COS 423	Theory of Algorithms	Spring 2018

Problem Set 3

This problem set is due Wednesday, February 28 at 11pm via electronic submission. Recall that when you are asked to design an efficient algorithm, you must (i) clearly describe the algorithm, (ii) rigorously prove that it is correct, and (iii) analyze its running time.

Collaboration is **not** *permitted, except with the course instructor and two preceptors.*

0. Read Section 4.4–4.7 in *Algorithm Design*.

1. (a) Let T and T' be two different spanning trees of a graph G. Prove that there exists an edge $e \in T$ and an edge $e' \in T'$ such that $e \notin T'$ and $T \cup \{e'\} - \{e\}$ is a spanning tree of G.

   (b) Let G be a connected graph with each edge colored either orange or black. Given an integer k, design an efficient algorithm to determine whether there exists a spanning tree of G that contains *exactly* k orange edges.

2. Consider a digraph $G = (V, E)$ in which each edge e has a length $\ell_e \geq 0$ and is colored either orange or black. Design an efficient algorithm to find a shortest path from s to t that uses *at most* k orange edges.

   For full credit, the running time of your algorithm should be $\mathcal{O}(k(m+n)\log n)$*, where* m *is the number of edges and* n *is the number of nodes in* G*.*

3. Consider an election with n candidates. Each voter ranks each of the n candidates in order of preference. Let $c(u, v)$ denote the number of voters who strictly prefer candidate u to candidate v, minus the number of voters who strictly prefer v to u. Consider some sequence of candidates $v_1, v_2, \ldots, v_r$. The *strength* of such a sequence is $\min_{1 \leq i < r} c(v_i, v_{i+1})$. Let $p(u, v)$ denote the maximum strength among all sequences that start with u and end with v. We say that candidate u *dominates* candidate v if $p(u, v) > p(v, u)$.

   (a) Prove that the dominates relation is *transitive*: if u dominates v and v dominates w, then u dominates w.

   (b) Design an $\mathcal{O}(n^2)$ time algorithm to determine whether one candidate s dominates another candidate t.

   Note: this path-voting method is used to determine the winner of elections in a various programming communities, including Debian, Gentoo, TopCoder, KDE, and Wikimedia.