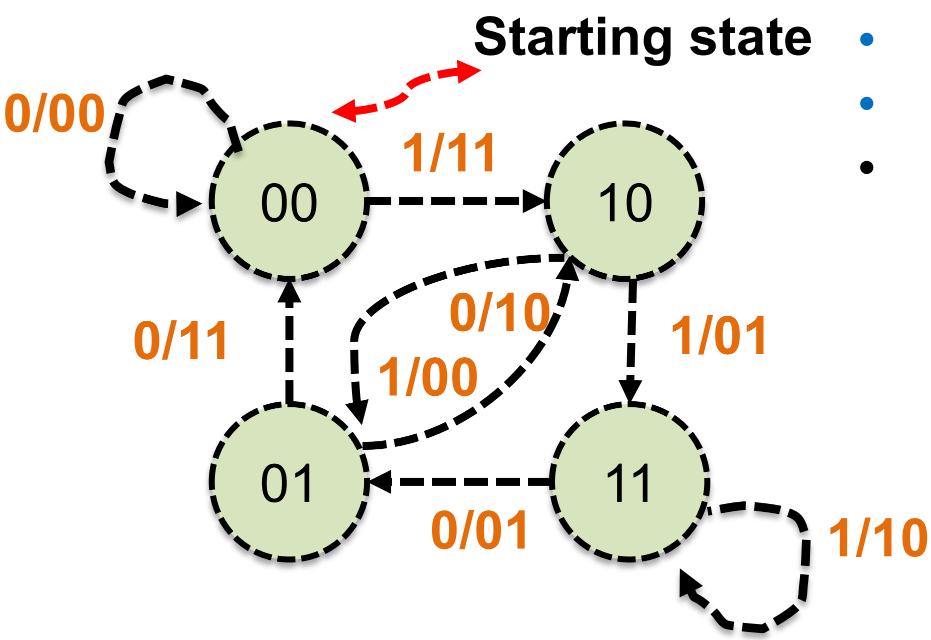
2. Decoding convolutional codes: Viterbi Algorithm

State-Machine View

- Example: **K = 3, code rate = ½, convolutional code**
  - There are 2^{K-1} state
  - **States** labeled with **(x[n-1], x[n-2])**
  - **Arcs** labeled with **$x[n]/p_0[n]p_1[n]$**
  - **Generator: g_0 = 111, g_1 = 101**
  - **msg** = 101100



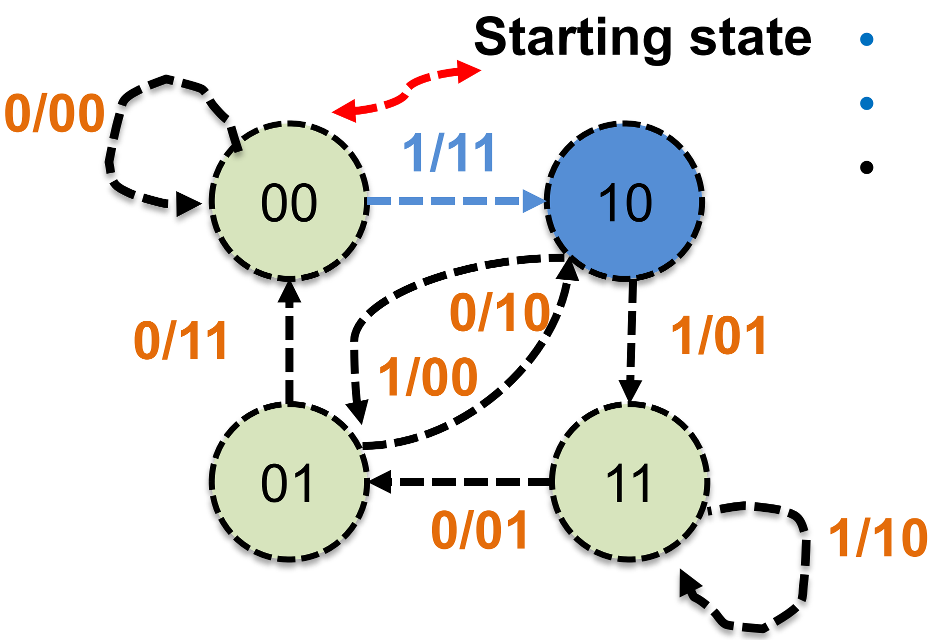
State-Machine View



- $P_0[n] = (1*x[n] + 1*x[n-1] + 1*x[n-2]) \bmod 2$
- $P_1[n] = (1*x[n] + 0*x[n-1] + 1*x[n-2]) \bmod 2$
- **Generators**: $g_0 = 111$, $g_1 = 101$

- **msg** = 101100
- **Transmit**:

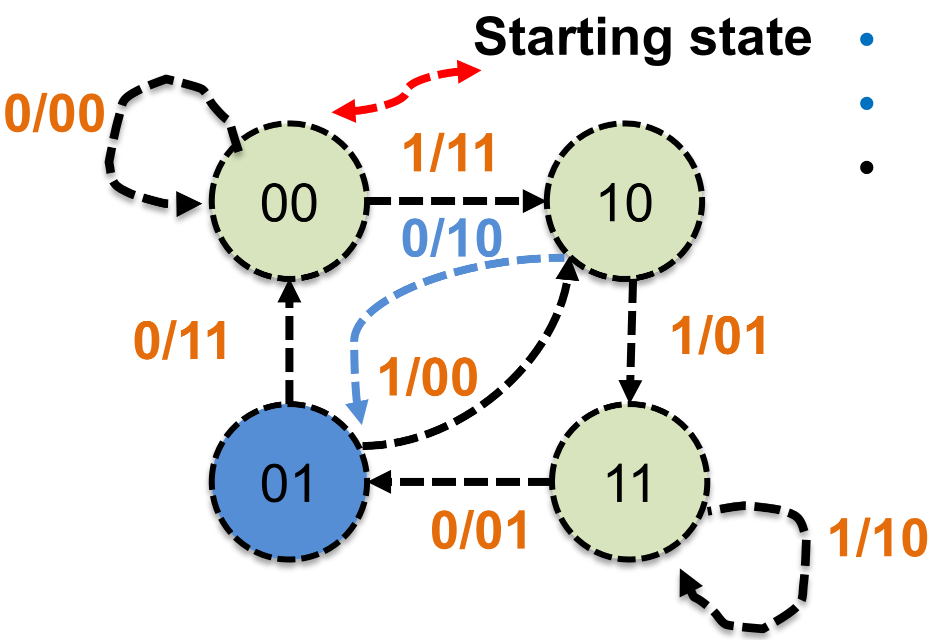
State-Machine View



- $P_0[n]$ = 1*1 + 1*0 + 1*0 mod 2
- $P_1[n]$ = 1*1 + 0*0 + 1*0 mod 2
- **Generators**: g_0 = 111, g_1 = 101

- **msg** = **1**01100
- **Transmit**: **11**

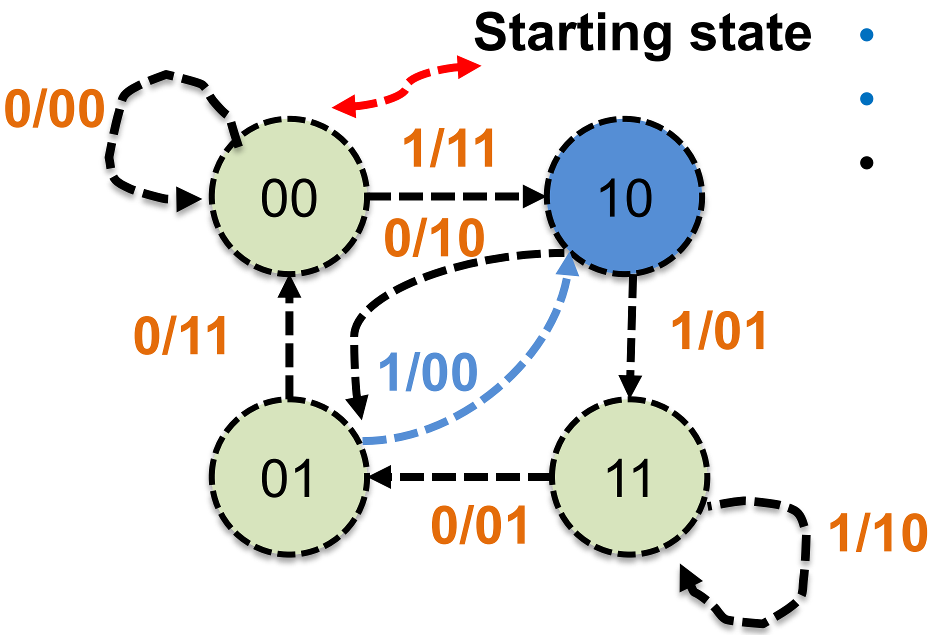
State-Machine View



- $P_0[n] = 1*0 + 1*1 + 1*0 \bmod 2$
- $P_1[n] = 1*0 + 0*1 + 1*0 \bmod 2$
- **Generators**: $g_0 = 111$, $g_1 = 101$

- **msg** = 1**0**1100
- **Transmit**: 11 **10**

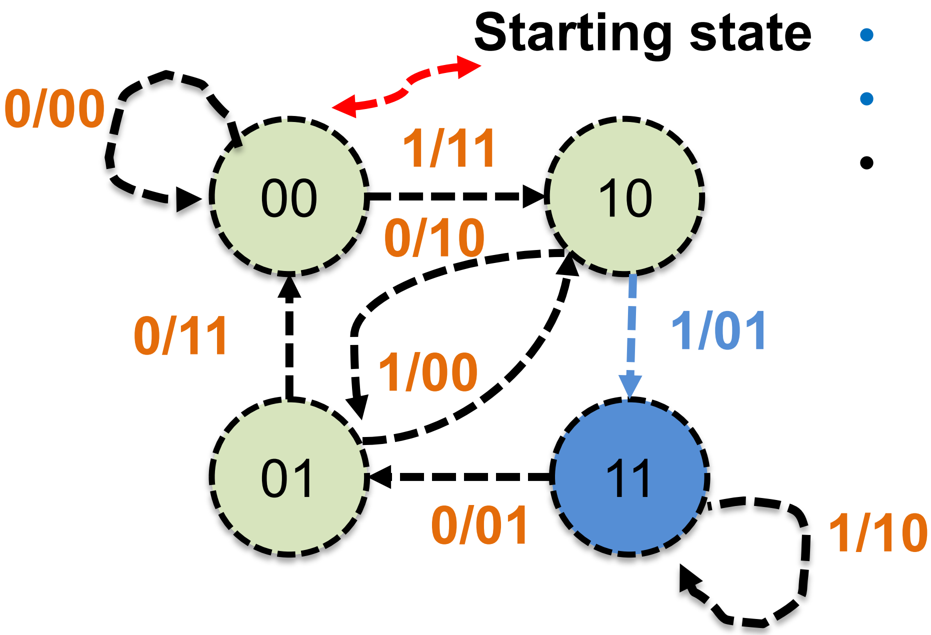
State-Machine View



- $P_0[n] = 1*1 + 1*0 + 1*1 \mod 2$
- $P_1[n] = 1*1 + 0*0 + 1*1 \mod 2$
- **Generators**: $g_0 = 111$, $g_1 = 101$

- **msg** = 10**1**100
- **Transmit**: 11 10 **00**

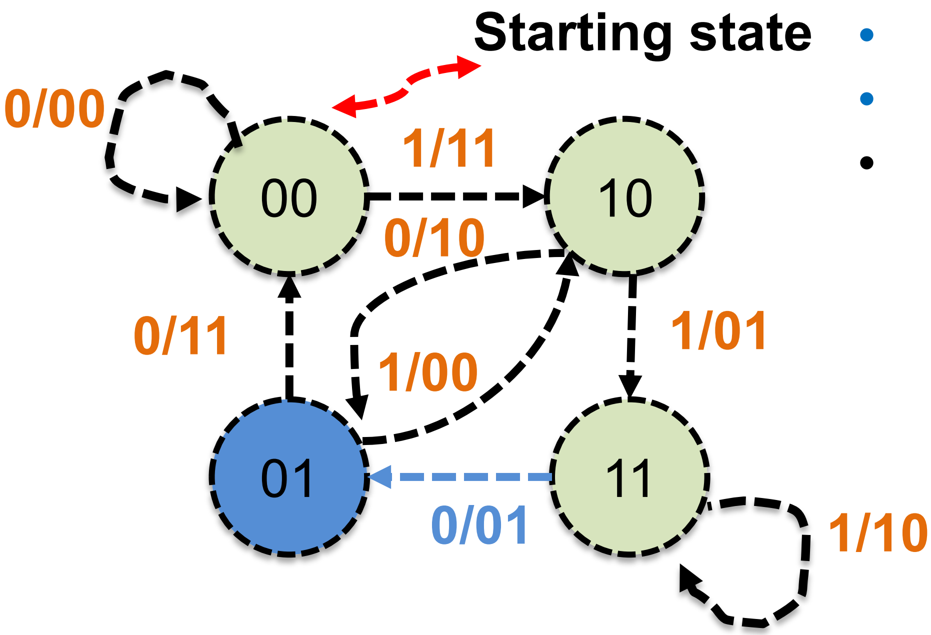
State-Machine View



- $P_0[n] = 1*1 + 1*1 + 1*0$
- $P_1[n] = 1*1 + 0*1 + 1*0$
- **Generators**: $g_0 = 111$, $g_1 = 101$

- **msg** = 101**1**00
- **Transmit**: 11 10 00 **01**

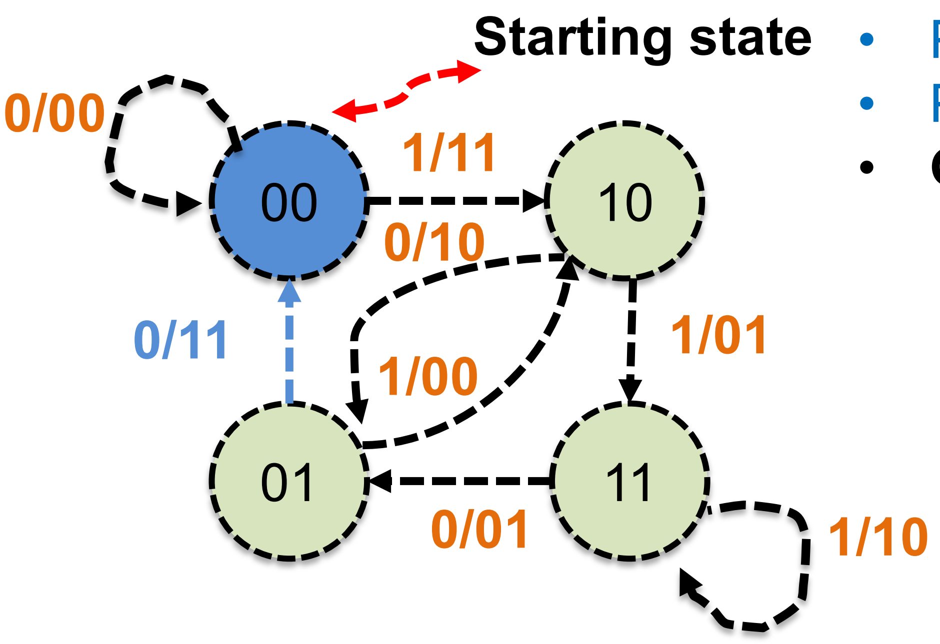
State-Machine View



- $P_0[n] = 1*0 + 1*1 + 1*1$
- $P_1[n] = 1*0 + 0*1 + 1*1$
- **Generators**: $g_0 = 111$, $g_1 = 101$

- **msg** = 1011**00**
- **Transmit**: 11 10 00 01 **01**

State-Machine View



- $P_0[n] = 1*0 + 1*0 + 1*1$
- $P_1[n] = 1*0 + 0*0 + 1*1$
- **Generators**: $g_0 = 111$, $g_1 = 101$

- **msg** = 10110**0**
- **Transmit**: 11 10 00 01 01 **11**

Today

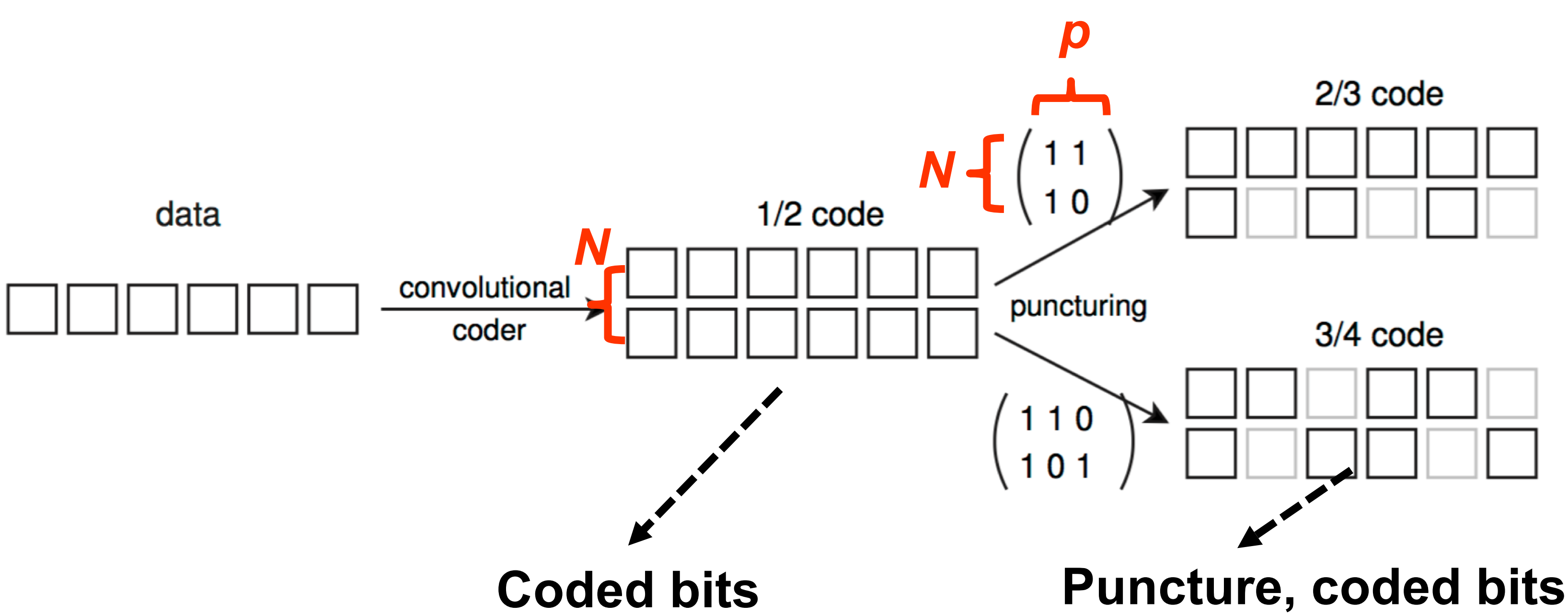
1. **Encoding data using convolutional codes**
   - Encoder state machine
   - **Changing code rate: Puncturing**

2. Decoding convolutional codes: Viterbi Algorithm

Varying the Code Rate

- How to increase/decrease rate?

- Transmitter and receiver agree on coded bits to **omit**
  - ***Puncturing table*** indicates which bits to include (**1**)
    - Contains ***p*** columns, ***N*** rows



Punctured convolutional codes: example

- **Coded bits =**

0	0	1	0	1
0	0	1	1	1

- **With Puncturing:**

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Puncturing table

Punctured convolutional codes: example

- **Coded bits =**

0	0	1	0	1
0	0	1	1	1

- **With Puncturing:**

3 out of 4 bits are used

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

2 out of 4 bits are used

Punctured convolutional codes: example

- **Coded bits =**

0	0	1	0	1
0	0	1	1	1

- **With Puncturing:**

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

- **Punctured, coded bits:**

0
0

Punctured convolutional codes: example

- **Coded bits =**

0	0	1	0	1
0	0	1	1	1

- **With Puncturing:**

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

- **Punctured, coded bits:**

0	0
0	

Punctured convolutional codes: example

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0	0	1	0	1
0	0	1	1	1

- **With Puncturing:**

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

- **Punctured, coded bits:**

0	0	1
0		

Punctured convolutional codes: example

- **Coded bits =**

0	0	1	0	1
0	0	1	1	1

- **With Puncturing:**

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

- **Punctured, coded bits:**

0	0	1	
0			1

Punctured convolutional codes: example

- **Coded bits =**

0	0	1	0	1
0	0	1	1	1

- **With Puncturing:**

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

- **Punctured, coded bits:**

0	0	1		1
0			1	1

Punctured convolutional codes: example

- **Coded bits =**

0	0	1	0	1
0	0	1	1	1

- **Punctured, coded bits:**

0	0	1		1
0			1	1

- **Punctured rate is**: **R = (1/2) / (5/8) = 4/5**

Today

1. Encoding data using convolutional codes
   – Changing code rate: Puncturing

2. **Decoding convolutional codes: Viterbi Algorithm**
   – Hard decision decoding
   – Soft decision decoding

Motivation: The Decoding Problem

- Received bits:
  000101100110

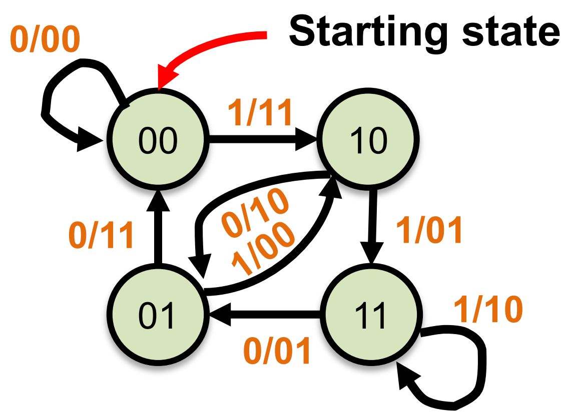
- Some errors have occurred

- *What's the 4-bit message?*

- **Most likely: 0111**
  – Message whose codeword is **closest to received bits** in Hamming distance

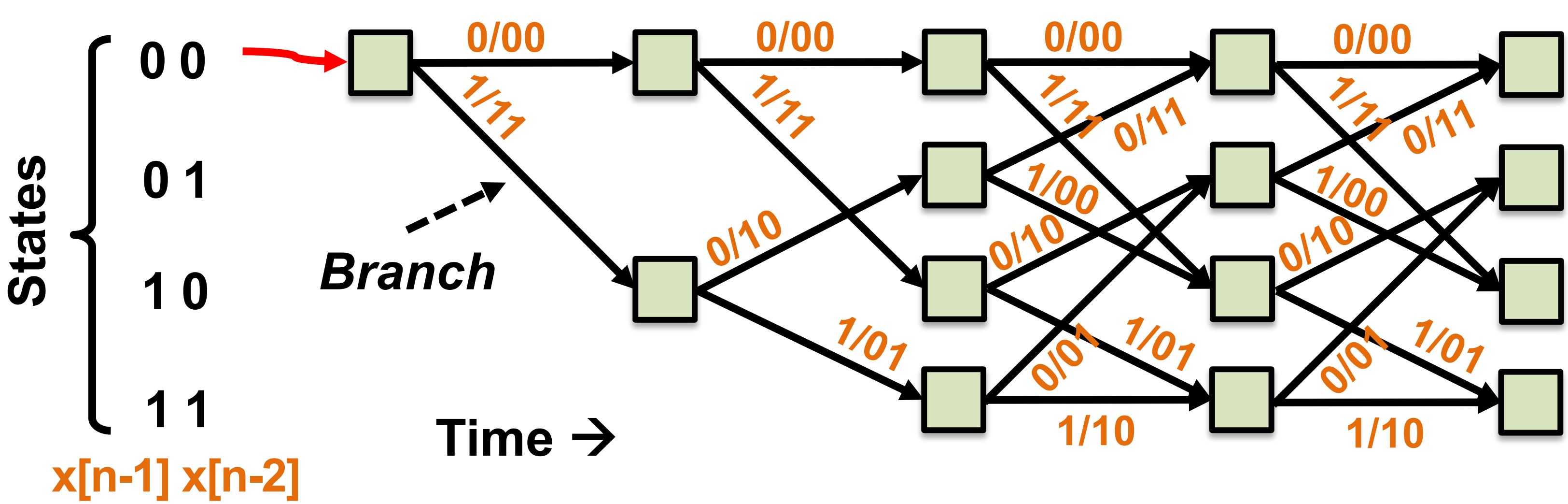
Message	Coded bits	Hamming distance
0000	000000000000	5
0001	000000111011	--
0010	000011101100	--
0011	000011010111	--
0100	001110110000	--
0101	001110001011	--
0110	001101011100	--
0111	001101100111	**2**
1000	111011000000	--
1001	111011111011	--
1010	111000101100	--
1011	111000010111	--
1100	110101110000	--
1101	110101001011	--
1110	110110011100	--
1111	110110100111	--

The Trellis



- **Vertically,** lists encoder **states**
- **Horizontally,** tracks **time steps**
- ***Branches*** connect states in successive time steps

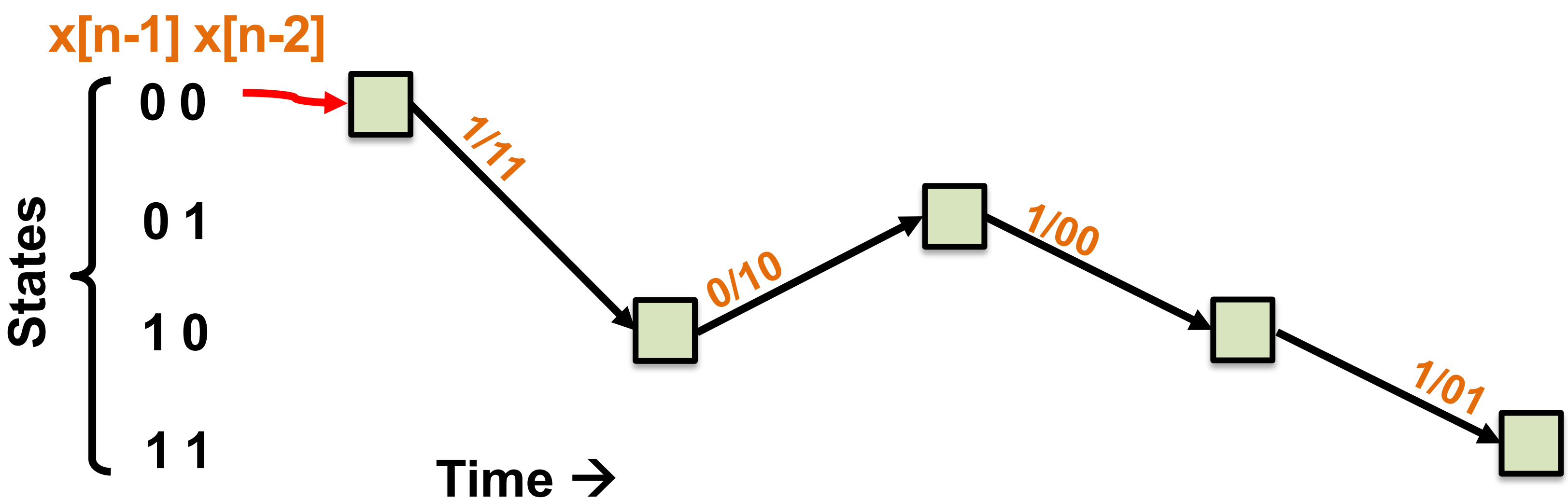
Trellis:



The Trellis: Sender's View

- **At the sender, transmitted bits** trace a unique, single ***path*** **of branches through the trellis**
  - *e.g.* transmitted data bits **1 0 1 1**

- **Recover transmitted bits ⟺ Recover path**



Viterbi algorithm

- Andrew Viterbi (USC)
- **Want**: Most likely **sent bit sequence**
- Calculates **most likely path** through **trellis**

1. ***Hard Decision* Viterbi algorithm:** Have **possibly-corrupted** encoded **bits**, after reception
2. ***Soft Decision* Viterbi algorithm:** Have **possibly-corrupted likelihoods** of each bit, after reception
   - *e.g.*: "this bit is 90% likely to be a 1."

Viterbi algorithm: Summary

- ***Branch metrics*** score **likelihood of each trellis branch**

- At any given time there are **$2^{K\text{-}1}$ most likely messages** we're tracking (one for each state)
  - One message ↔ one trellis path
  - ***Path metrics*** score **likelihood of each trellis path**

- **Most likely message** is the one that produces the **smallest path metric**

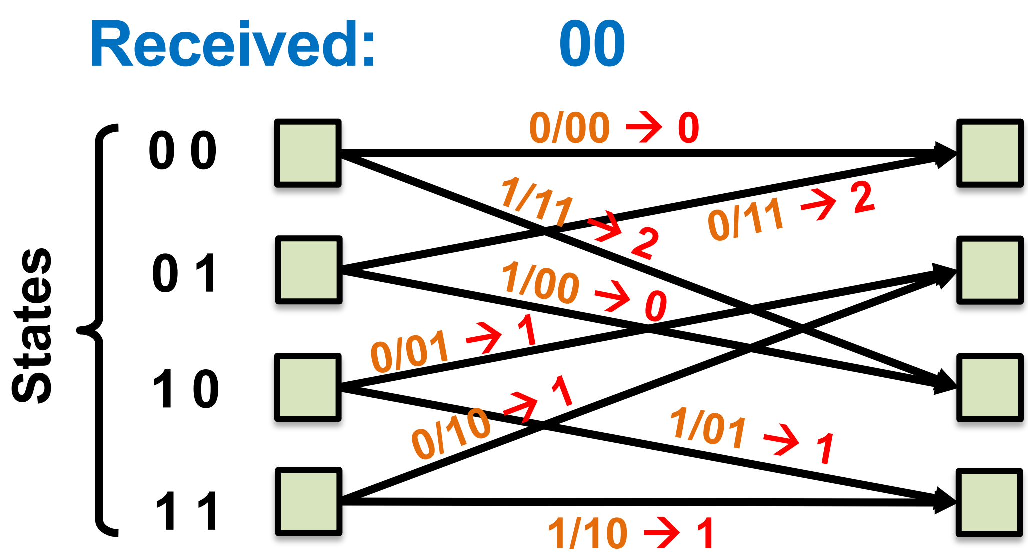
Today

1. Encoding data using convolutional codes
   – Changing code rate: Puncturing

2. **Decoding convolutional codes: Viterbi Algorithm**
   – **Hard decision decoding**
   – Soft decision decoding

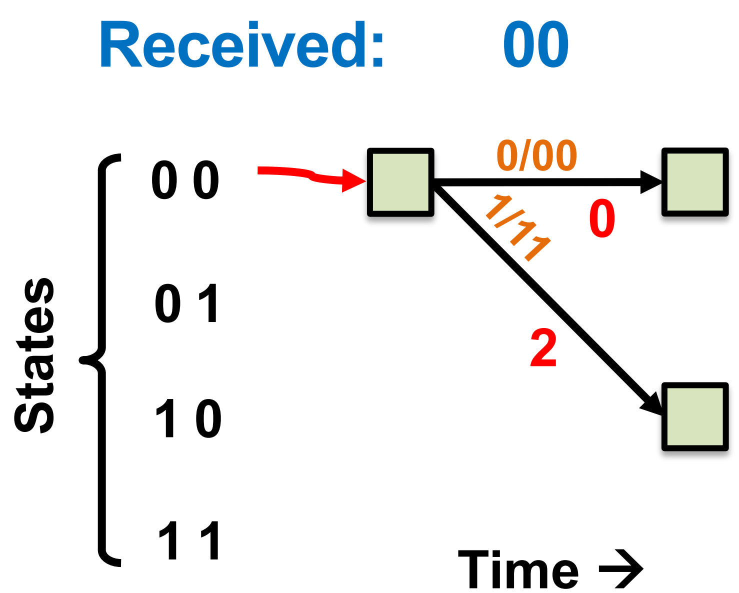
Hard-decision branch metric

- Hard decisions → **input is bits**
- **Label every branch** of trellis with branch metrics
  - ***Hard Decision Branch metric:*** **Hamming Distance** between **received** and **transmitted** bits



Hard-decision branch metric

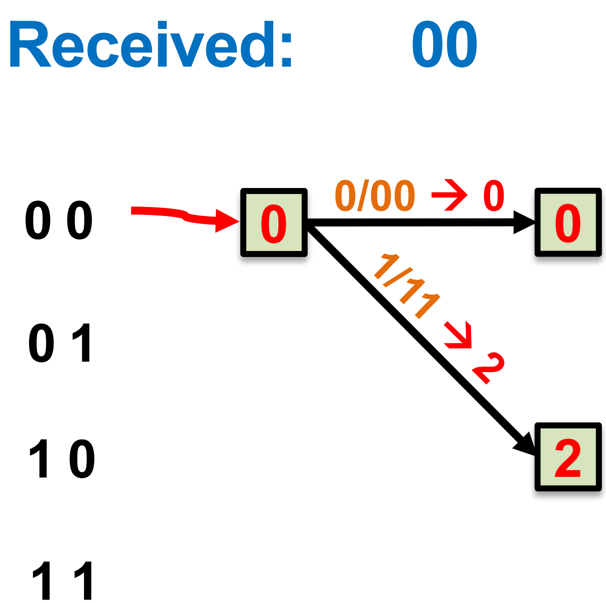
- Suppose we know encoder is in **state 00,** **receive bits: 00**



Hard-decision path metric

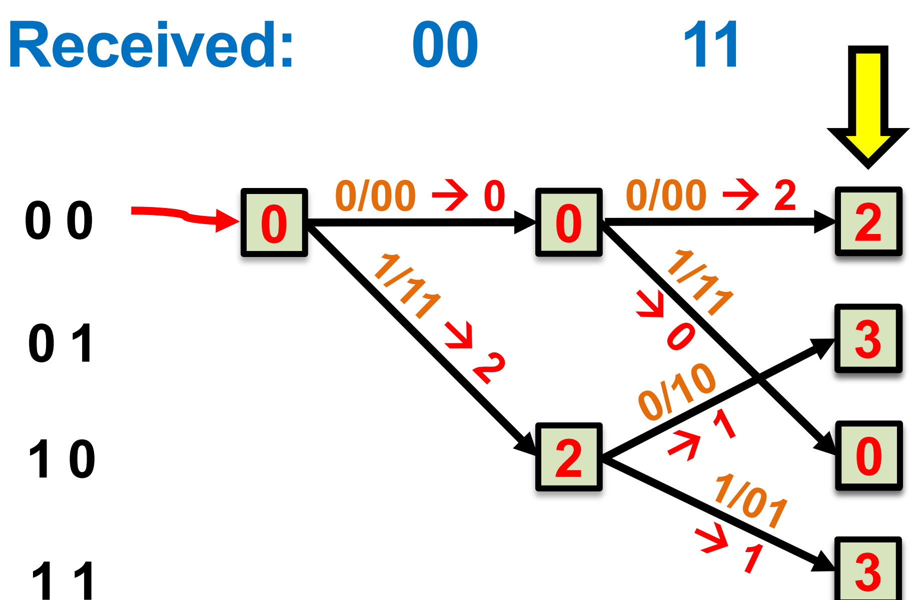
- ***Hard-decision path metric:*** **Sum** Hamming distance between **sent** and **received bits** **along path**

- Encoder is initially in **state 00,** **receive bits: 00**



Hard-decision path metric

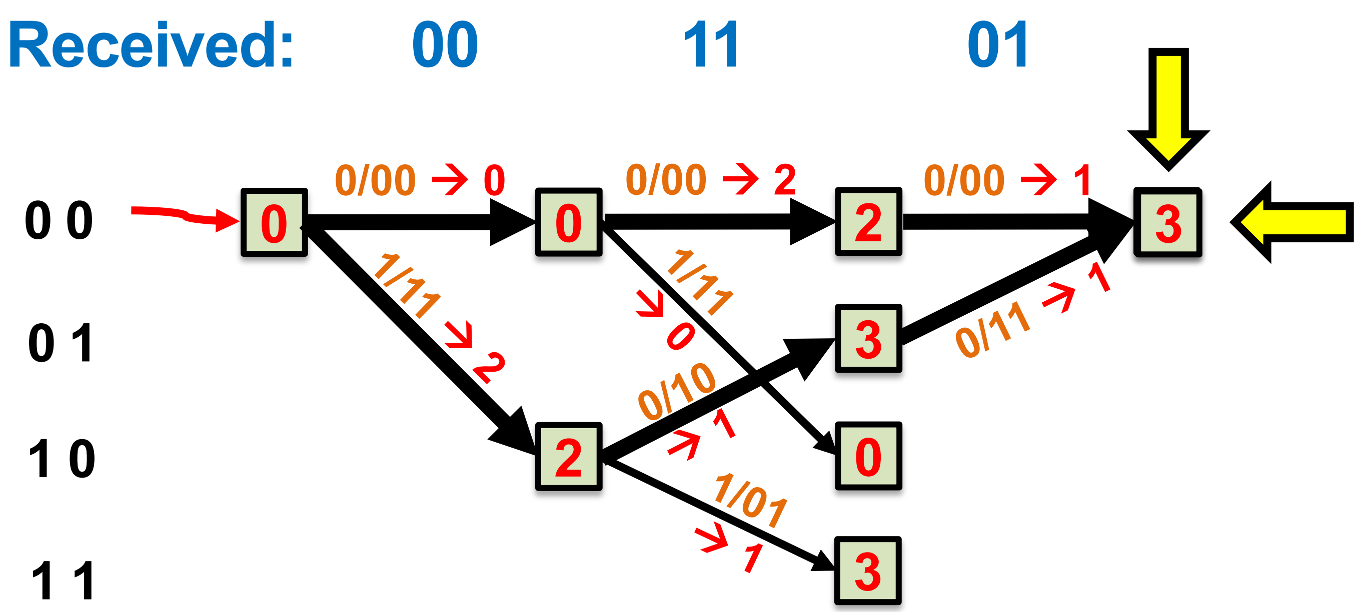
- Right now, each state has a **unique** **predecessor** state
- Path metric: Total bit errors **along path ending at state**
  - Path metric of **predecessor** + **branch metric**



Hard-decision path metric

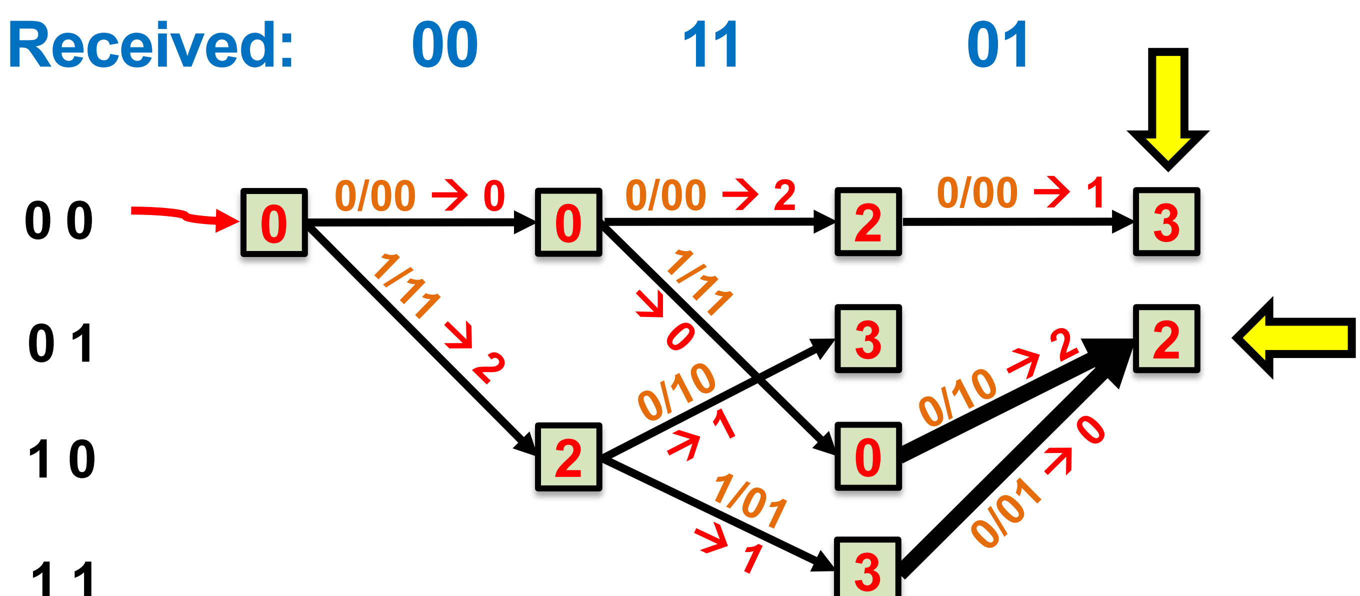
- Each state has **two predecessor states, two** ***predecessor paths*** (which to use?)

- **Winning** branch has **lower** path metric (**fewer** bit errors): ***Prune*** **losing** branch



Hard-decision path metric

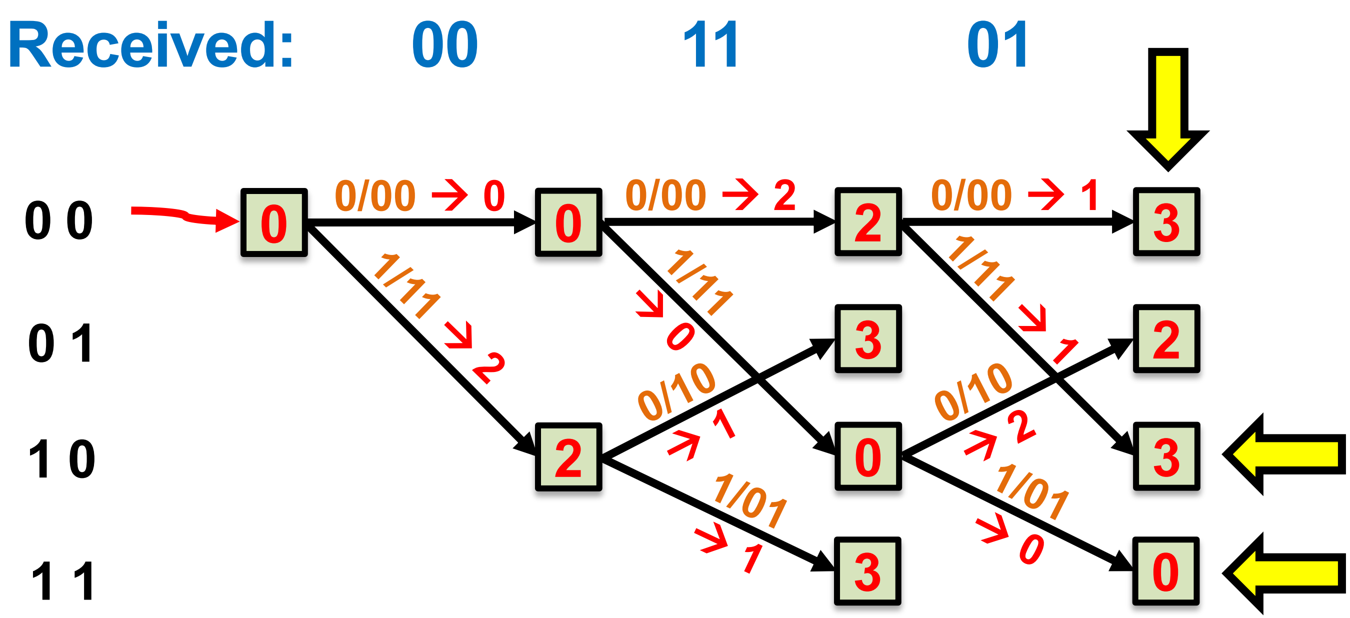
- Prune losing branch **for each state** in trellis



Pruning non-surviving branches

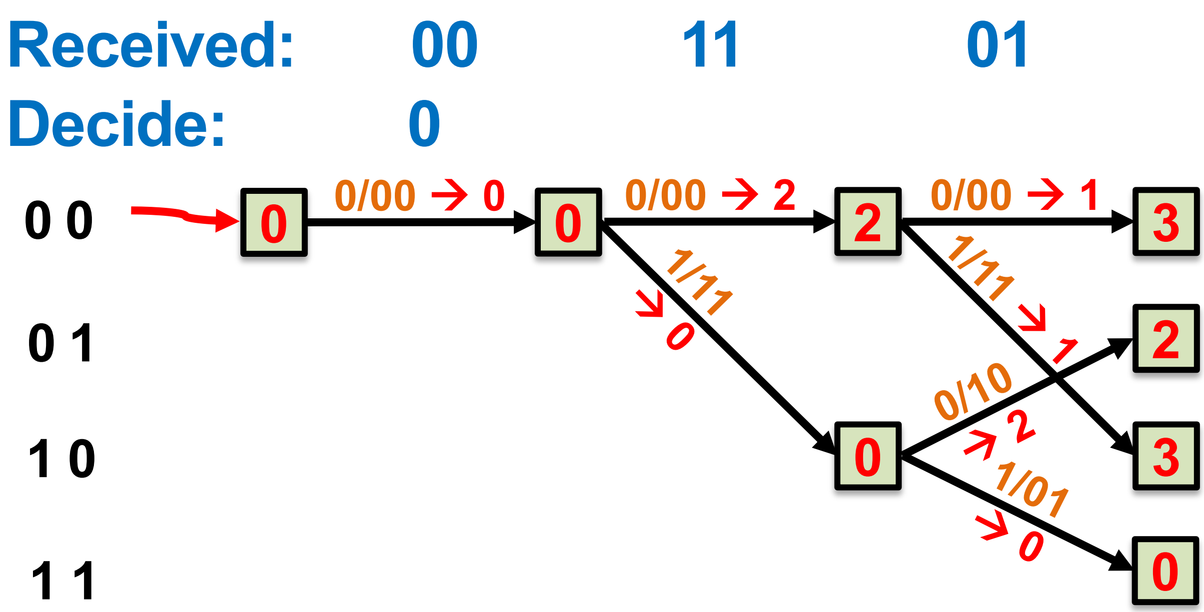
- ***Survivor path*** begins at each state, traces **unique path** back to **beginning** of trellis
  - **Correct path** is one of **four** survivor paths

- Some branches are not part of any survivor: **prune them**



Making bit decisions

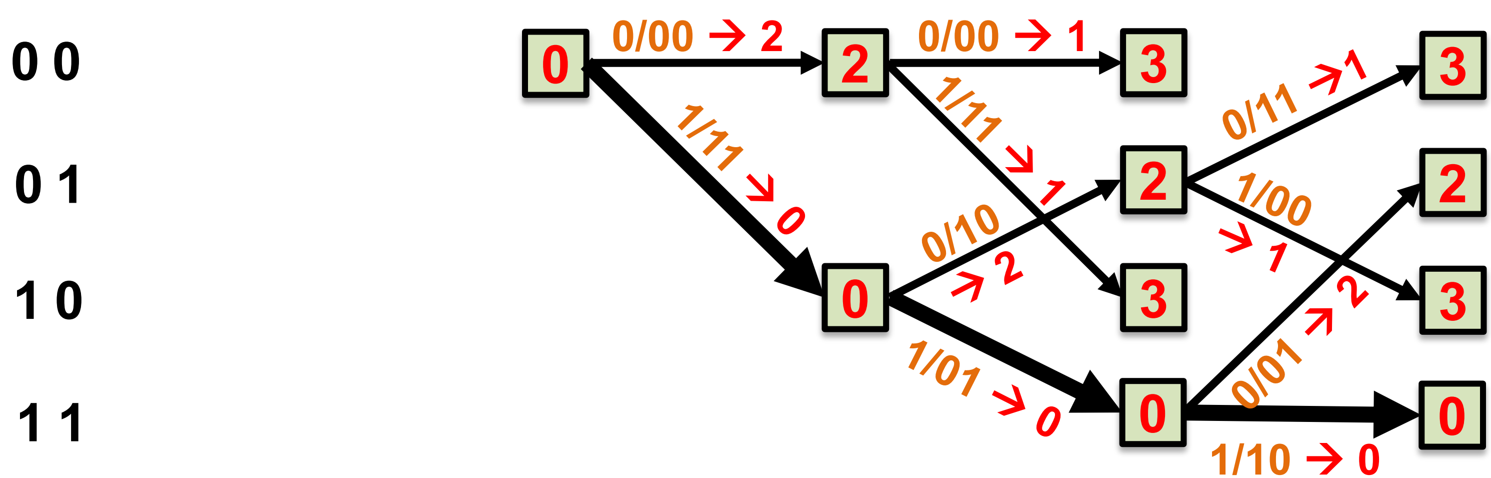
- When **only one branch remains** at a stage, the Viterbi algorithm **decides** that branch's **input bits:**



End of received data

- **Trace back** the survivor with **minimal path metric**
- Later stages **don’t get benefit** of future error correction, had data not ended

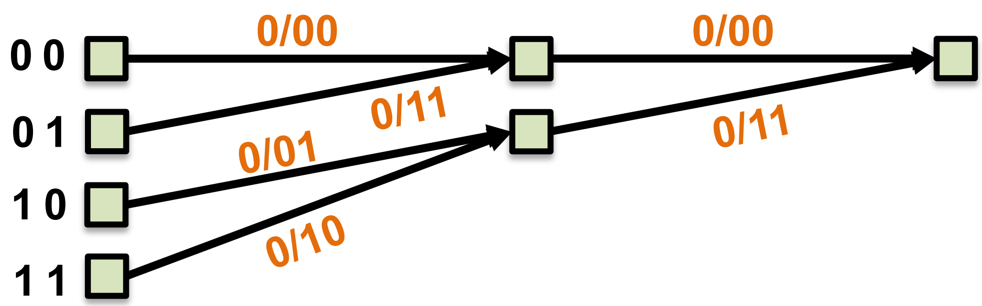
Received:	**00**	**11**	**01**	**10**
Decide:	**0**	**1**	**1**	**1**



Terminating the code

- **Sender** transmits **two 0 data bits** at end of data

- **Receiver** uses the following trellis at end:

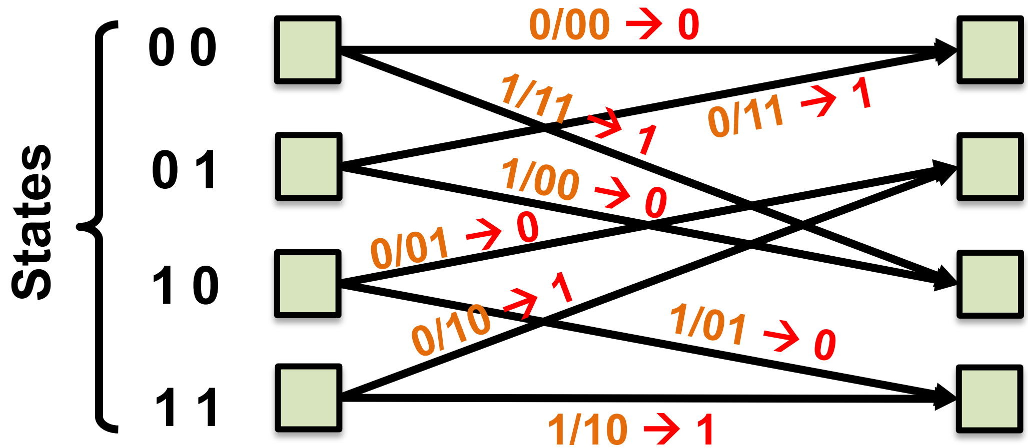


- **After termination only one trellis survivor path** remains
  - Can **make better bit decisions at end of data** based on this sole survivor

Viterbi with a Punctured Code

- Punctured bits are never transmitted
- Branch metric measures dissimilarity only between **received and transmitted unpunctured bits**
  - Same path metric, same Viterbi algorithm
  - **Lose** some **error correction capability**

Received: 0-



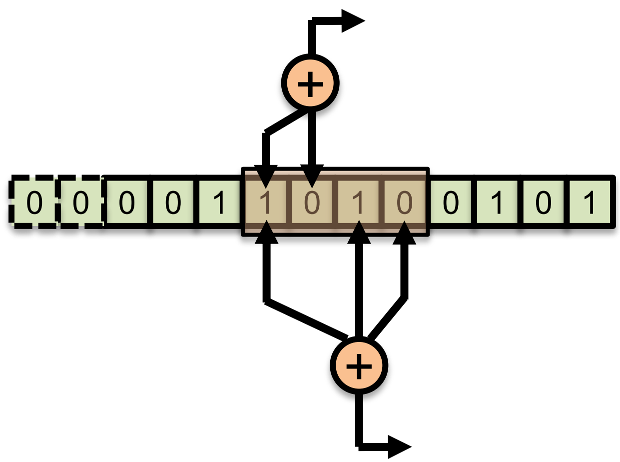
Today

1. Encoding data using convolutional codes
   – Changing code rate: Puncturing

2. **Decoding convolutional codes: Viterbi Algorithm**
   – **Hard decision decoding**
      • **Error correcting capability**
   – Soft decision decoding

How many bit errors can we correct?

- Think back to the encoder; **linearity property:**
  - Message m_1 → Coded bits c_1
  - Message m_2 → Coded bits c_2
  - Message $m_1 \oplus m_2$ → Coded bits $c_1 \oplus c_2$

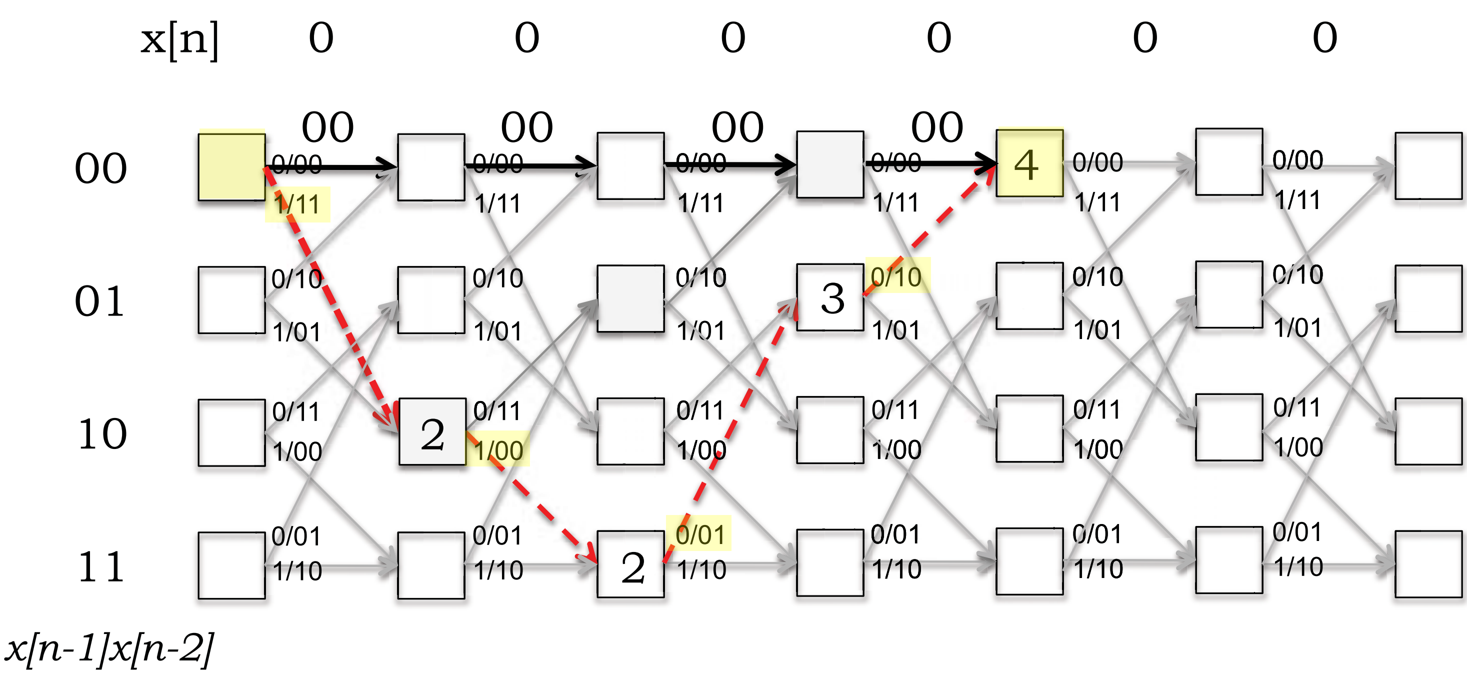


- So, d_{min} = minimum distance between **000...000** codeword and **codeword with fewest 1s**

Calculating d_{min} for the convolutional code

- Find path with **smallest non-zero path metric** going from **first 00** state to a **future 00 state**

- Here, d_{min} = 4, so can correct **1 error in 8 bits:**



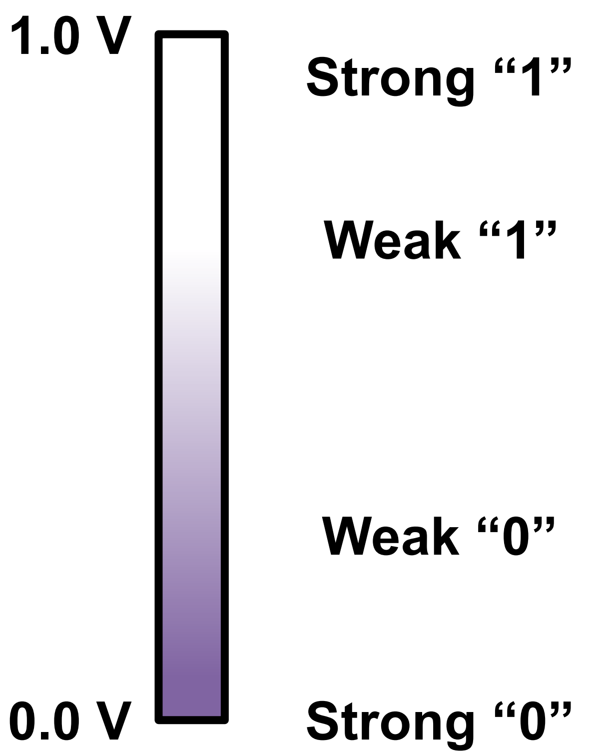
Today

1. Encoding data using convolutional codes
   – Changing code rate: Puncturing

2. **Decoding convolutional codes: Viterbi Algorithm**
   – Hard decision decoding
   – **Soft decision decoding**

Model for Today

- **Coded bits** are actually **continuously-valued "voltages"** between **0.0 V and 1.0 V:**



On Hard Decisions

- Hard decisions digitize each voltage to "**0**" or "**1**" by comparison against ***threshold voltage* 0.5 V**
  - **Lose information** about how "good" the bit is
    - Strong "1" (0.99 V) **treated equally to** weak "1" (0.51 V)

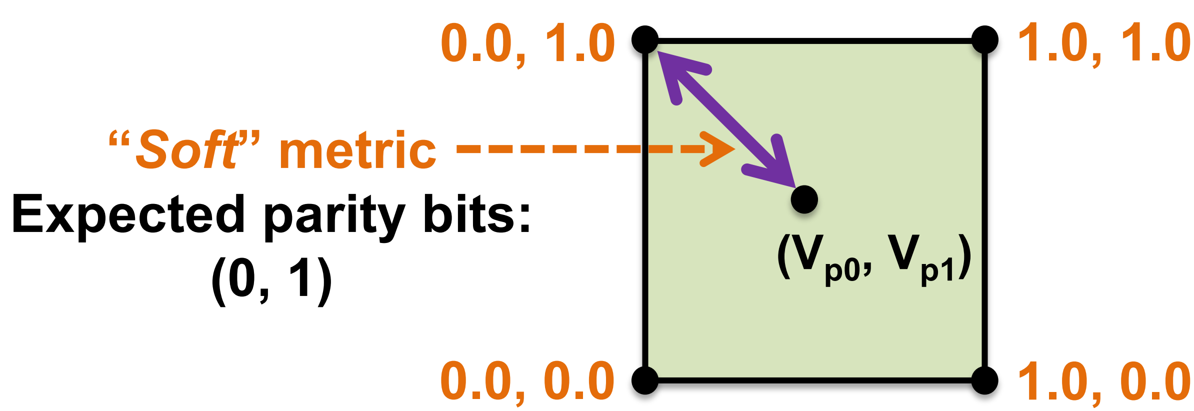
- **Hamming distance** for branch metric computation

- But **throwing away information** is almost never a good idea when making decisions
  - *Find a **better branch metric** that **retains information** about the received voltages?*

Soft-decision decoding

- **Idea**: **Pass received voltages to decoder before digitizing**
  - **Problem:** Hard branch metric was Hamming distance

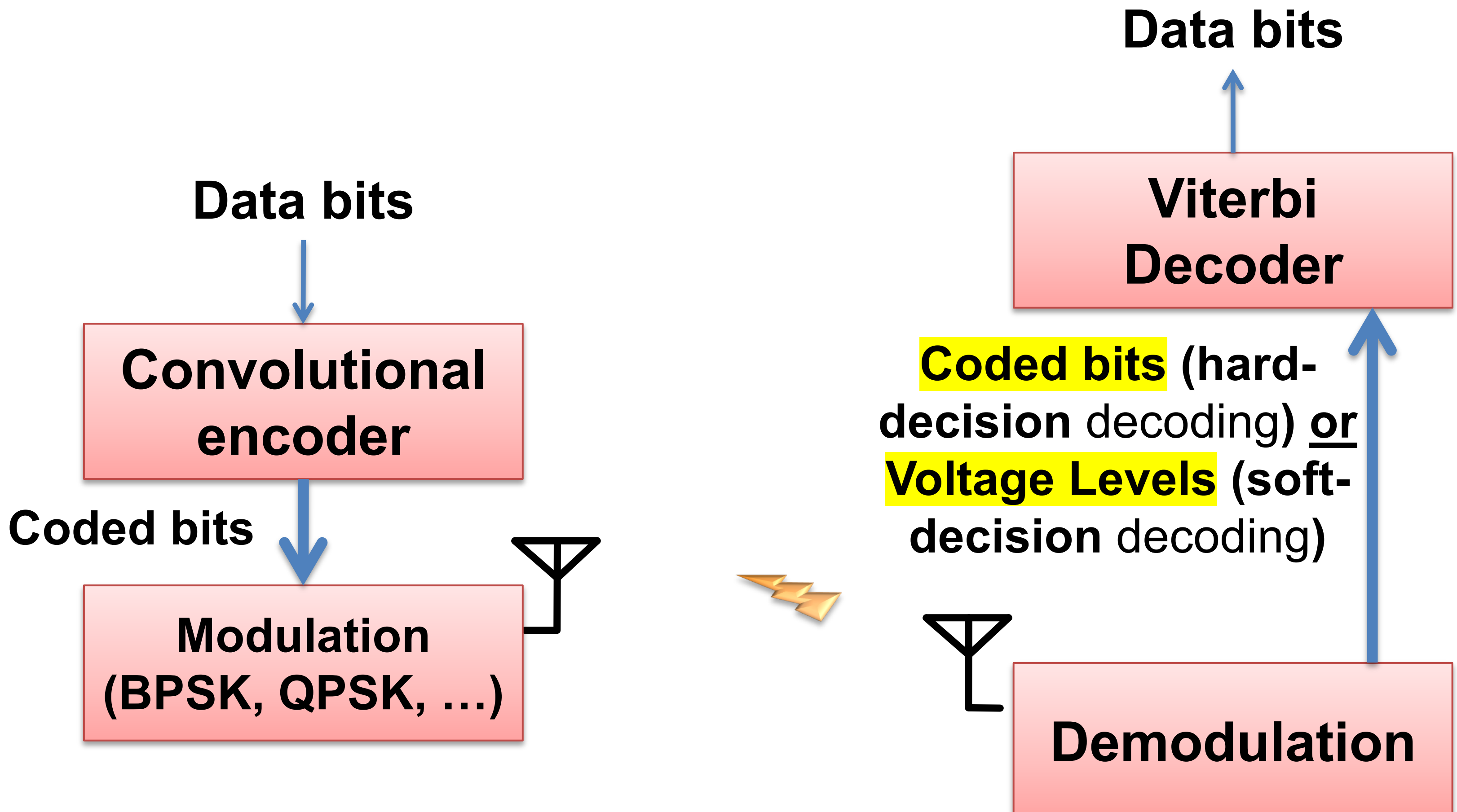
- ***"Soft" branch metric***
  - **Euclidian distance** between received voltages and voltages of expected bits:



Soft-decision decoding

- **Different** branch metric, hence **different** path metric
- **Same** path metric computation
- **Same** Viterbi algorithm
- **Result:** Choose **path** that minimizes sum of squares of Euclidean distances between received, expected voltages

Putting it together: Convolutional coding in Wi-Fi



Thursday Topic:
Rateless Codes

Friday Precept:
Midterm Review