7. **Network Flow I**

- Ford–Fulkerson demo
- pathological example
7. **Network Flow I**

- *Ford–Fulkerson demo*
- *pathological example*
Ford–Fulkerson algorithm demo

network $G$ and flow $f$

residual network $G_f$
Ford–Fulkerson algorithm demo

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network $G$ and flow $f$

residual network $G_f$

fixes mistake from second augmenting path
Ford–Fulkerson algorithm demo

network $G$ and flow $f$

residual network $G_f$
Ford–Fulkerson algorithm demo

network $G$ and flow $f$

![Diagram of network $G$ and flow $f$ with min cut and residual network $G_f$.]

- Capacity $= 10 + 9 = 19$

residual network $G_f$

- Nodes reachable from $s$

value of max flow

9
7. **Network Flow I**

- Ford–Fulkerson demo
- pathological example
Ford–Fulkerson pathological example

**Intuition.** Let $r$ satisfy $r^2 = 1 - r$.

- Initially, some residual capacities are 1 and $r$.
- After two augmenting paths, some residual capacities are $r$ and $r^2$.
- After two more augmenting paths, some residual capacities are $r^2$ and $r^3$.
- After two more, some residual capacities are $r^3$ and $r^4$.
- By carefully choreographing the augmenting paths, infinitely many residual capacities arise!

\[
r = \frac{\sqrt{5} - 1}{2} \implies r^2 = 1 - r
\]

\[
r \approx 0.618 \implies r^4 < r^3 < r^2 < r < 1
\]
Ford–Fulkerson pathological example

flow network $G$

$C$ sufficiently large that it won't ever be a bottleneck (we'll suppress)

$r^2 = 1 - r$
Ford–Fulkerson pathological example

augmenting path 1: $s \rightarrow w \rightarrow v \rightarrow t$ (bottleneck capacity = 1)

$r^2 = 1 - r$
Ford–Fulkerson pathological example

augmenting path 2: $s \rightarrow u \rightarrow v \rightarrow w \rightarrow x \rightarrow t$ (bottleneck capacity = $r$)

$r^2 = 1 - r$
augmenting path 3: \( s \rightarrow w \rightarrow v \rightarrow u \rightarrow t \) (bottleneck capacity = \( r \))

\[ r^2 = 1 - r \]
Ford–Fulkerson pathological example

augmenting path 4: \( s \rightarrow u \rightarrow v \rightarrow w \rightarrow x \rightarrow t \) (bottleneck capacity = \( r^2 \))

\[
r^2 = 1 - r
\]
Ford–Fulkerson pathological example

**augmenting path 5:** \( s \to x \to w \to v \to t \) (bottleneck capacity = \( r^2 \))

\[
r^2 = 1 - r
\]
Ford–Fulkerson pathological example

augmenting path 6: \( s \rightarrow u \rightarrow v \rightarrow w \rightarrow x \rightarrow t \) (bottleneck capacity = \( r^3 \))

\[
r^2 = 1 - r
\]
Ford–Fulkerson pathological example

augmenting path 7: $s \rightarrow w \rightarrow v \rightarrow u \rightarrow t$ (bottleneck capacity = $r^3$)

$r^2 = 1 - r$
Ford–Fulkerson pathological example

augmenting path 8: $s \rightarrow u \rightarrow v \rightarrow w \rightarrow x \rightarrow t$ (bottleneck capacity = $r^4$)

$r^2 = 1 - r$
Ford–Fulkerson pathological example

augmenting path 9: $s \rightarrow x \rightarrow w \rightarrow v \rightarrow t$ (bottleneck capacity = $r^4$)

\[ r^2 = 1 - r \]
Ford–Fulkerson pathological example

flow after augmenting path 1: \{ r - r^1, 1, 1 - r^0 \} \ (value of flow = 1) 

flow after augmenting path 5: \{ r - r^3, 1, 1 - r^2 \} \ (value of flow = 1 + 2r + 2r^2) 

flow after augmenting path 9: \{ r - r^5, 1, 1 - r^4 \} \ (value of flow = 1 + 2r + 2r^2 + 2r^3 + 2r^4) 

\[ r^2 = 1 - r \]
Ford–Fulkerson pathological example

**Theorem.** The Ford–Fulkerson algorithm may not terminate; moreover, it may converge to a value not equal to the value of the maximum flow.

**Pf.**

- After \((1 + 4k)\) augmenting paths of the form just described, the value of the flow

\[
1 + 2 \sum_{i=1}^{2k} r^i 
\leq 1 + 2 \sum_{i=1}^{\infty} r^i 
= 3 + 2r 
< 5
\]

- Value of maximum flow = \(2C + 1\). □