COS 423	Theory of Algorithms	Spring 2018

Precept 3

These problems will be solved in precept.

1. Let $G = (V, E)$ be a connected graph with positive edge weights. Let $x_1 \leq x_2 \leq \ldots \leq x_{n-1}$ be the weights of the edges in any MST T^* of G and let Let $y_1 \leq y_2 \leq \ldots \leq y_{n-1}$ be the weights of the edges in any spanning tree T of G. Prove that $x_i \leq y_i$ for every i. In other words, the i^{th} smallest edge in any MST is no larger than the i^{th} smallest edge in any spanning tree.

   Using this property, quickly prove that 9 of the following 10 statements are corollaries (and identity the one that is false).

   (a) T^* is a spanning tree that minimizes the *minimum* edge weight.
   (b) T^* is a spanning tree that minimizes the *maximum* edge weight.
   (c) T^* is a spanning tree that minimizes the *median* edge weight.
   (d) T^* is a spanning tree that minimizes the *arithmetic mean* of the edge weights.
   (e) T^* is a spanning tree that minimizes the *geometric mean* of the edge weights.
   (f) T^* is a spanning tree that minimizes the *harmonic mean* of the edge weights.
   (g) T^* is a spanning tree that minimizes the *variance* of the edge weights.
   (h) T^* is a spanning tree that minimizes the *sum of the squares* of the edge weights.
   (i) T^* is a spanning tree that minimizes the *product* of the edge weights.
   (j) T^* is a spanning tree that minimizes the *sum of the logarithms* of the edge weights.

   Prove these two corollaries as well.

   (k) Any two MSTs have the same multiset of edge weights.
   (l) If the edge weights are distinct, then there is a unique MST.

2. Given a directed acyclic graph G with edge costs $c_e > 0$, design a greedy algorithm to find a minimum-cost arborescence. The running time should be $O(m + n)$.