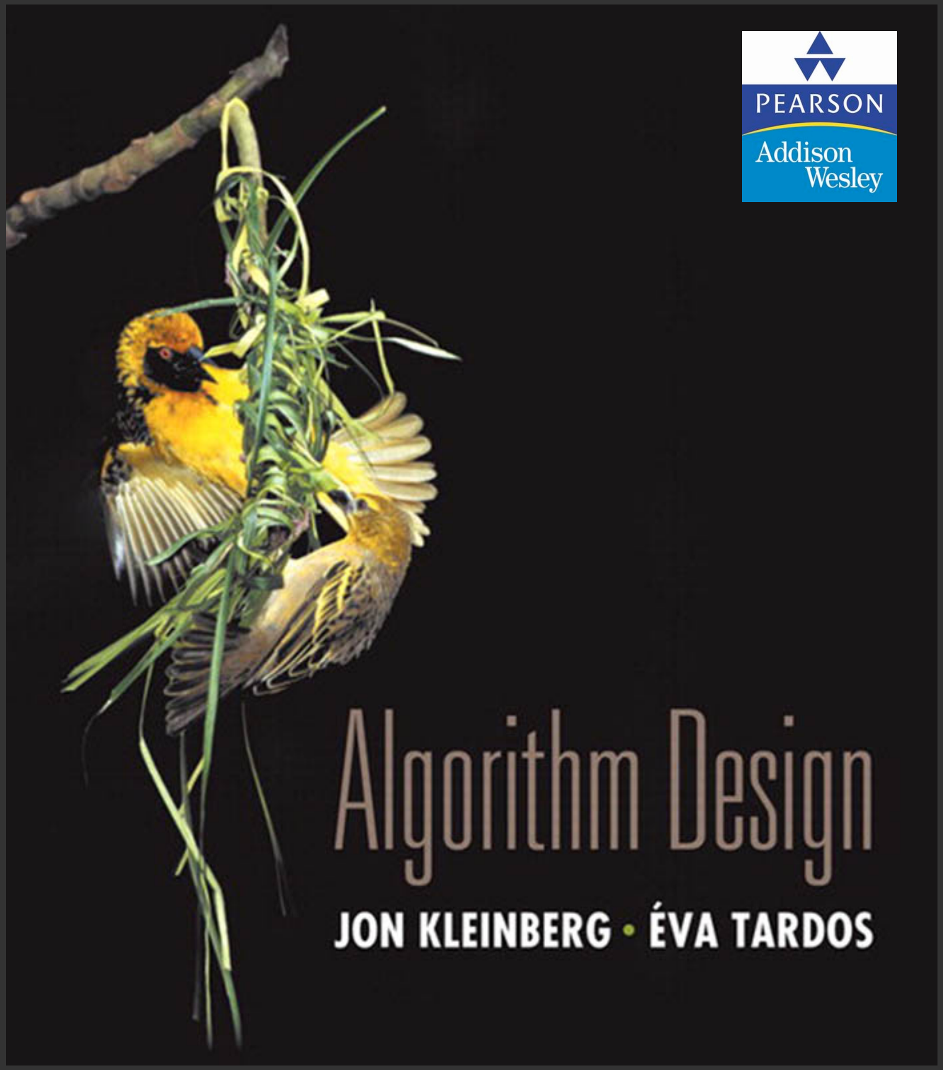


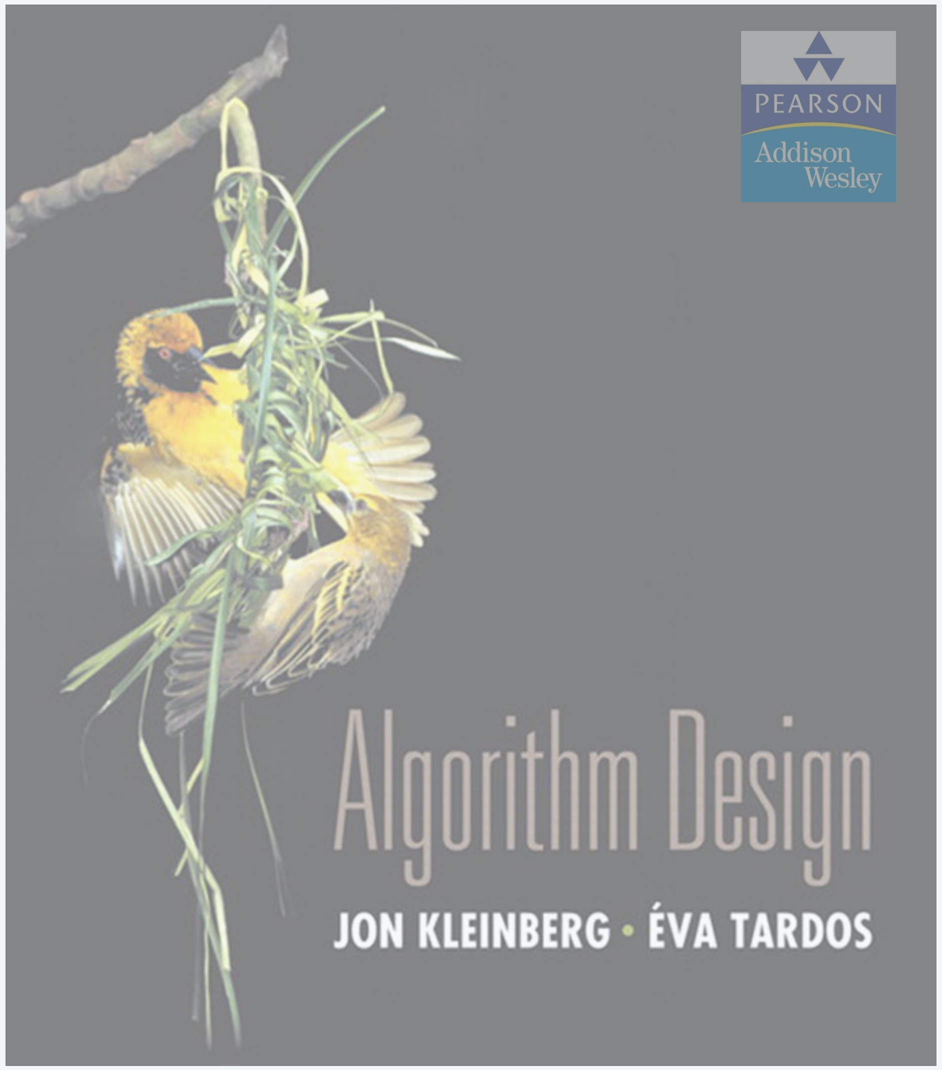
7. Network Flow I

- *Ford–Fulkerson demo*
- *exponential-time example*
- *pathological example*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on 4/17/18 9:07 AM



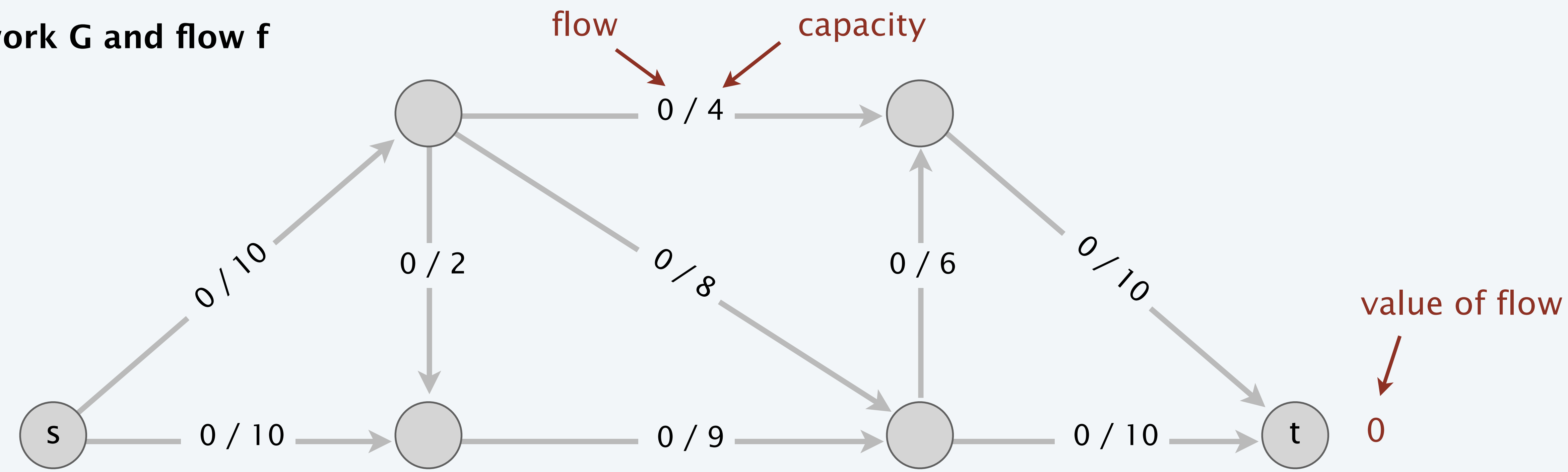
SECTION 7.1

7. NETWORK FLOW I

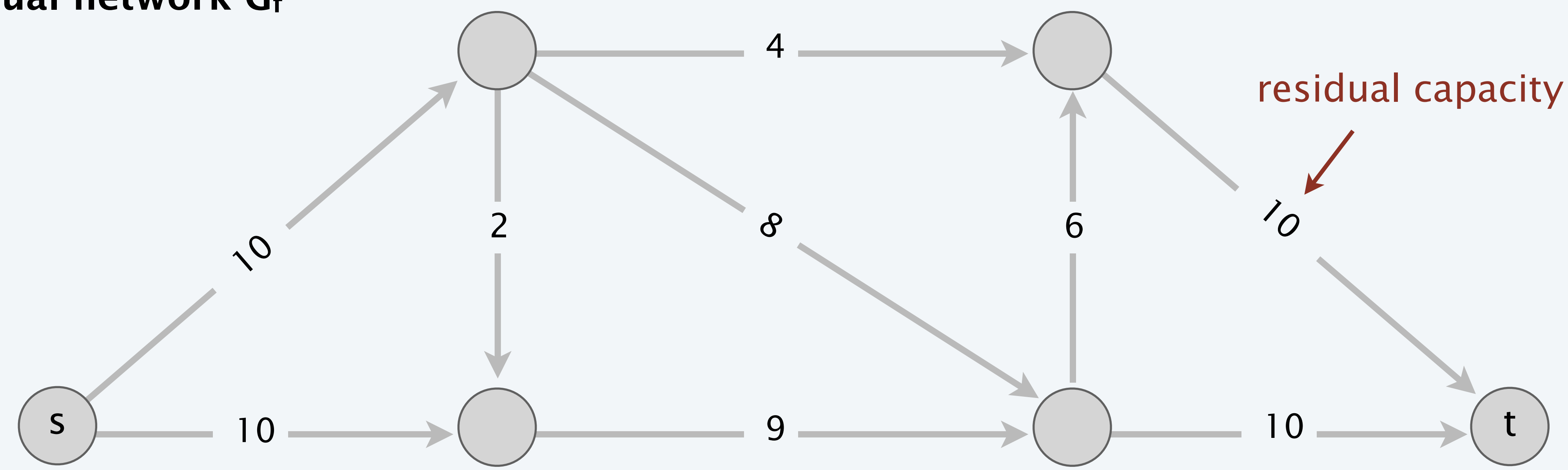
- *Ford–Fulkerson demo*
- *exponential-time example*
- *pathological example*

Ford-Fulkerson algorithm demo

network G and flow f

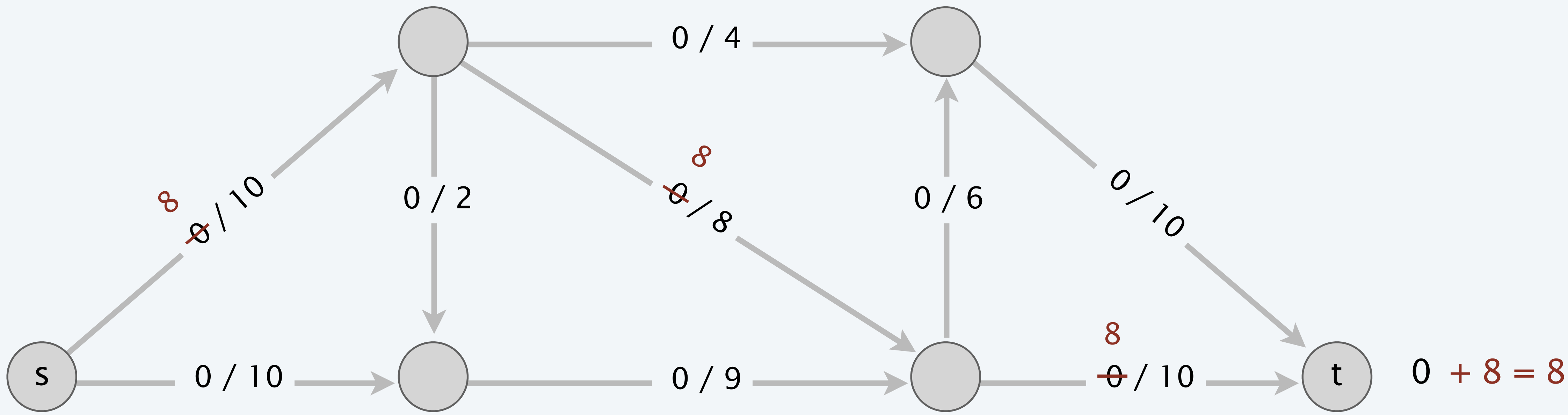


residual network G_f

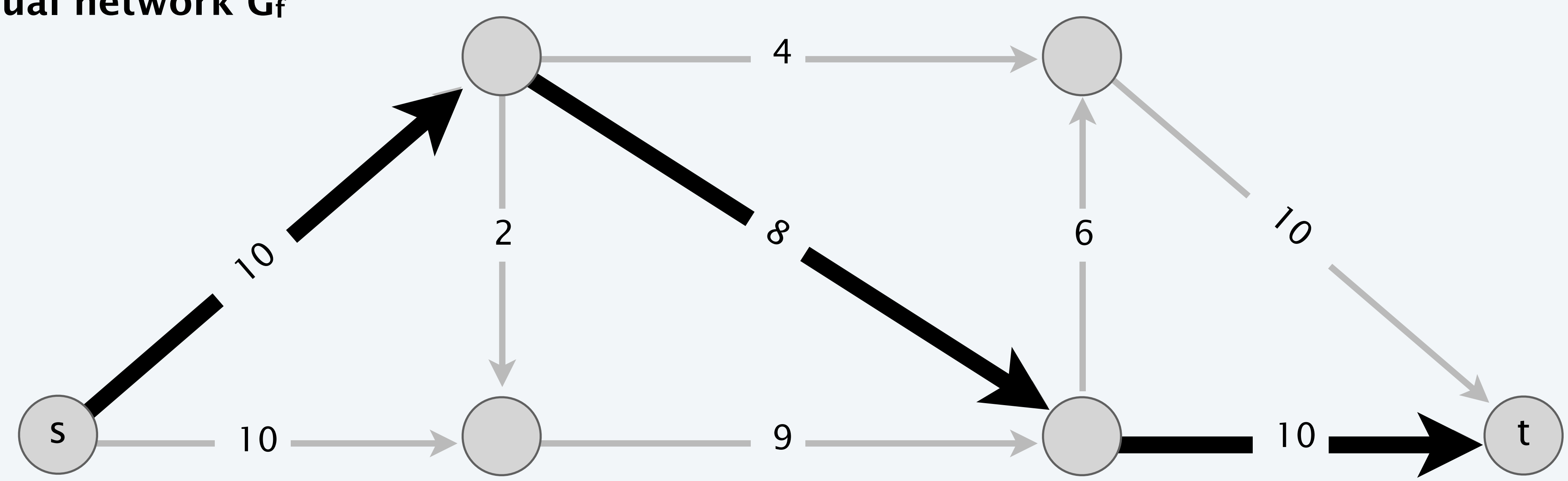


Ford–Fulkerson algorithm demo

network G and flow f

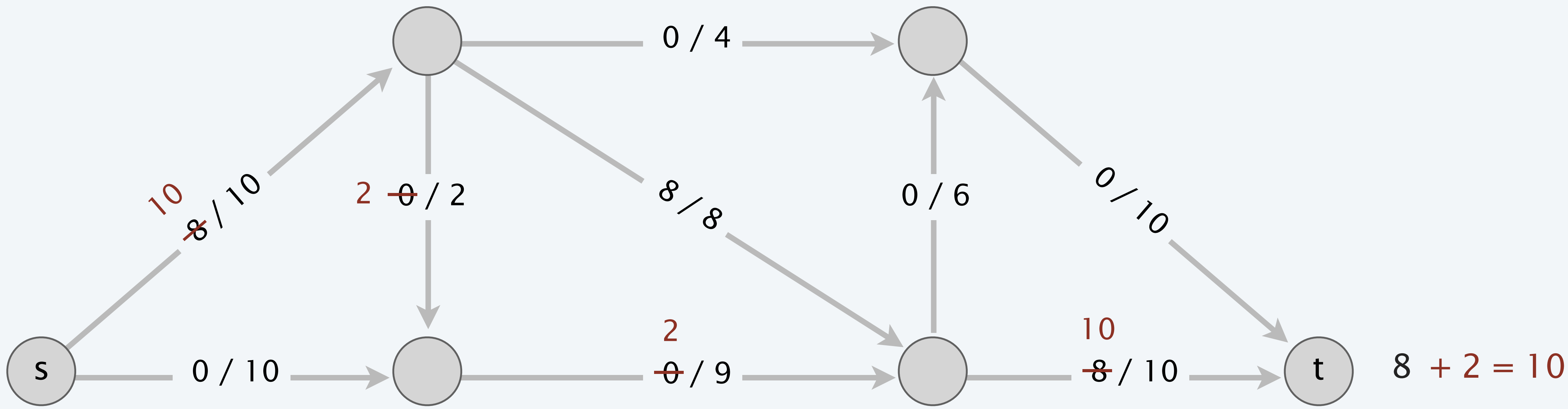


residual network G_f

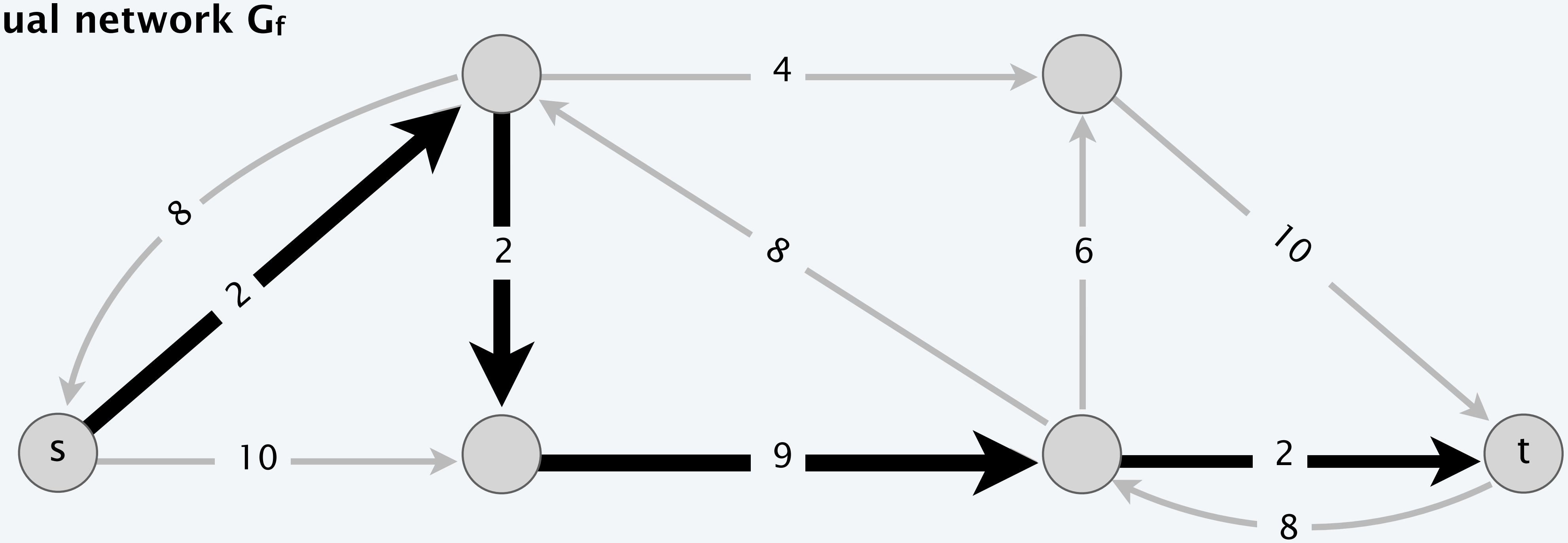


Ford–Fulkerson algorithm demo

network G and flow f

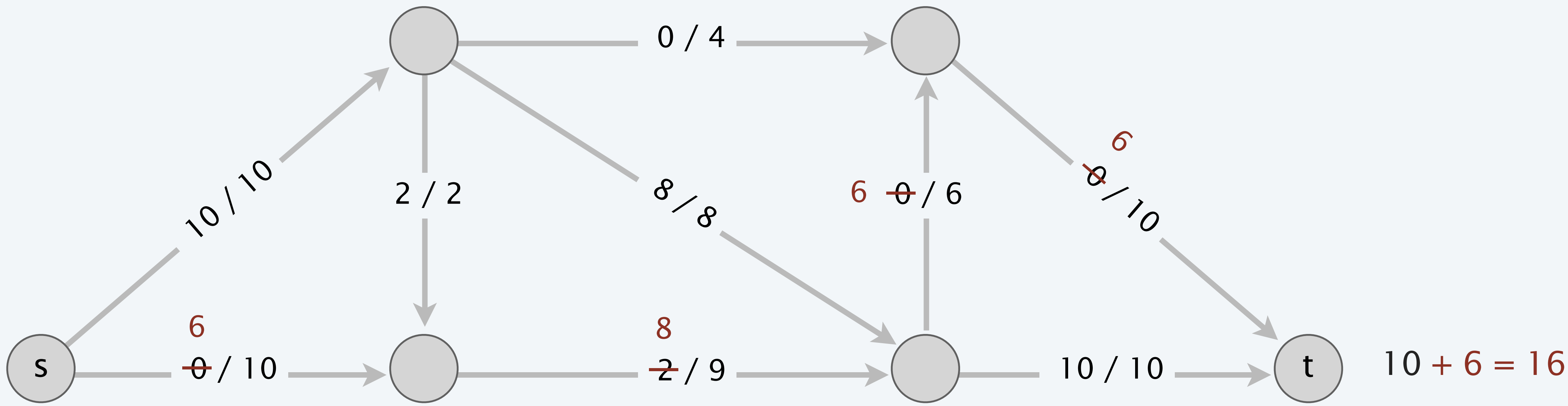


residual network G_f

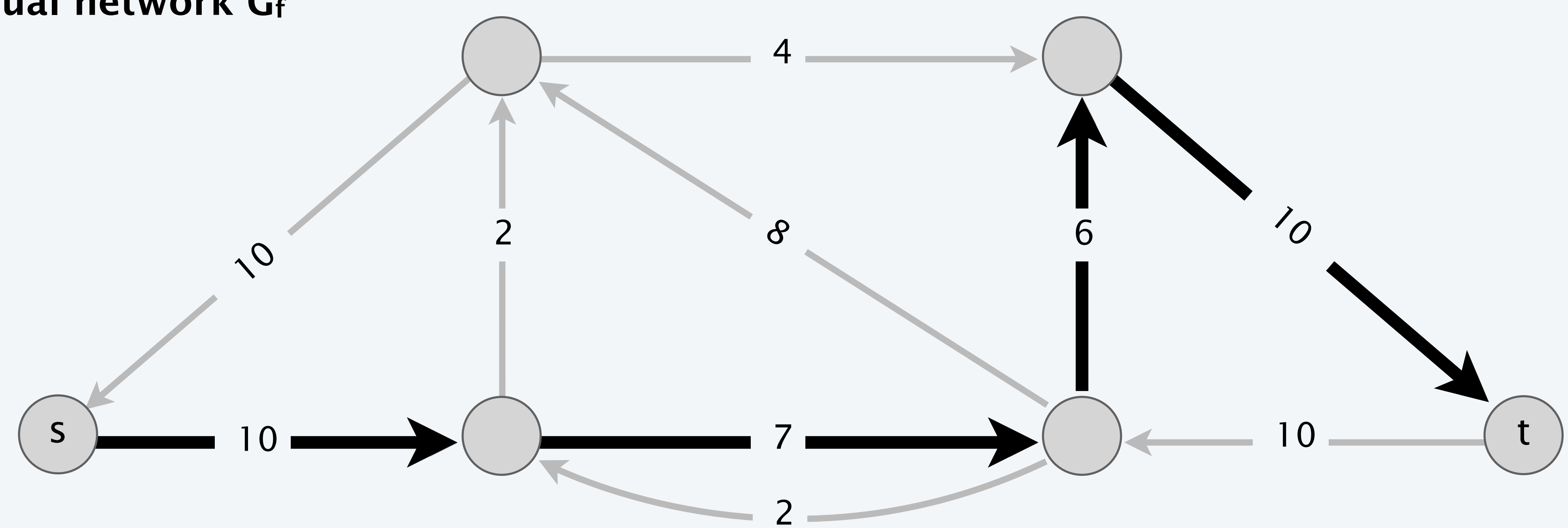


Ford–Fulkerson algorithm demo

network G and flow f

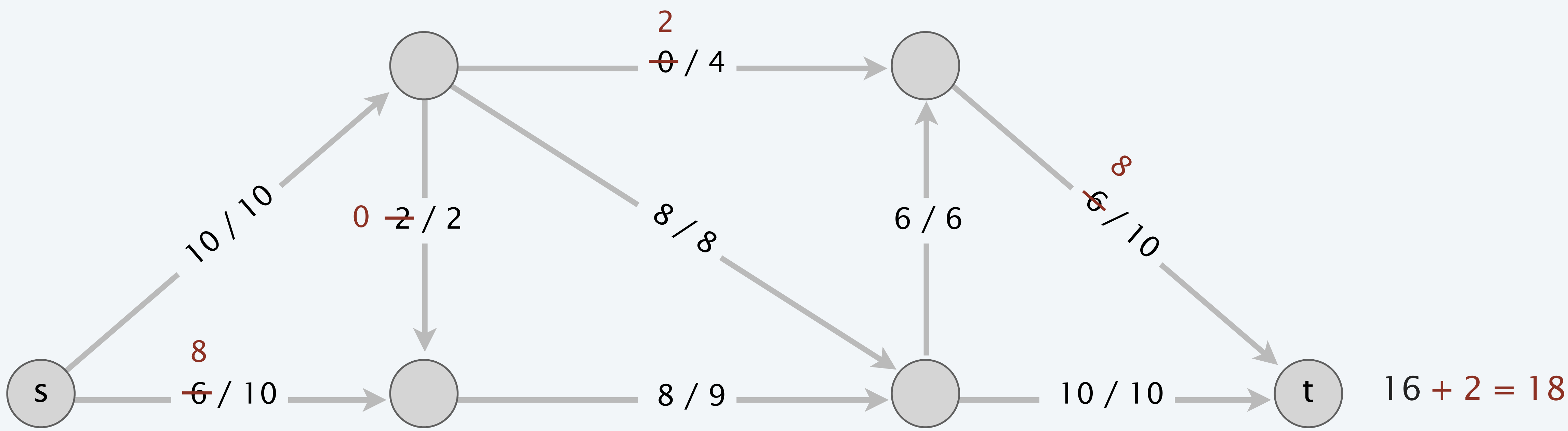


residual network G_f



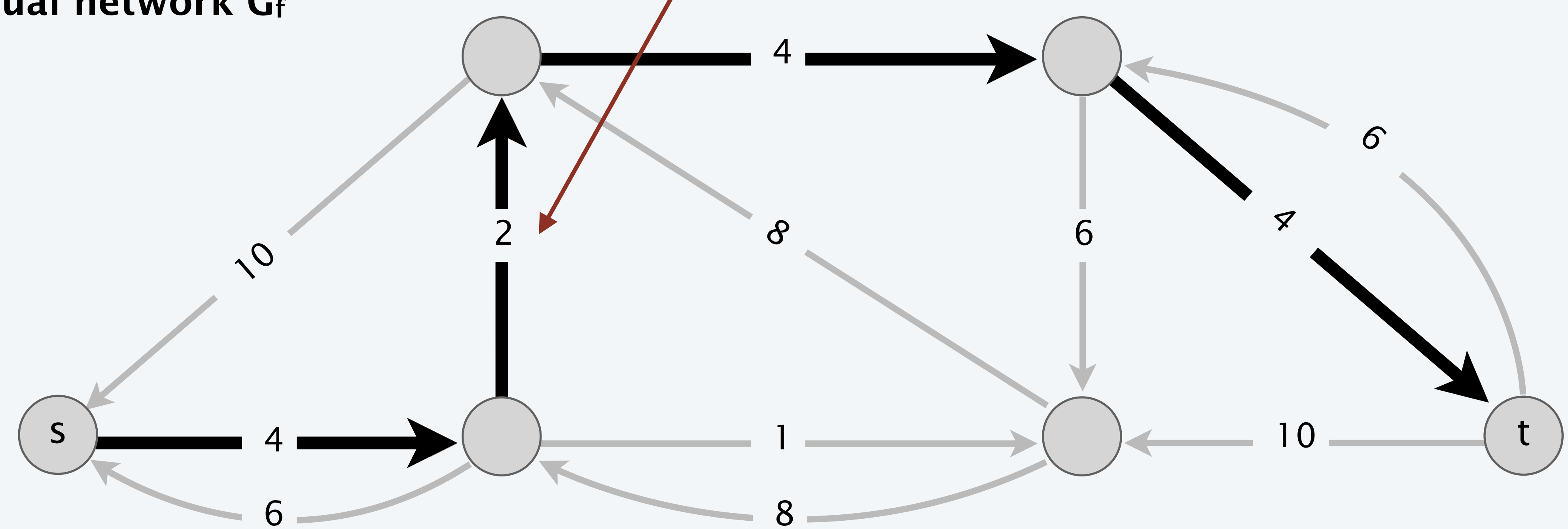
Ford–Fulkerson algorithm demo

network G and flow f



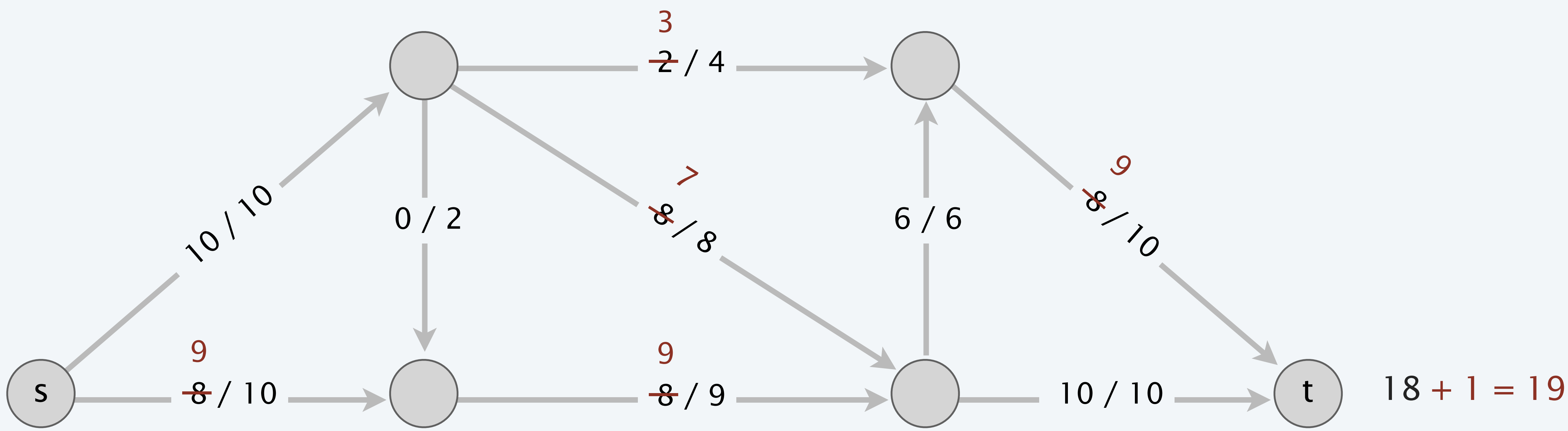
fixes mistake from
second augmenting path

residual network G_f

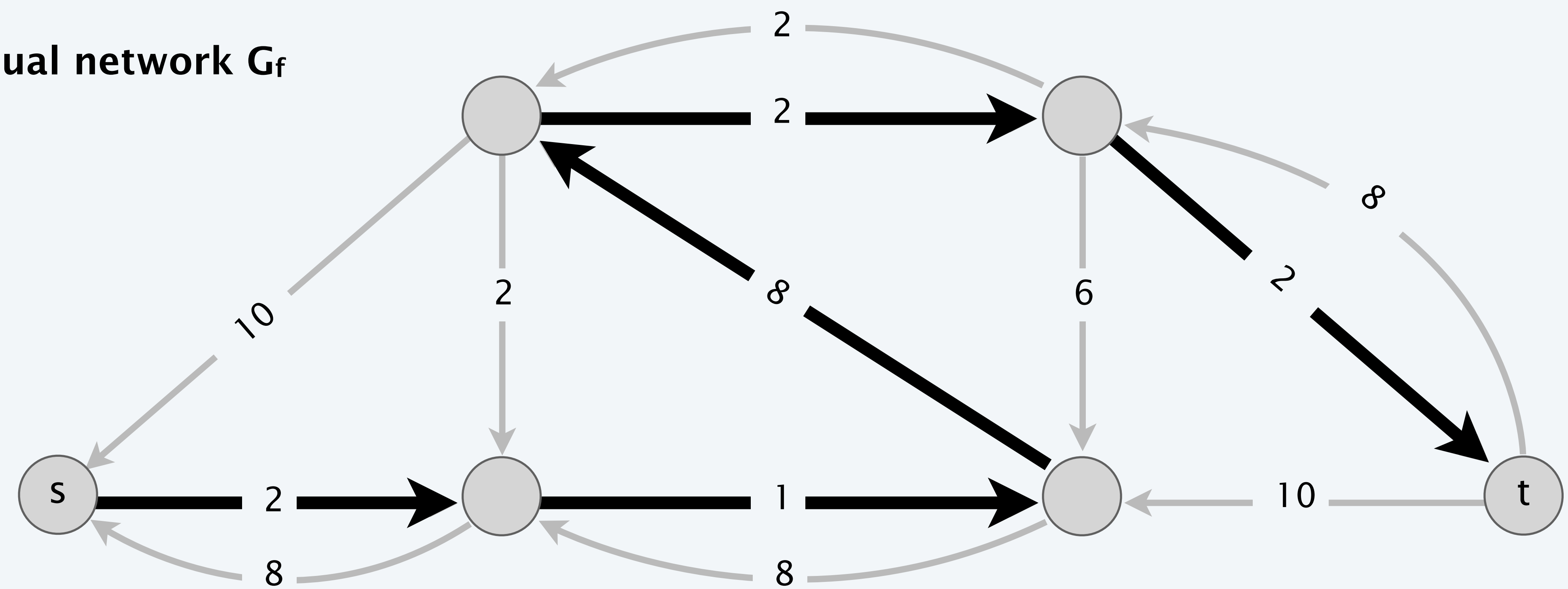


Ford–Fulkerson algorithm demo

network G and flow f

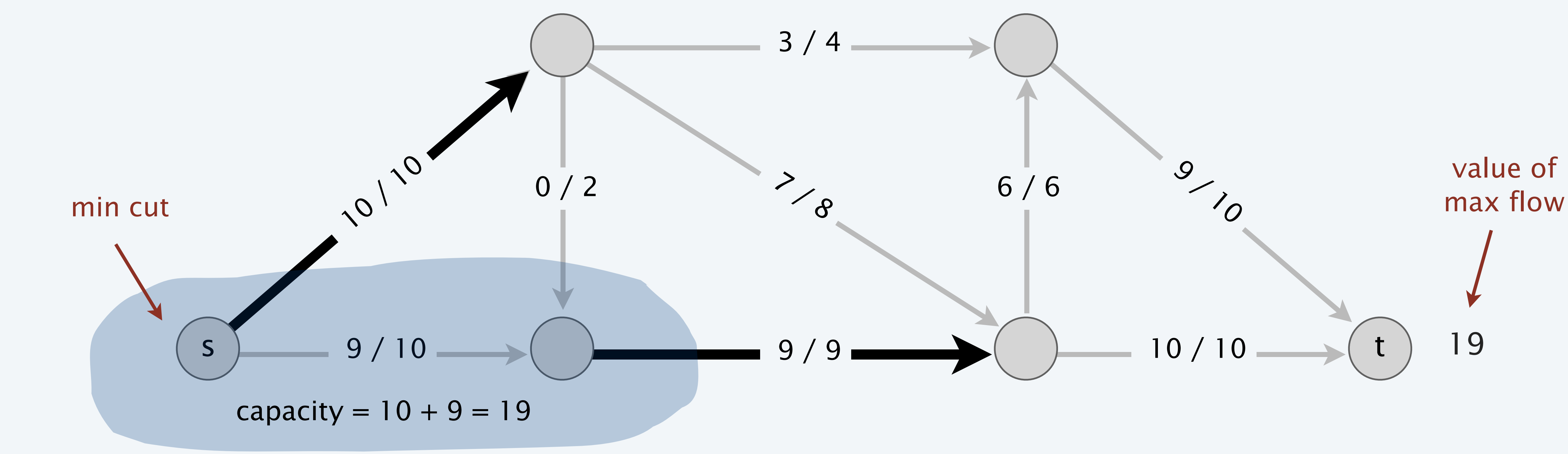


residual network G_f

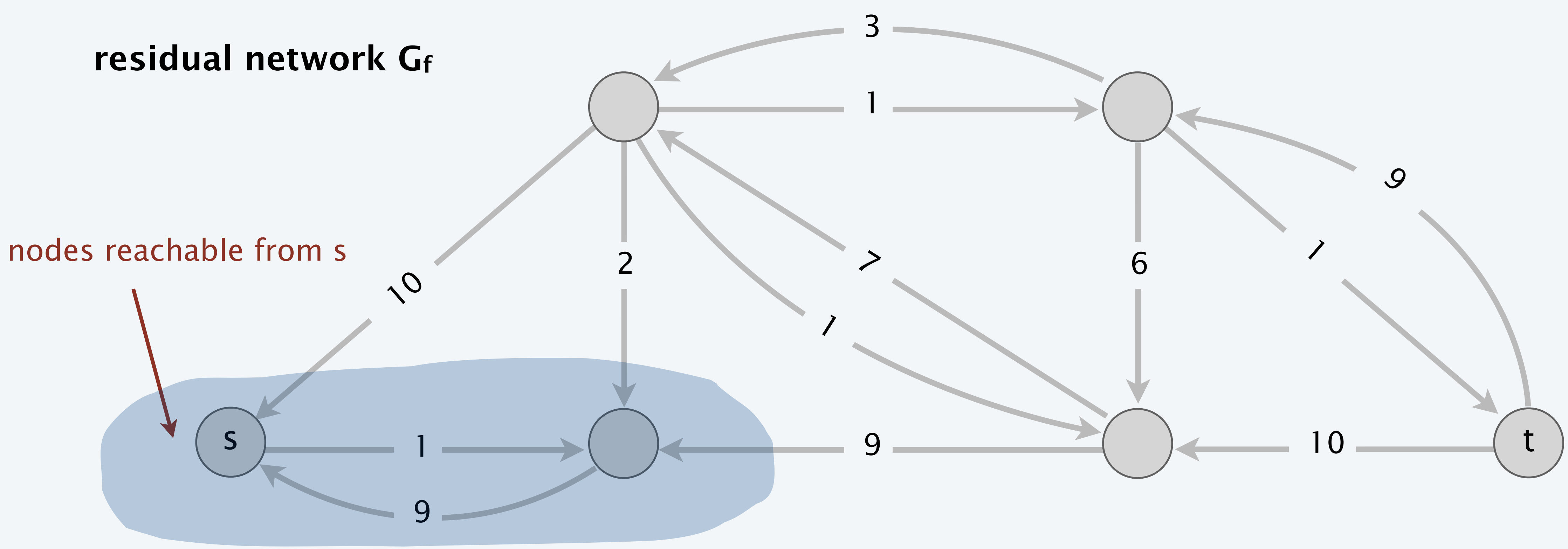


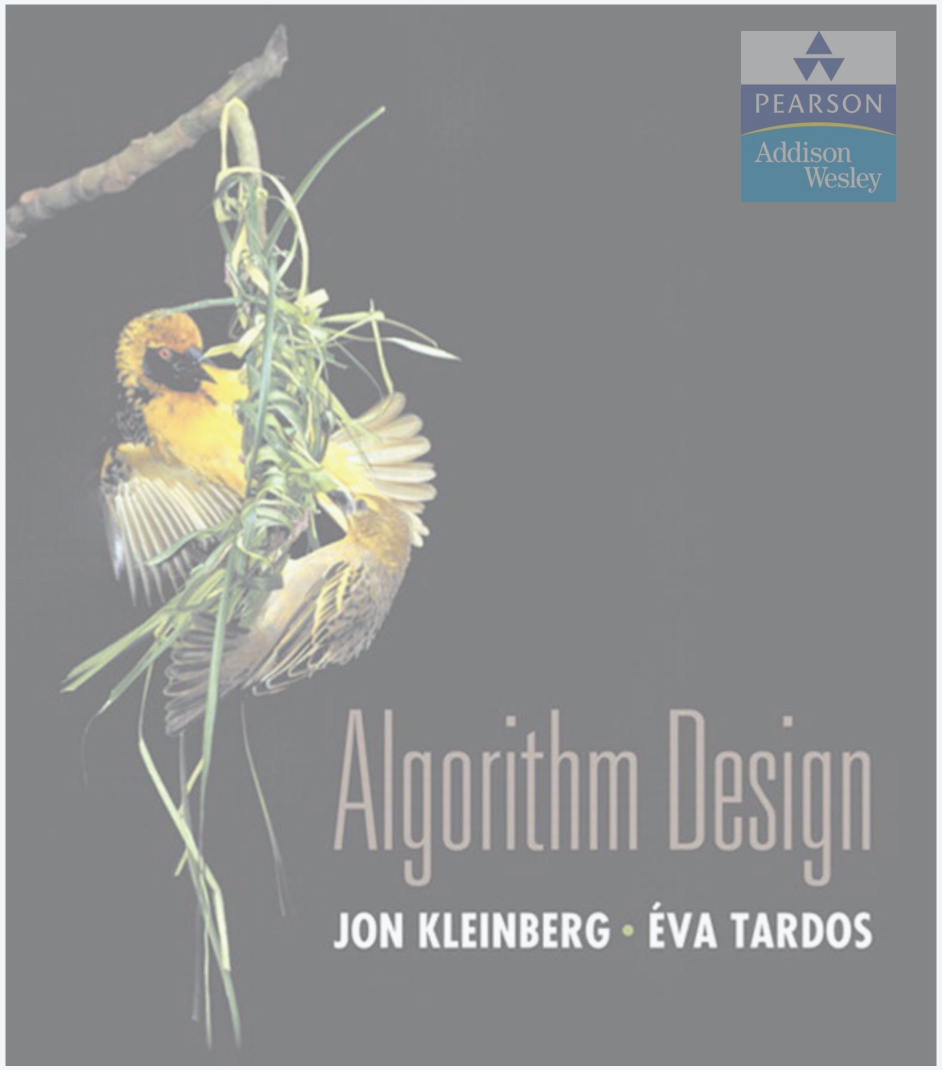
Ford–Fulkerson algorithm demo

network G and flow f



residual network G_f





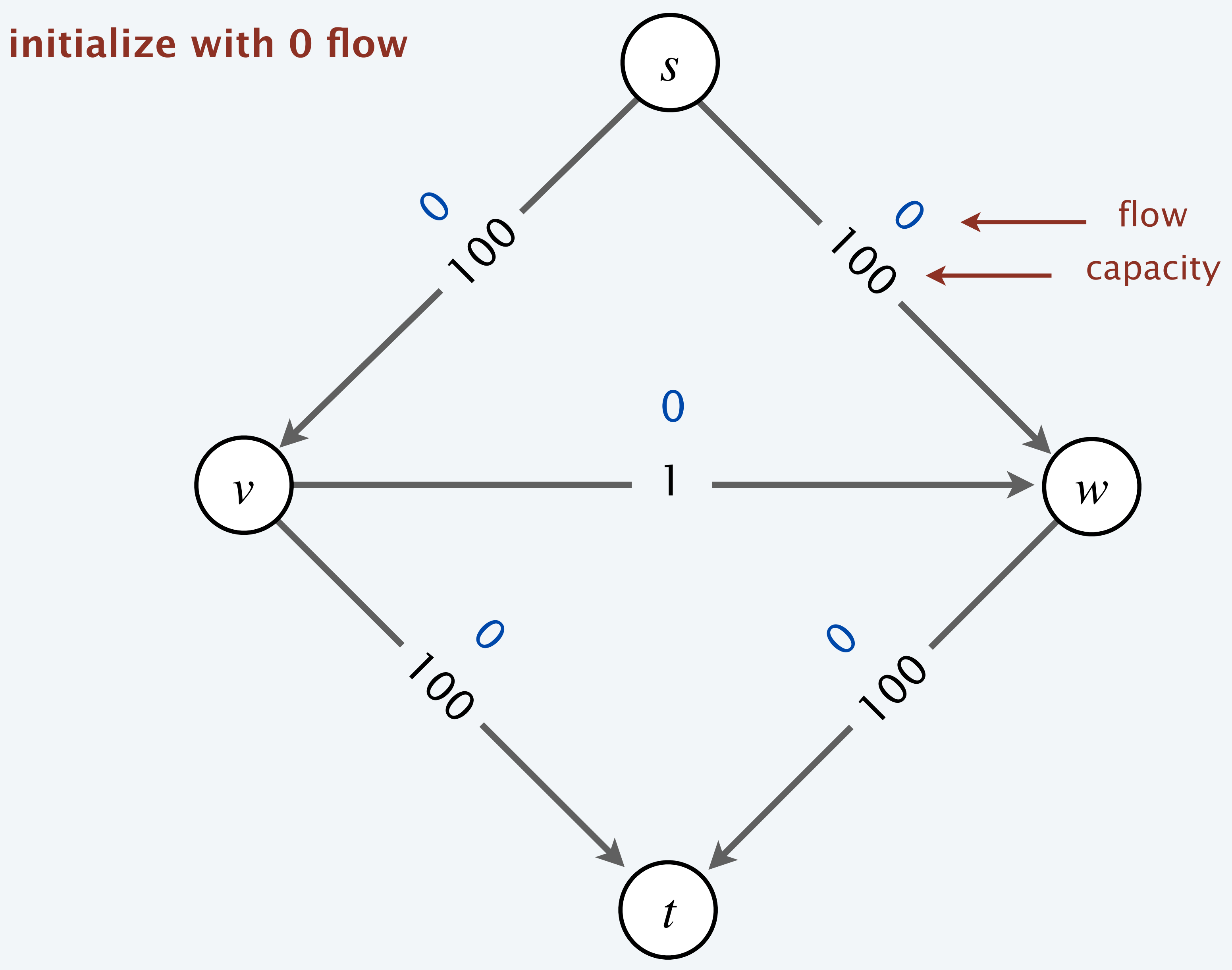
SECTION 7.1

7. NETWORK FLOW I

- Ford–Fulkerson demo
- *exponential-time example*
- pathological example

Ford–Fulkerson algorithm: exponential-time example

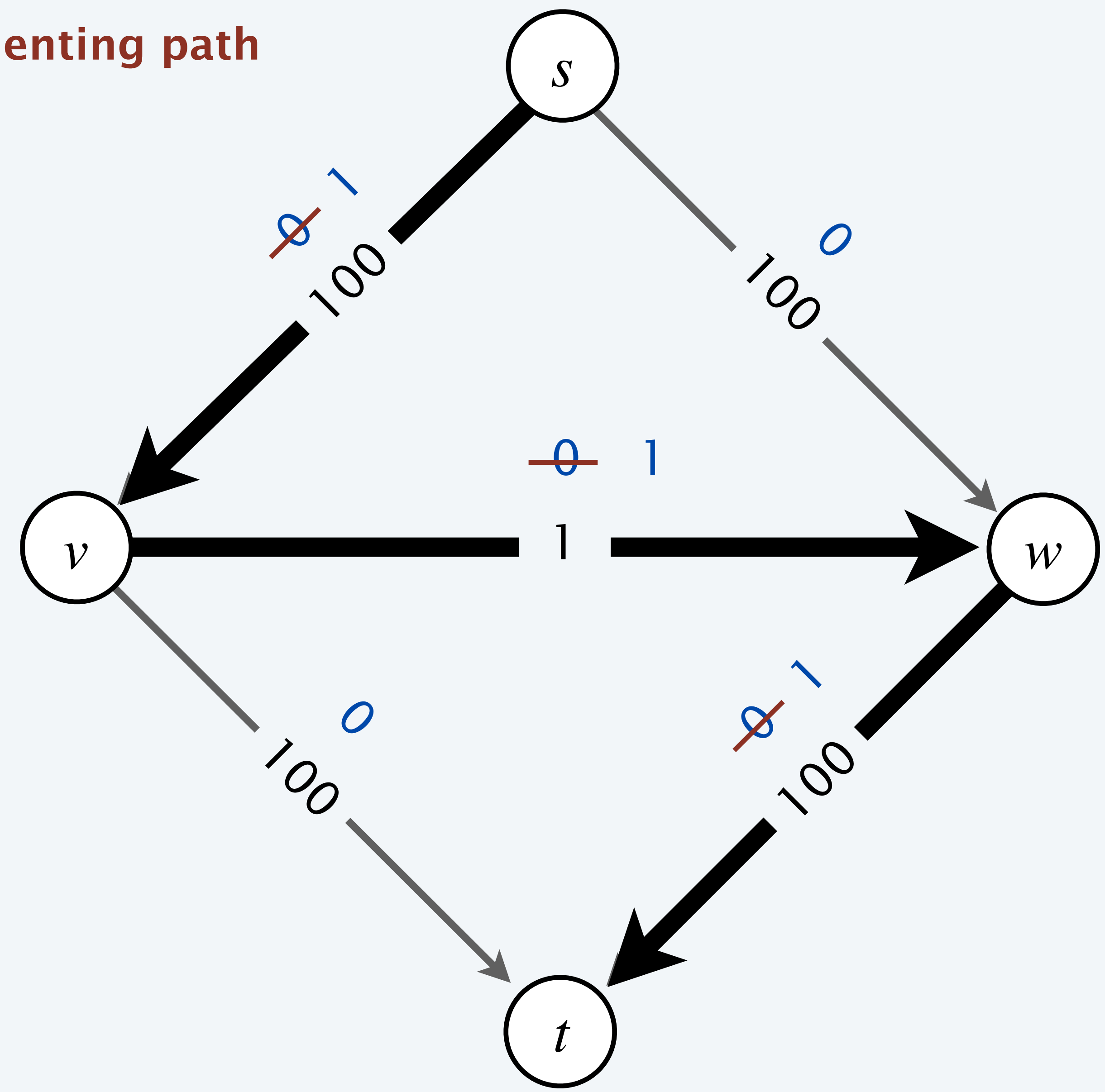
Bad news. Number of augmenting paths can be exponential in input size.



Ford–Fulkerson algorithm: exponential-time example

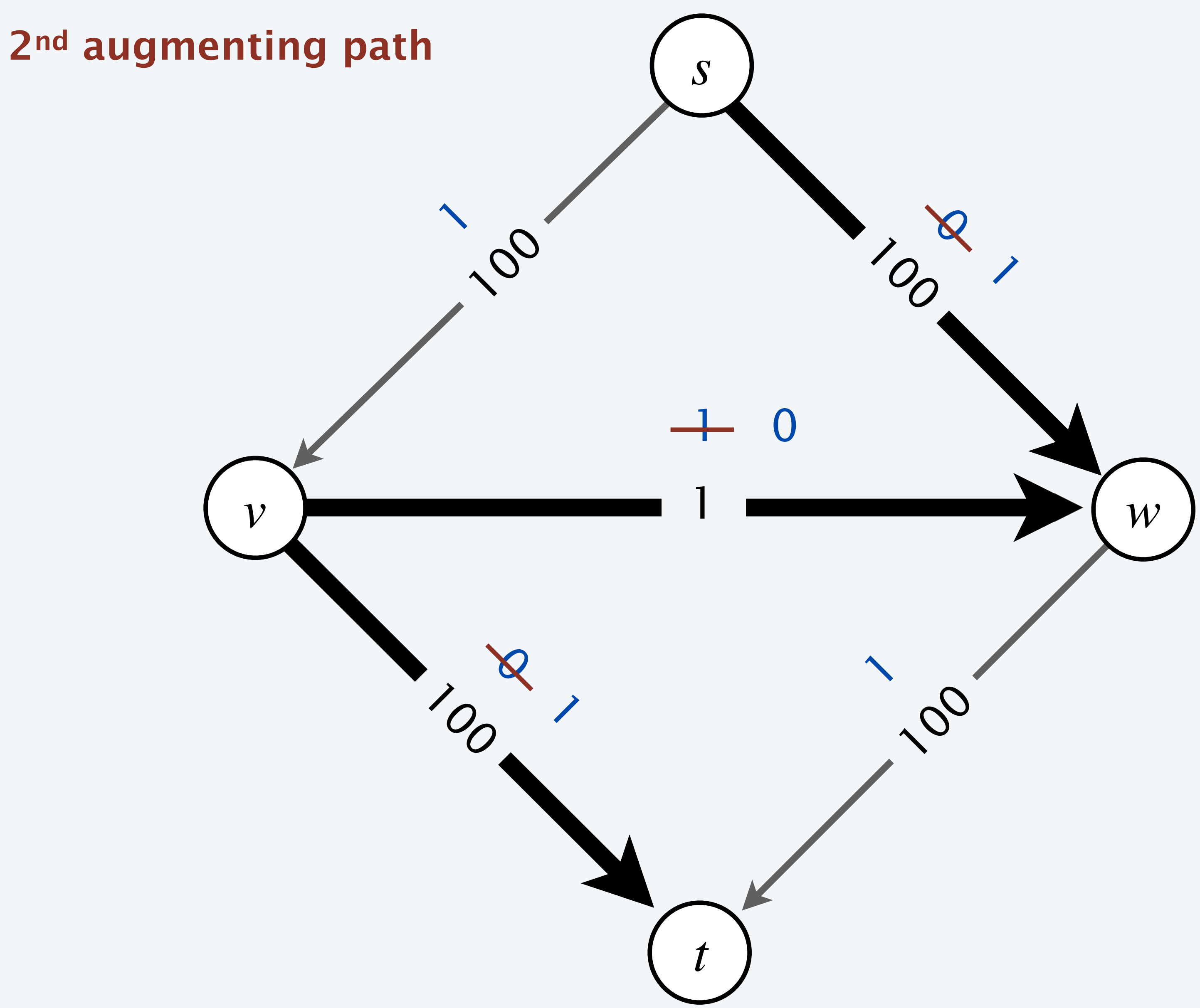
Bad news. Number of augmenting paths can be exponential in input size.

1st augmenting path



Ford–Fulkerson algorithm: exponential-time example

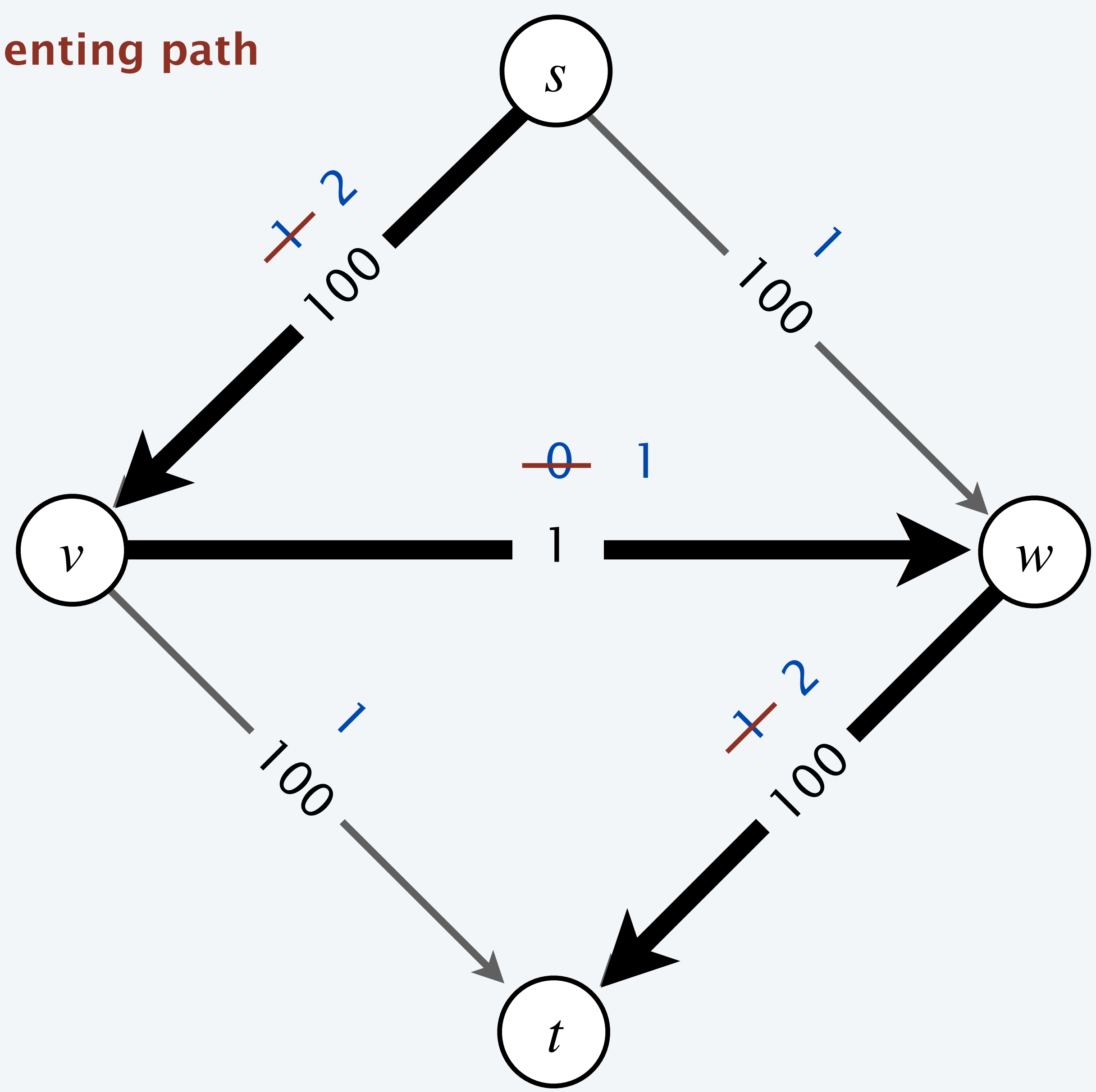
Bad news. Number of augmenting paths can be exponential in input size.



Ford–Fulkerson algorithm: exponential-time example

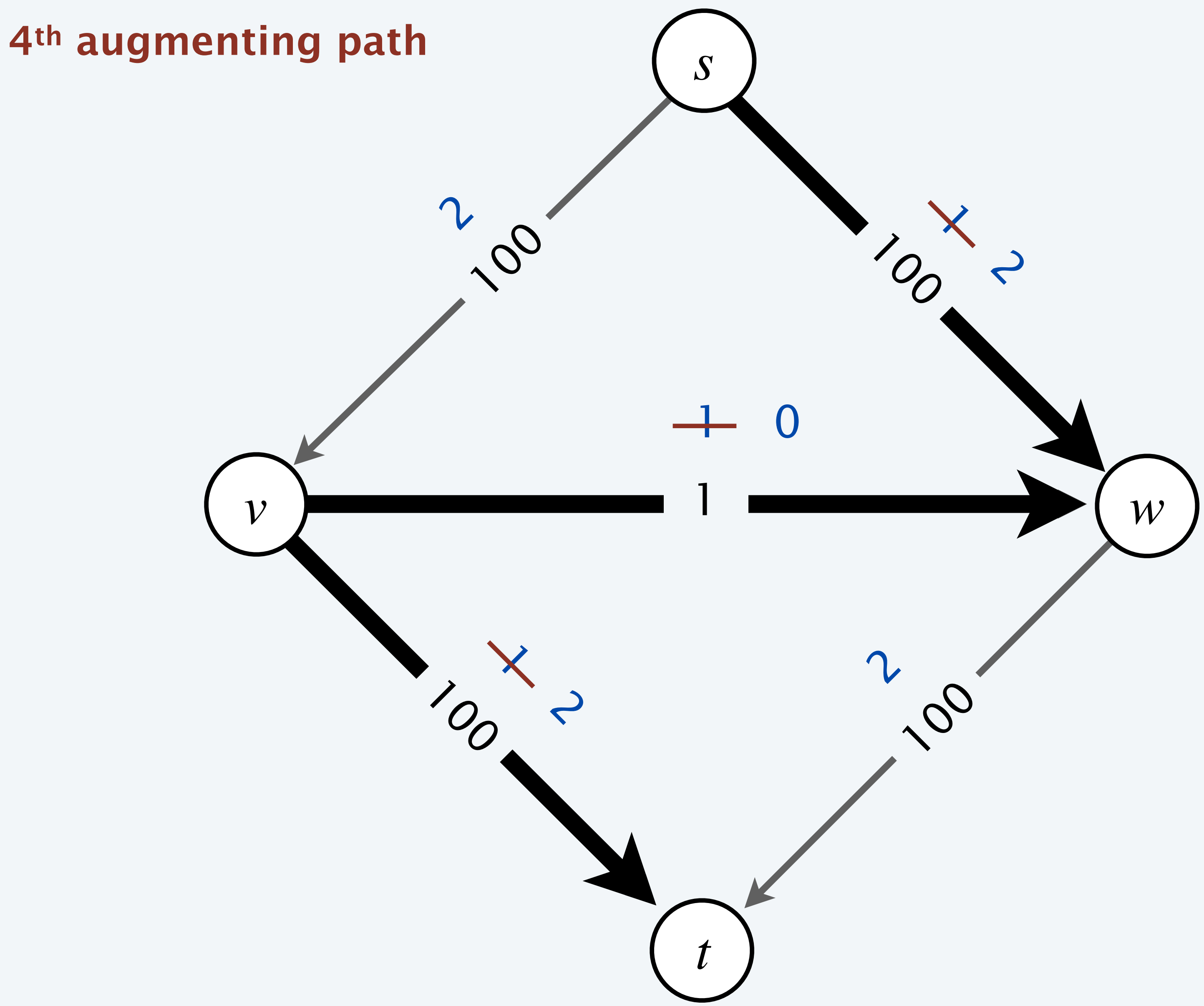
Bad news. Number of augmenting paths can be exponential in input size.

3rd augmenting path



Ford–Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.



Ford–Fulkerson algorithm: exponential-time example

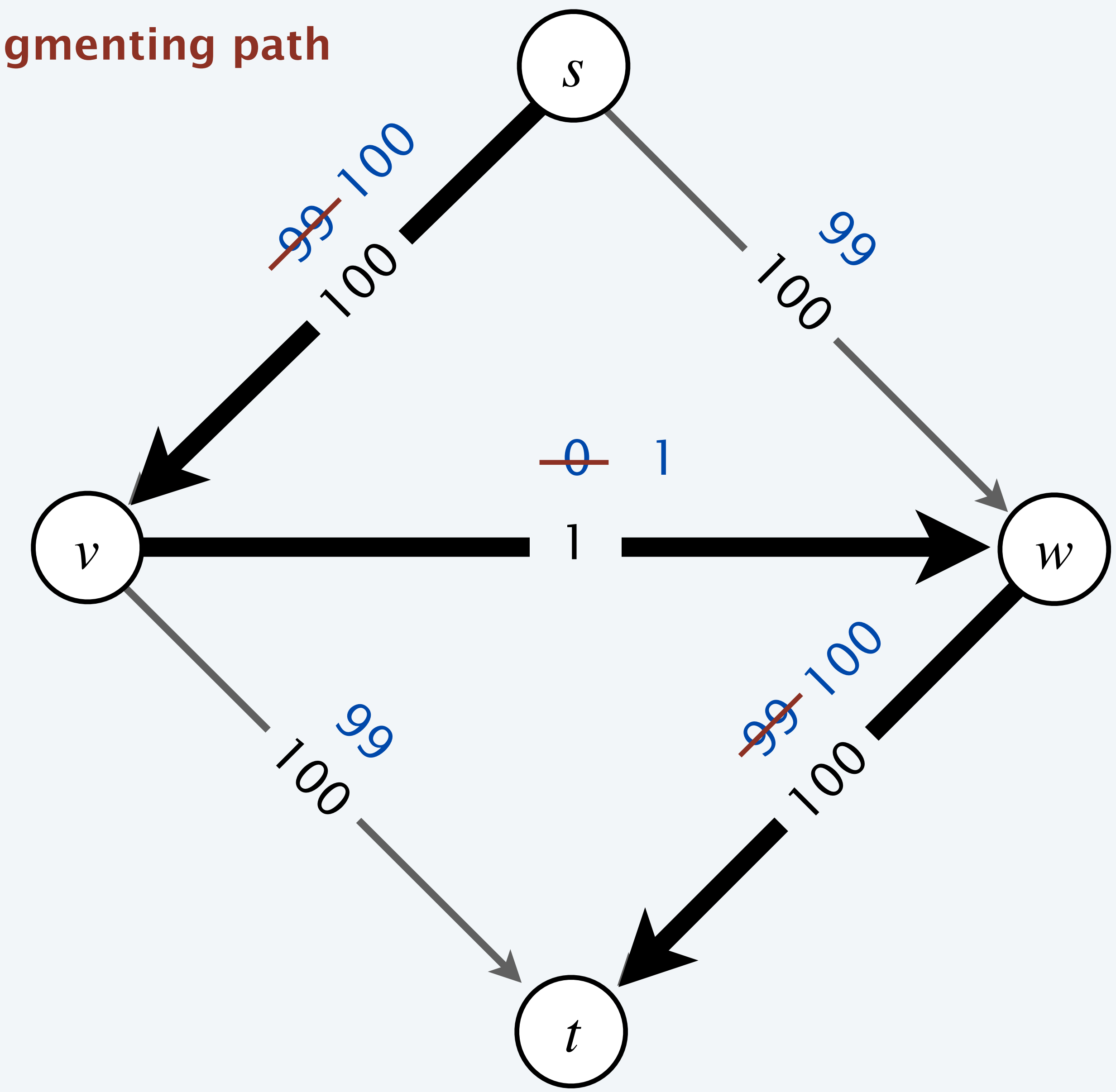
Bad news. Number of augmenting paths can be exponential in input size.

• • •

Ford–Fulkerson algorithm: exponential-time example

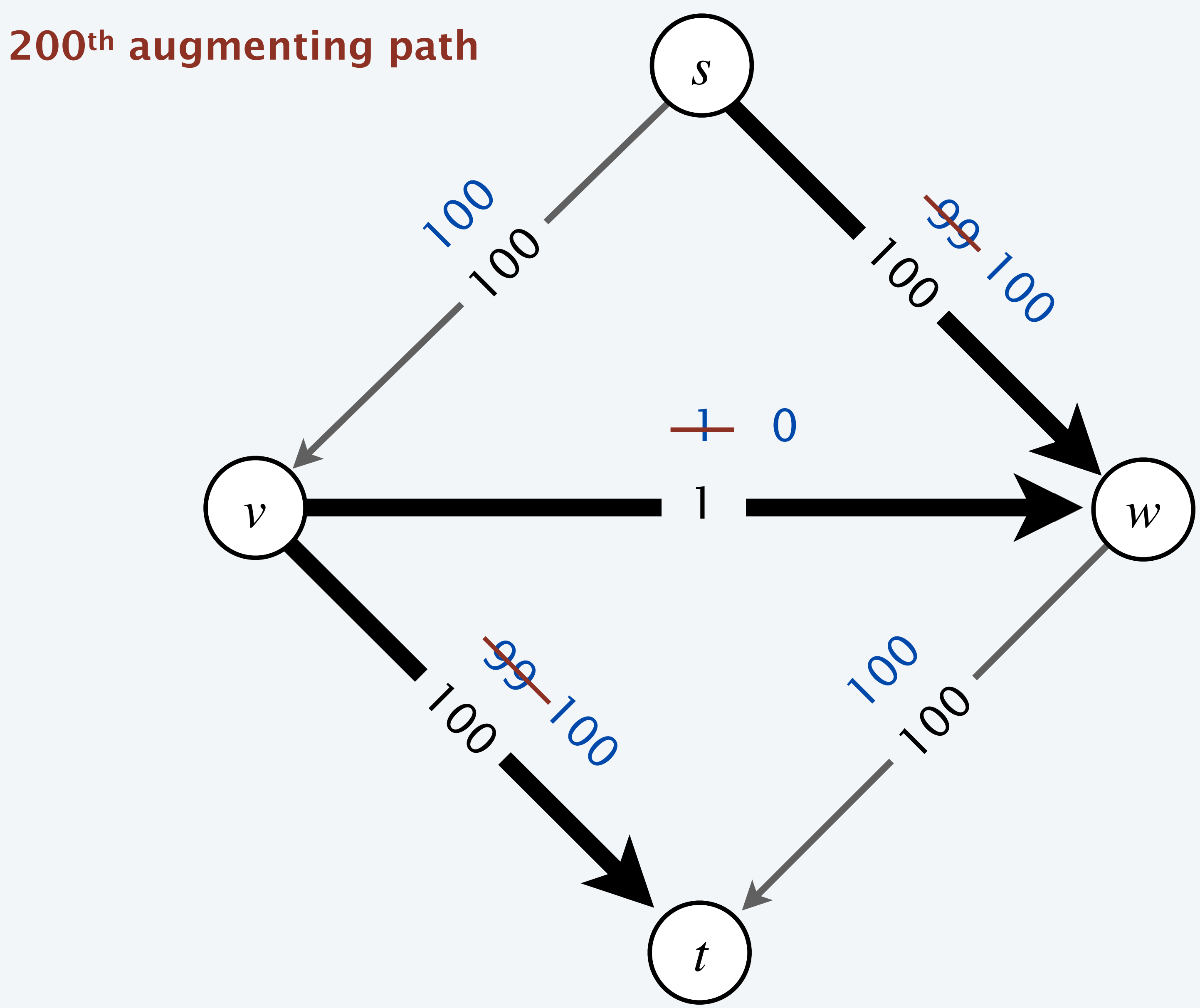
Bad news. Number of augmenting paths can be exponential in input size.

199th augmenting path



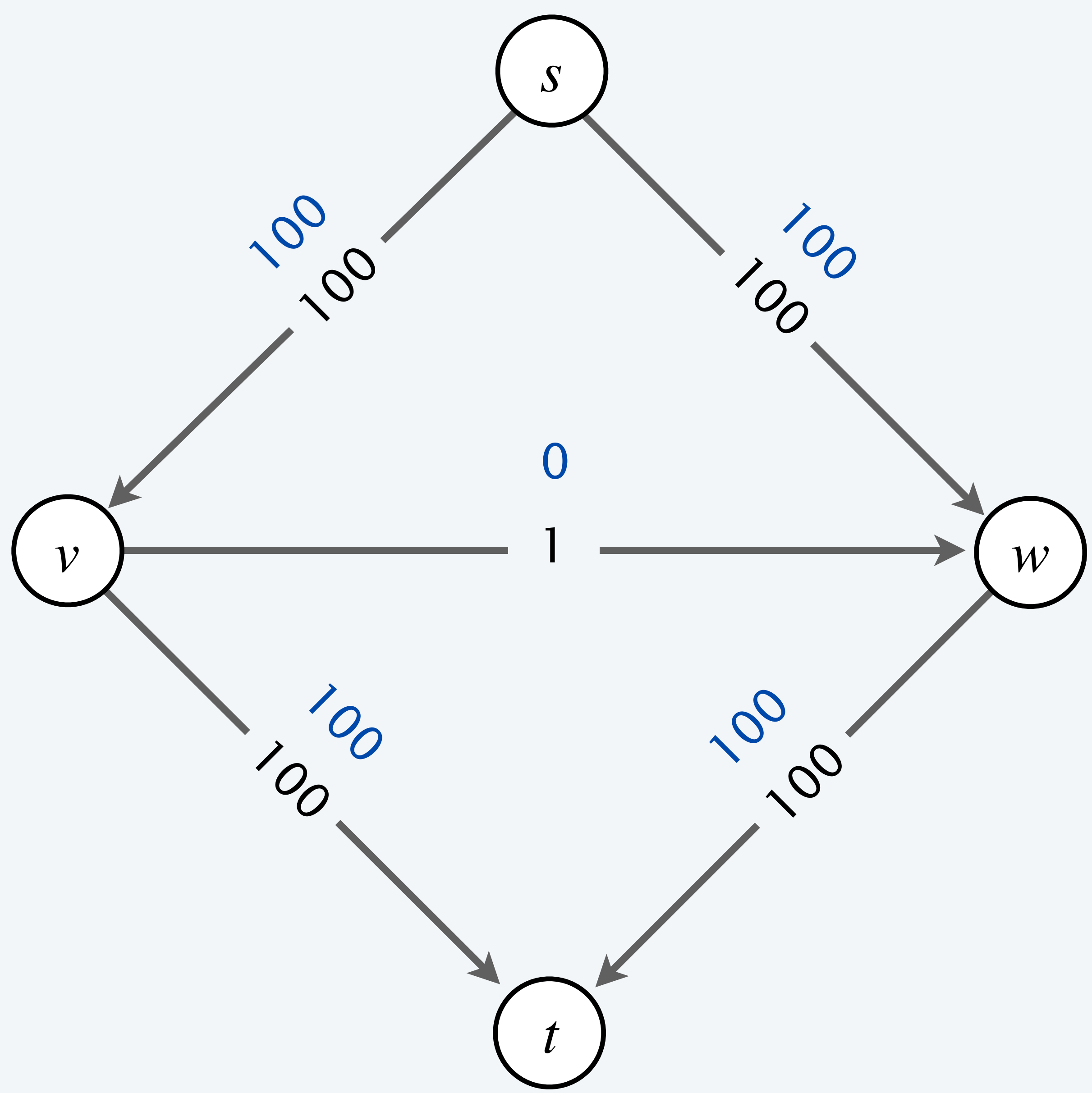
Ford–Fulkerson algorithm: exponential-time example

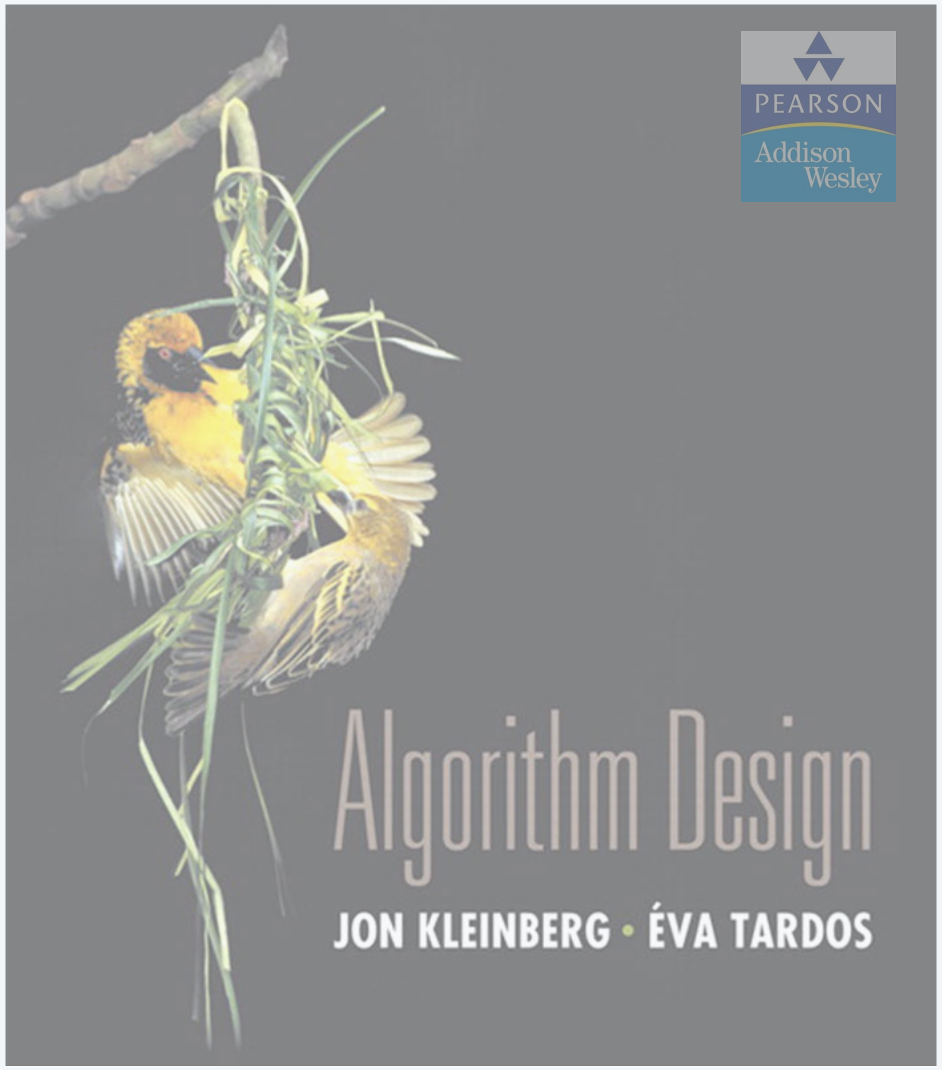
Bad news. Number of augmenting paths can be exponential in input size.



Ford–Fulkerson algorithm: exponential-time example

Bad news. Number of augmenting paths can be exponential in input size.





SECTION 7.1

7. NETWORK FLOW I

- *Ford–Fulkerson demo*
- *exponential-time example*
- *pathological example*

Ford–Fulkerson algorithm: pathological example

$1 - r$

Intuition. Let $r > 0$ satisfy $r^2 = 1 - r$.

- Initially, some residual capacities are 1 and r.
- After two augmenting paths, some residual capacities are r and r^2.
- After two more augmenting paths, some residual capacities are r^2 and r^3.
- After two more, some residual capacities are r^3 and r^4.
- By carefully choreographing the augmenting paths, infinitely many residual capacities arise!

$r - r^2$

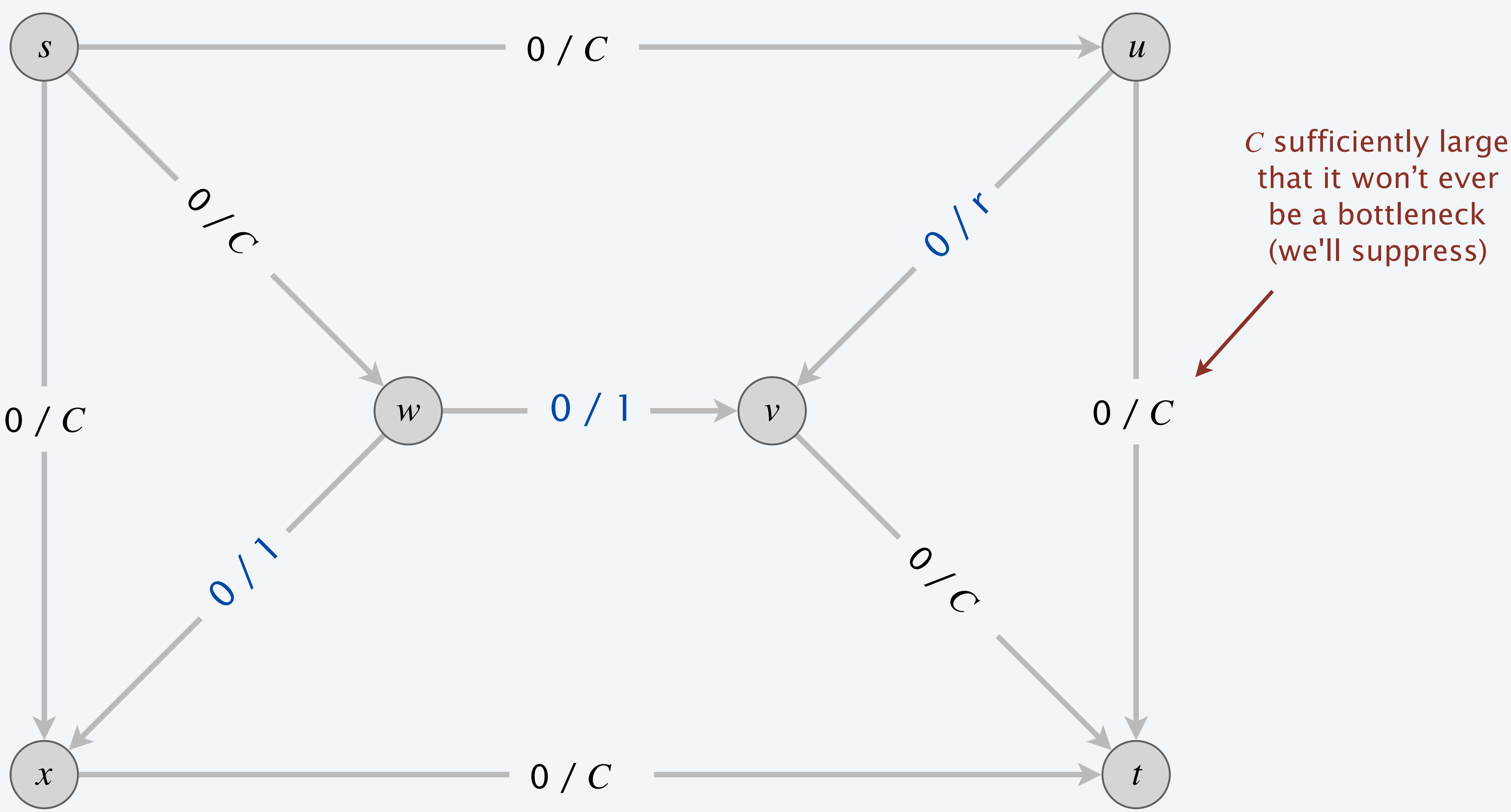
$r^2 - r^3$

$$r = \frac{\sqrt{5} - 1}{2} \implies r^2 = 1 - r$$

$$r \approx 0.618 \implies r^4 < r^3 < r^2 < r < 1$$

Ford–Fulkerson algorithm: pathological example

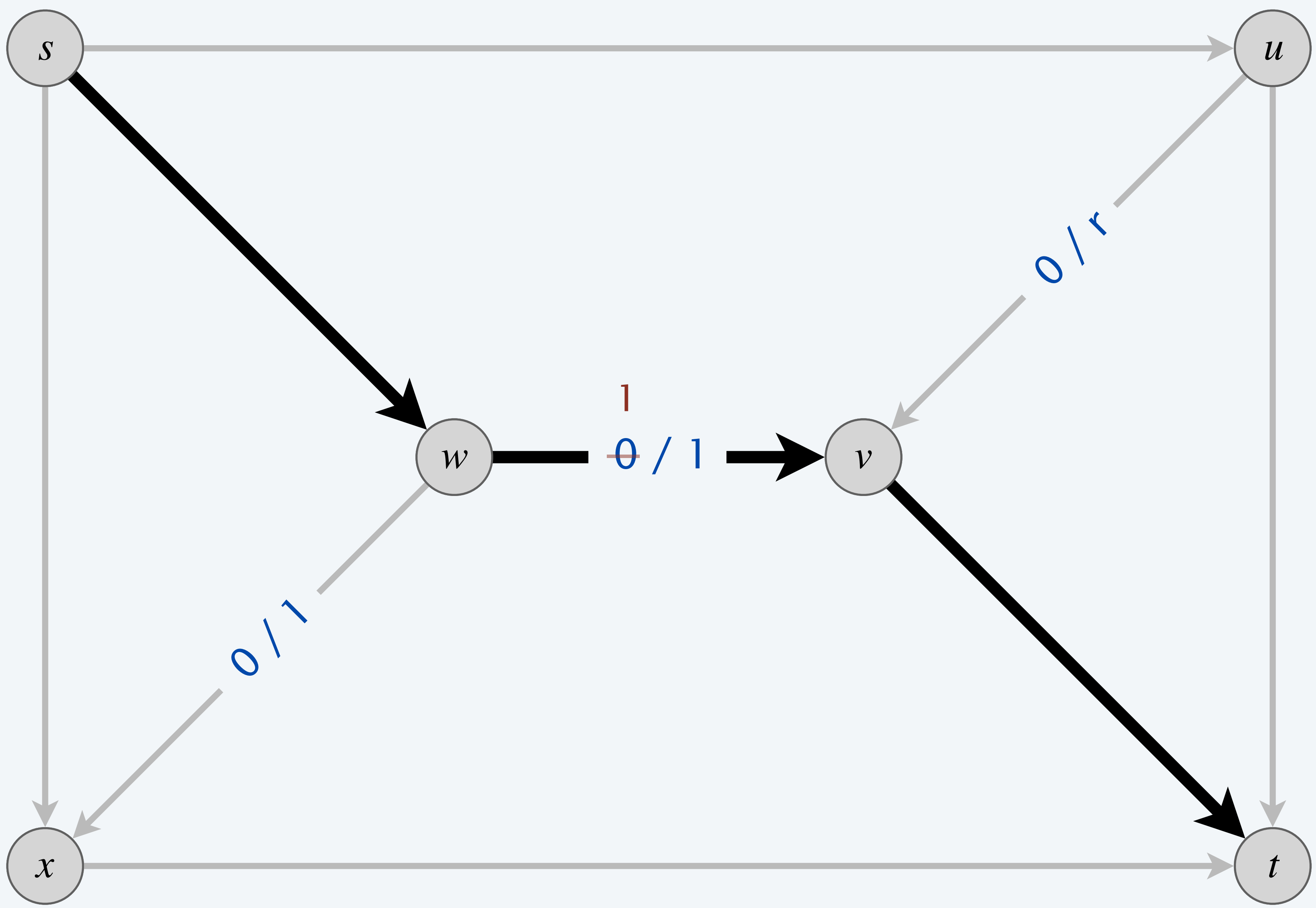
flow network G



$$r^2 = 1 - r$$

Ford–Fulkerson algorithm: pathological example

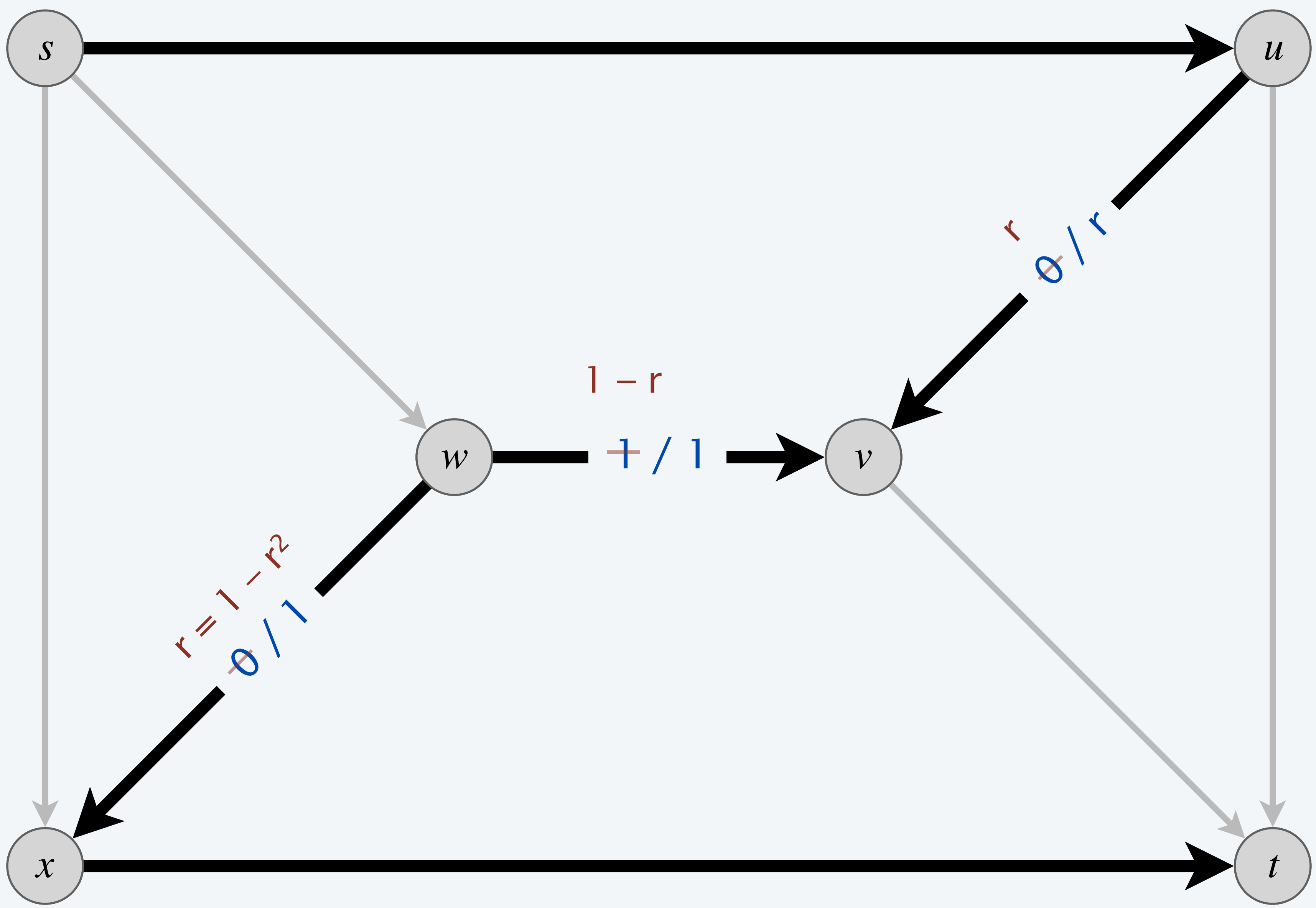
augmenting path 1: s→w→v→t (bottleneck capacity = 1)



$$r^2 = 1 - r$$

Ford–Fulkerson algorithm: pathological example

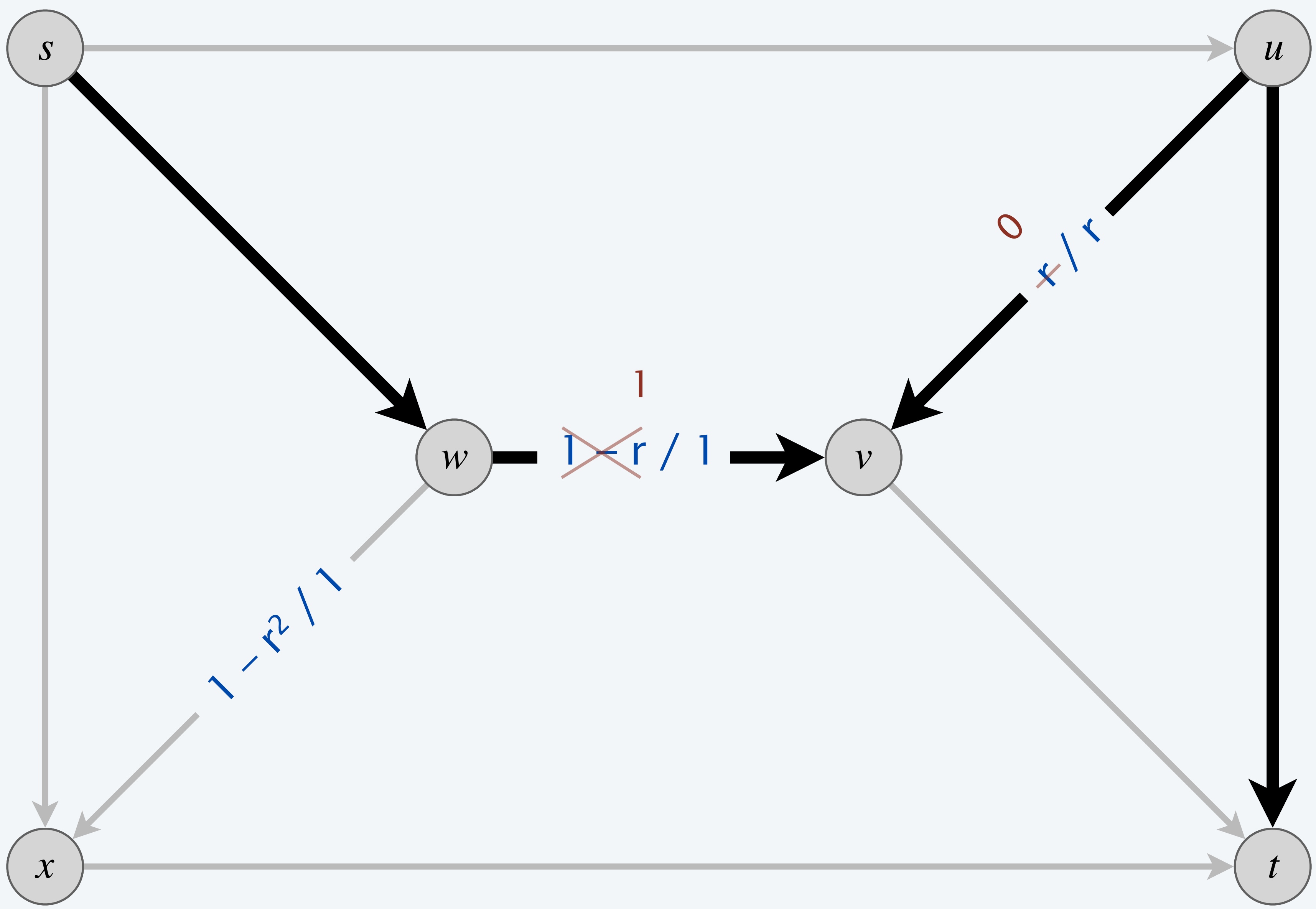
augmenting path 2: s→u→v→w→x→t (bottleneck capacity = r)



$$r^2 = 1 - r$$

Ford–Fulkerson algorithm: pathological example

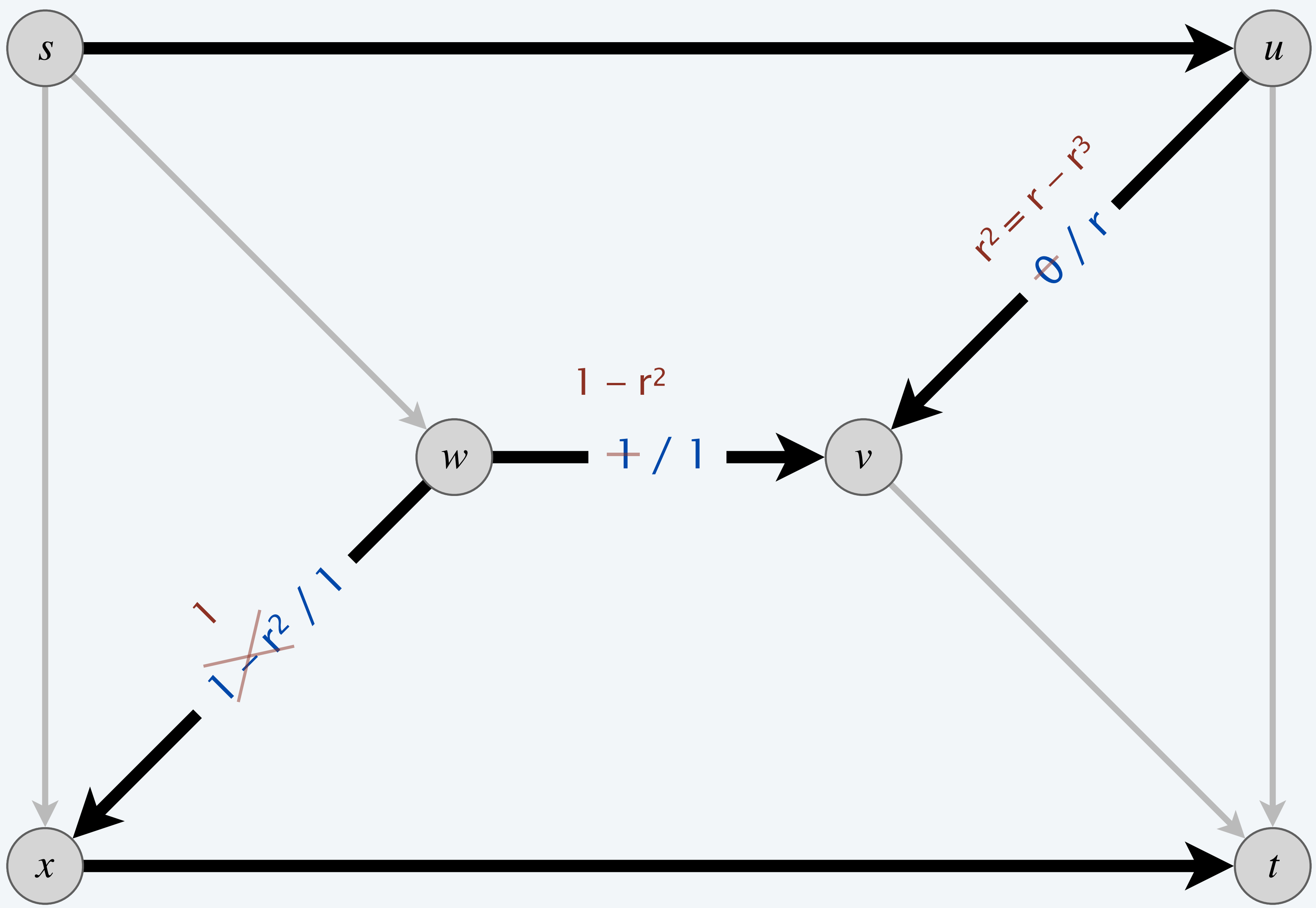
augmenting path 3: s→w→v→u→t (bottleneck capacity = r)



$$r^2 = 1 - r$$

Ford–Fulkerson algorithm: pathological example

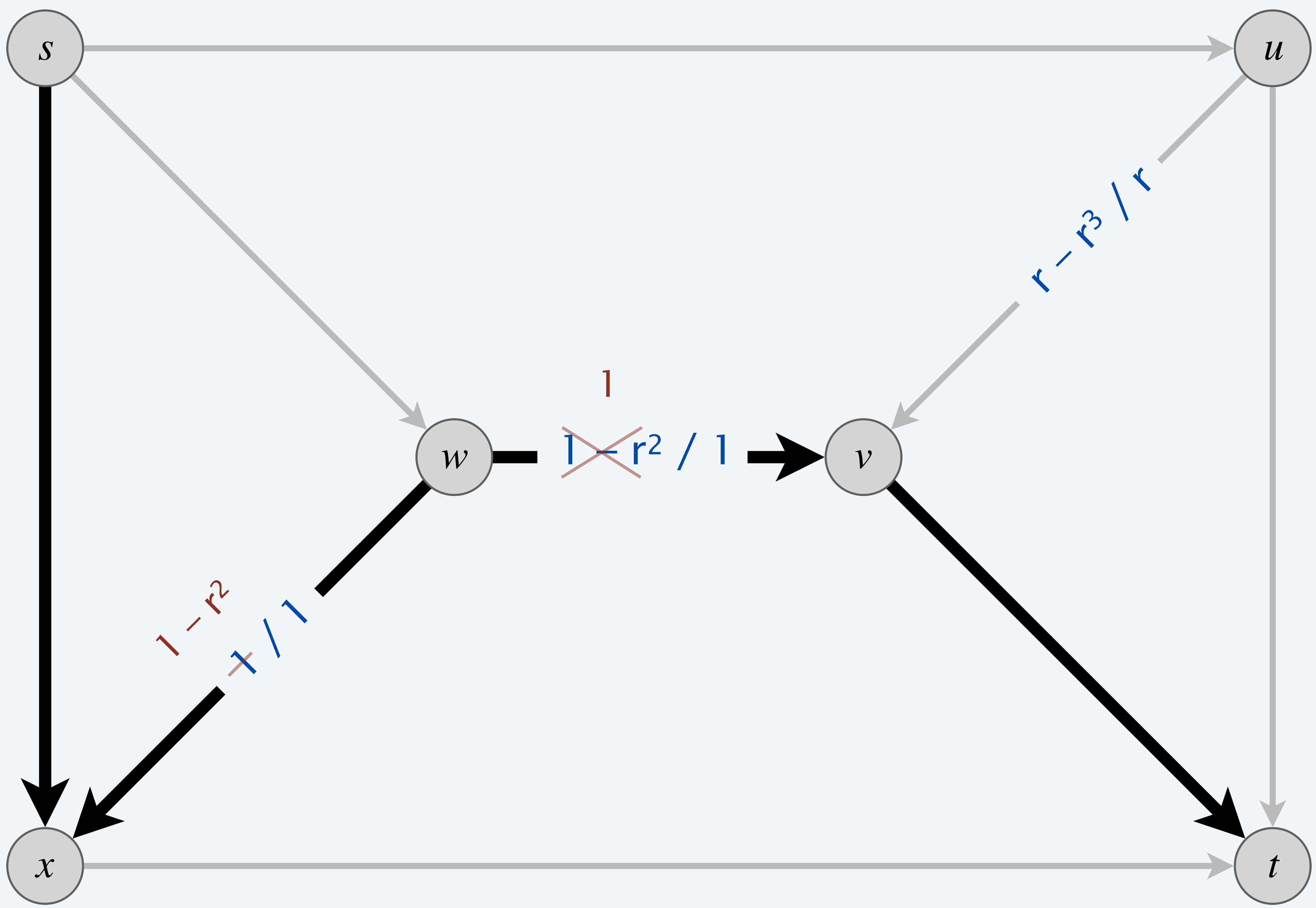
augmenting path 4: s→u→v→w→x→t (bottleneck capacity = r^2)



$r^2 = 1 - r$

Ford–Fulkerson algorithm: pathological example

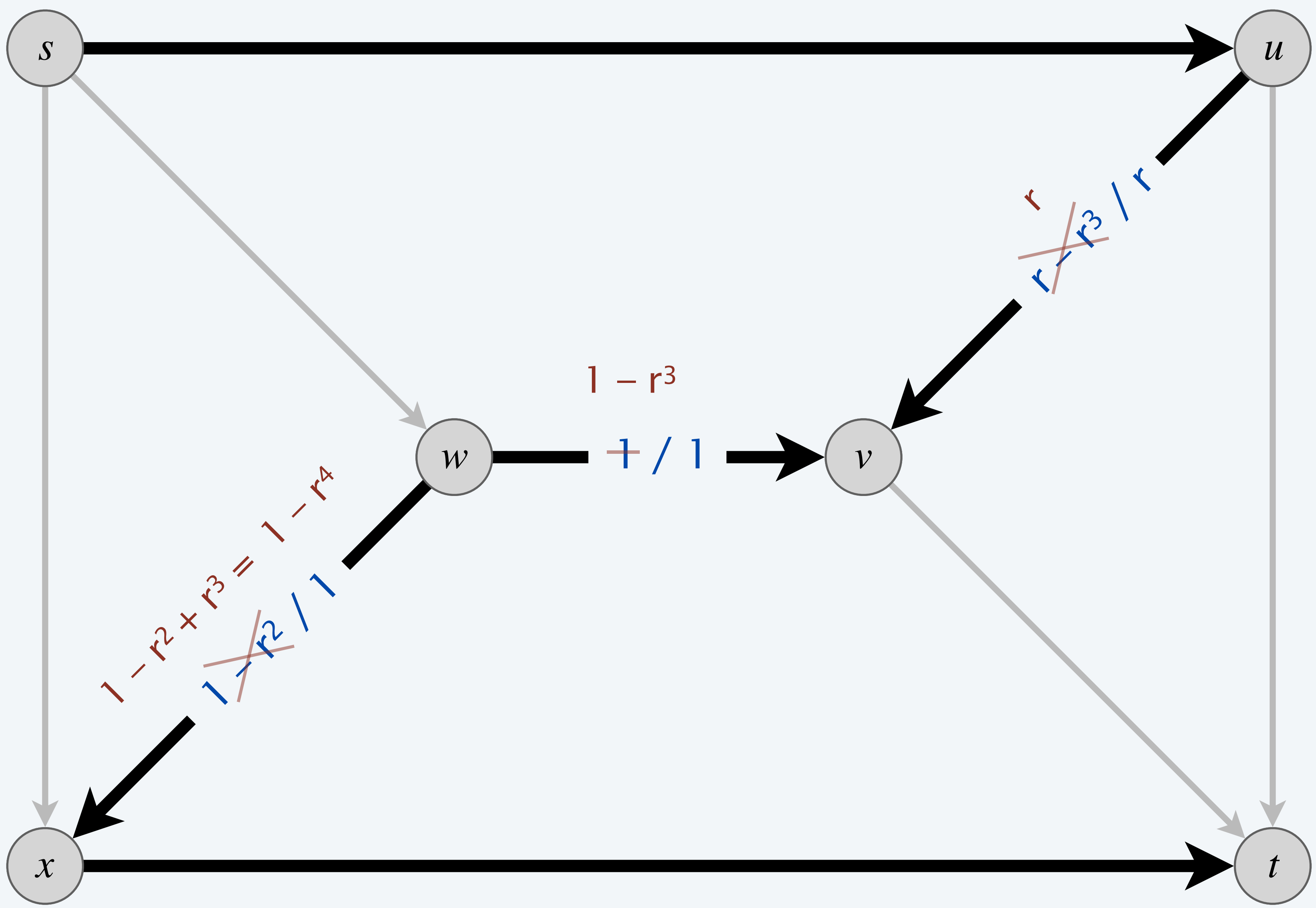
augmenting path 5: s→x→w→v→t (bottleneck capacity = r²)



$$r^2 = 1 - r$$

Ford–Fulkerson algorithm: pathological example

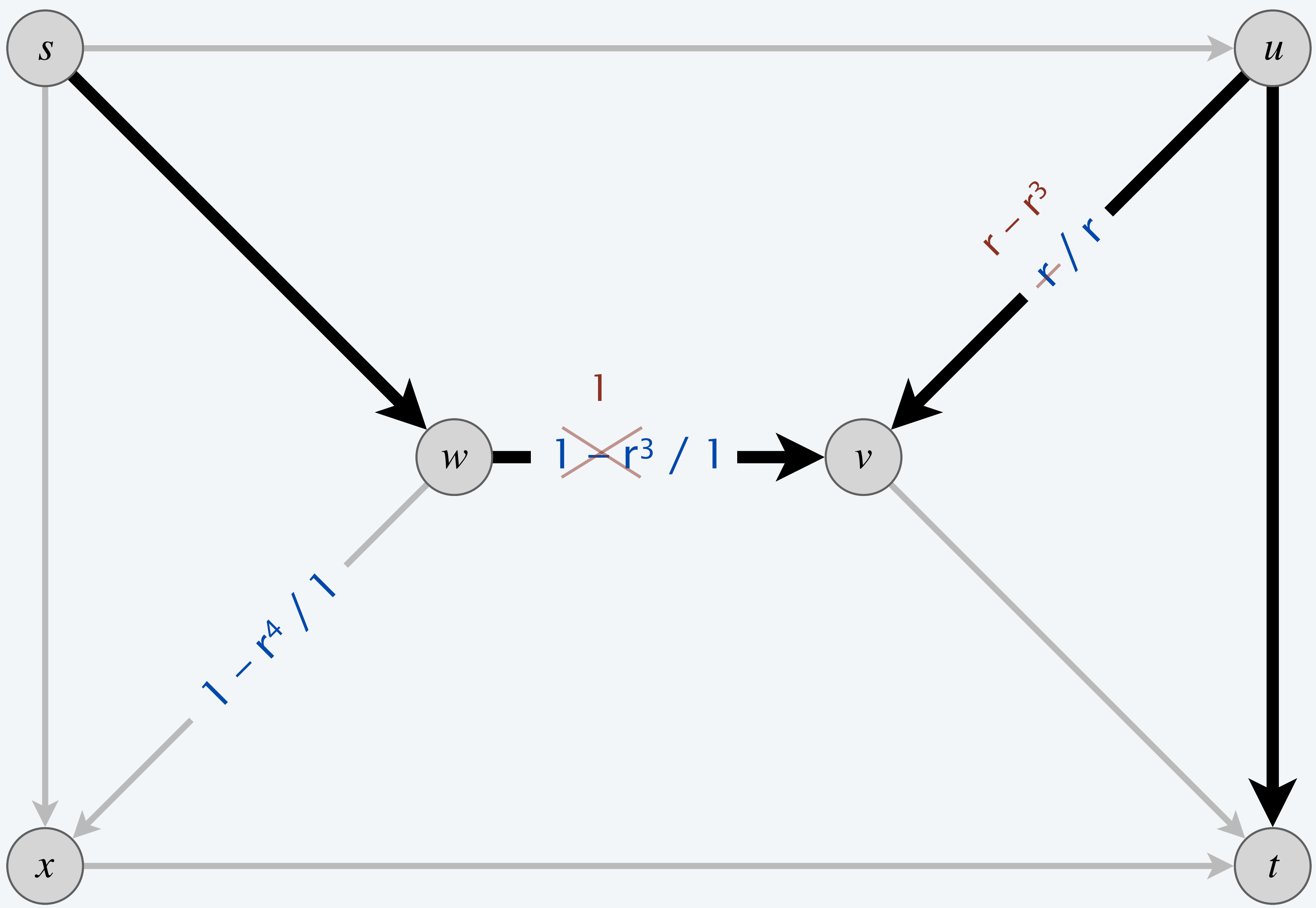
augmenting path 6: s→u→v→w→x→t (bottleneck capacity = r^3)



$$r^2 = 1 - r$$

Ford–Fulkerson algorithm: pathological example

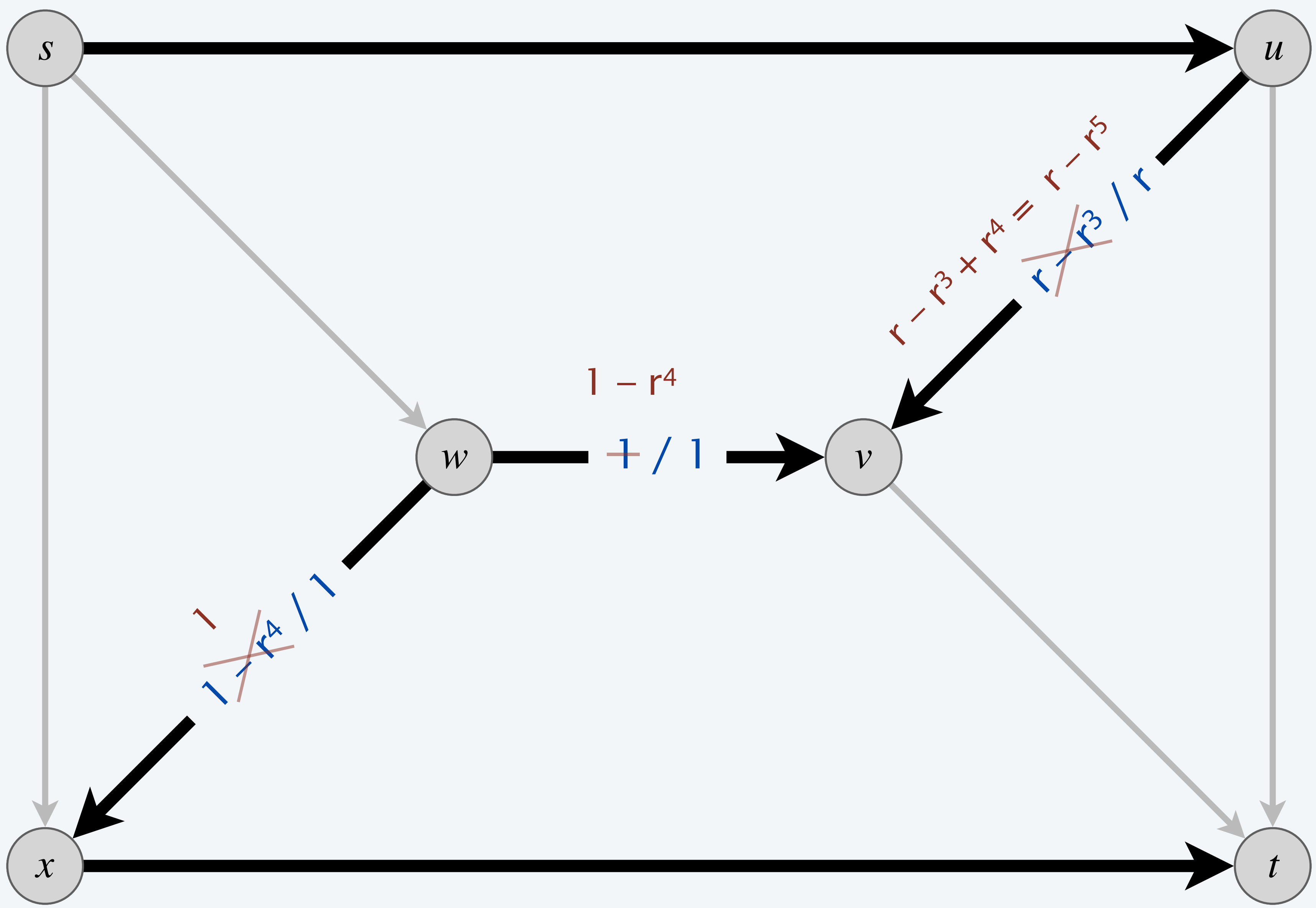
augmenting path 7: s→w→v→u→t (bottleneck capacity = r³)



$$r^2 = 1 - r$$

Ford–Fulkerson algorithm: pathological example

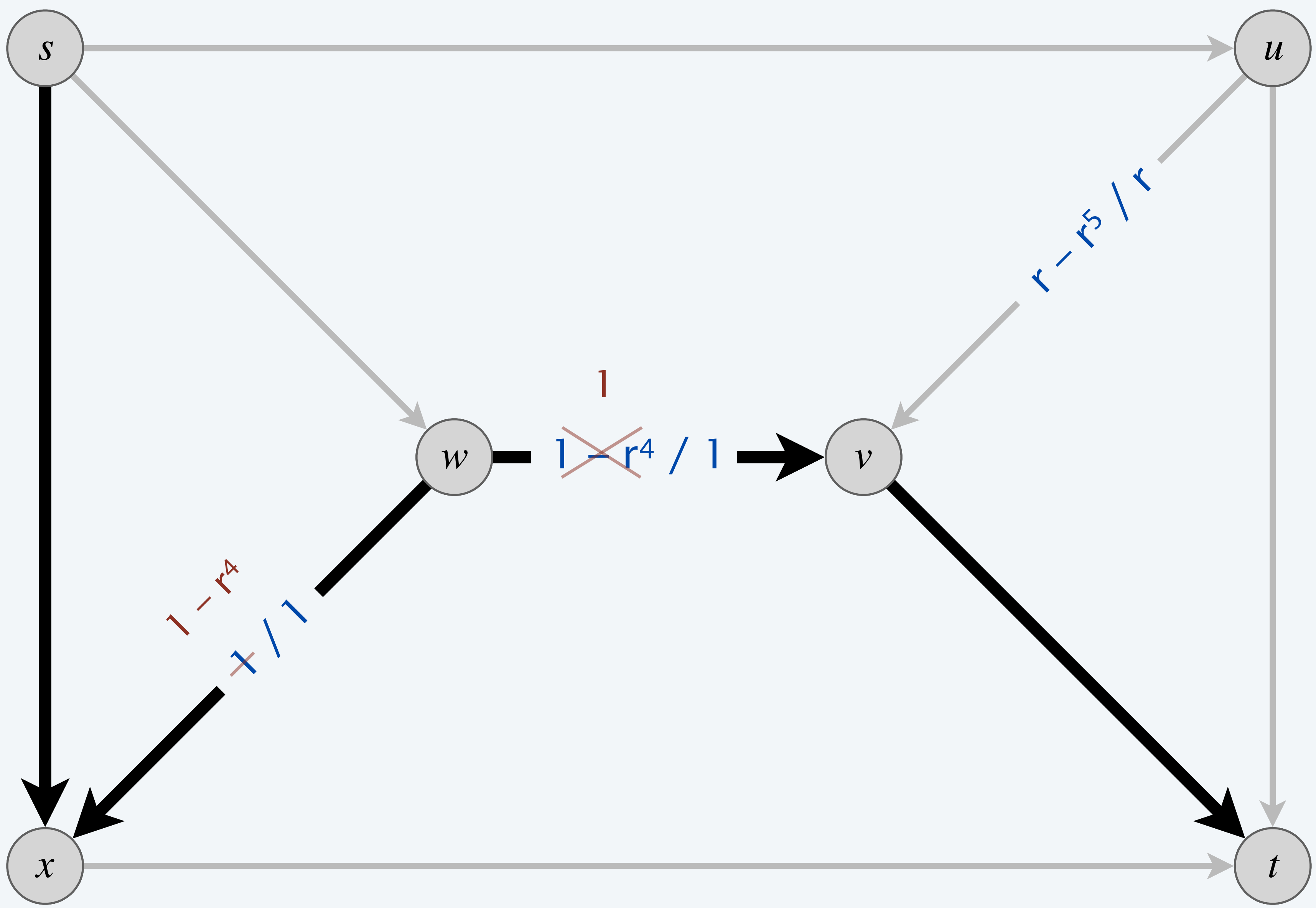
augmenting path 8: s→u→v→w→x→t (bottleneck capacity = r^4)



$$r^2 = 1 - r$$

Ford–Fulkerson algorithm: pathological example

augmenting path 9: s→x→w→v→t (bottleneck capacity = r⁴)



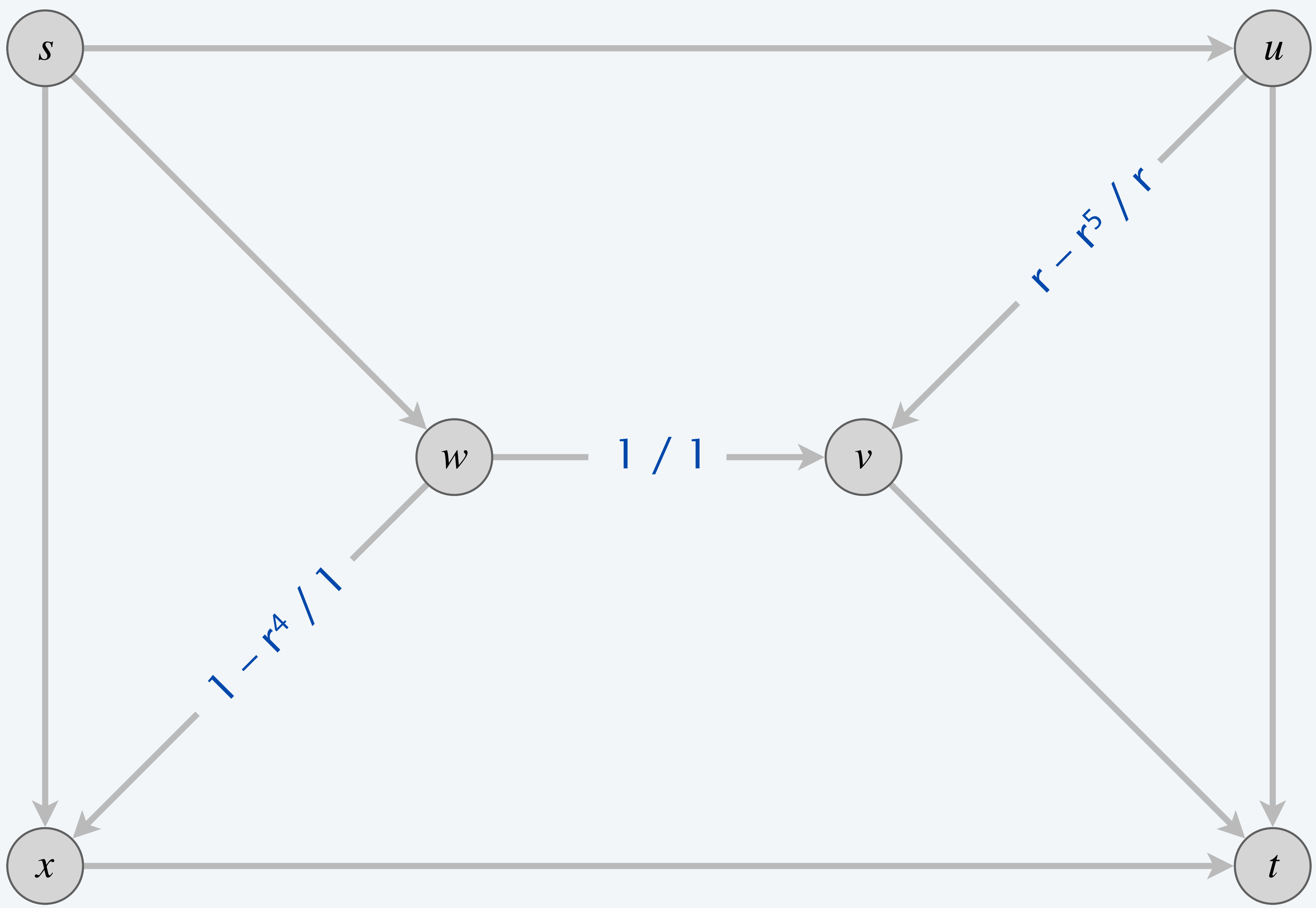
$$\boxed{r^2 = 1 - r}$$

Ford–Fulkerson algorithm: pathological example

flow after augmenting path 1: { $r - r^1$, 1, $1 - r^0$ } (value of flow = 1)

flow after augmenting path 5: { $r - r^3$, 1, $1 - r^2$ } (value of flow = $1 + 2r + 2r^2$)

flow after augmenting path 9: { $r - r^5$, 1, $1 - r^4$ } (value of flow = $1 + 2r + 2r^2 + 2r^3 + 2r^4$)



$$r^2 = 1 - r$$

Ford–Fulkerson algorithm: pathological example

Theorem. The Ford–Fulkerson algorithm may not terminate; moreover, it may converge to a value not equal to the value of the maximum flow.

Pf.

- After $(1 + 4k)$ augmenting paths of the form just described, the value of the flow

$$
\begin{aligned}
&= 1 + 2\sum_{i=1}^{2k} r^i \\
&\le 1 + 2\sum_{i=1}^{\infty} r^i \\
&= 1 + \frac{2r}{1-r} \\
&< 5
\end{aligned}
$$

$$\boxed{r = \frac{\sqrt{5}-1}{2}}$$

- Value of maximum flow = $2C + 1$. ▪

Reference

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Theoretical Computer Science

Note

The smallest networks on which the Ford–Fulkerson maximum flow procedure may fail to terminate

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Abstract

It is widely known that the Ford–Fulkerson procedure for finding the maximum flow in a network need not terminate if some of the capacities of the network are irrational. Ford and Fulkerson gave as an example a network with 10 vertices and 48 edges on which their procedure may fail to halt. We construct much smaller and simpler networks on which the same may happen. Our smallest network has only 6 vertices and 8 edges. We show that it is the smallest example possible.