Class Meeting #20

COS 226 — Spring 2018

Mark Braverman

(based on slides by Robert Sedgewick and Kevin Wayne)

Linear programming

- A “Swiss army knife” for optimization algorithms.
- Can solve a large fraction of optimization problems efficiently.
- Good libraries in most languages.
- “Duality” an important concept with connections to Game Theory and other areas.

Linear programming

What is it? Problem-solving model for optimal allocation of scarce resources, among a number of competing activities that encompasses:

- Shortest paths, maxflow, MST, matching, assignment, ...
- 2-person zero-sum games, ...

$$
\begin{array}{lrcrcr}
\text{maximize} & 13A & + & 23B & & \\
\text{subject to the constraints} & 5A & + & 15B & \leq & 480 \\
 & 4A & + & 4B & \leq & 160 \\
 & 35A & + & 20B & \leq & 1190 \\
 & A & , & B & \geq & 0
\end{array}
$$

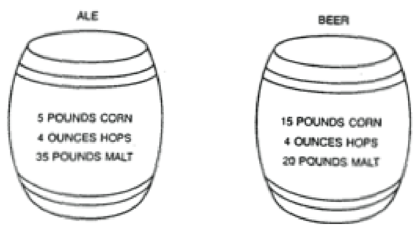
Toy LP example: brewer's problem

Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.

corn (480 lbs) hops (160 oz) malt (1190 lbs)

- Recipes for ale and beer require different proportions of resources.



$13 profit per barrel $23 profit per barrel

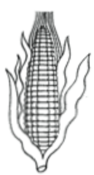
Brewer's problem: choose product mix to maximize profits.

34 barrels × 35 lbs malt = 1190 lbs

[amount of available malt]

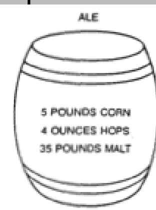
ale	beer				
34	0	170	136	1190	$442
0	32	480	128	640	$736
19.5	20.5	405	160	1092.5	$725
12	28	480	160	980	$800
?	?				> $800 ?

goods are divisible

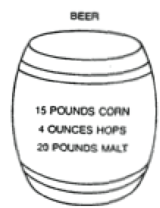

corn (480 lbs)


hops (160 oz)


malt (1190 lbs)



$13 profit per barrel

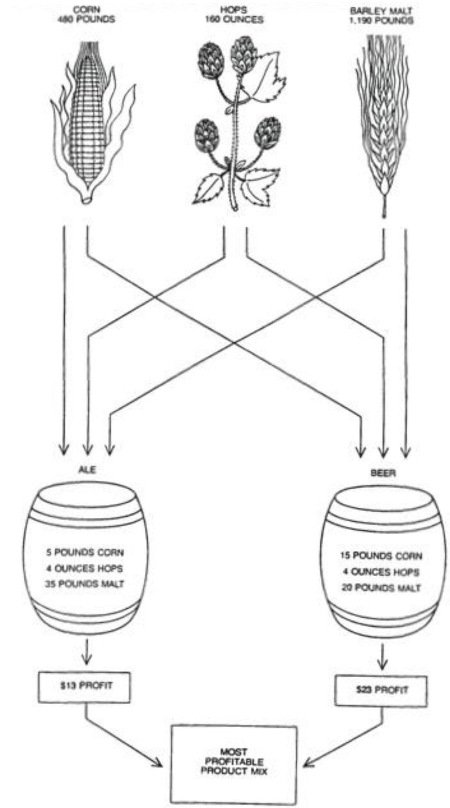


$23 profit per barrel

Linear programming formulation.

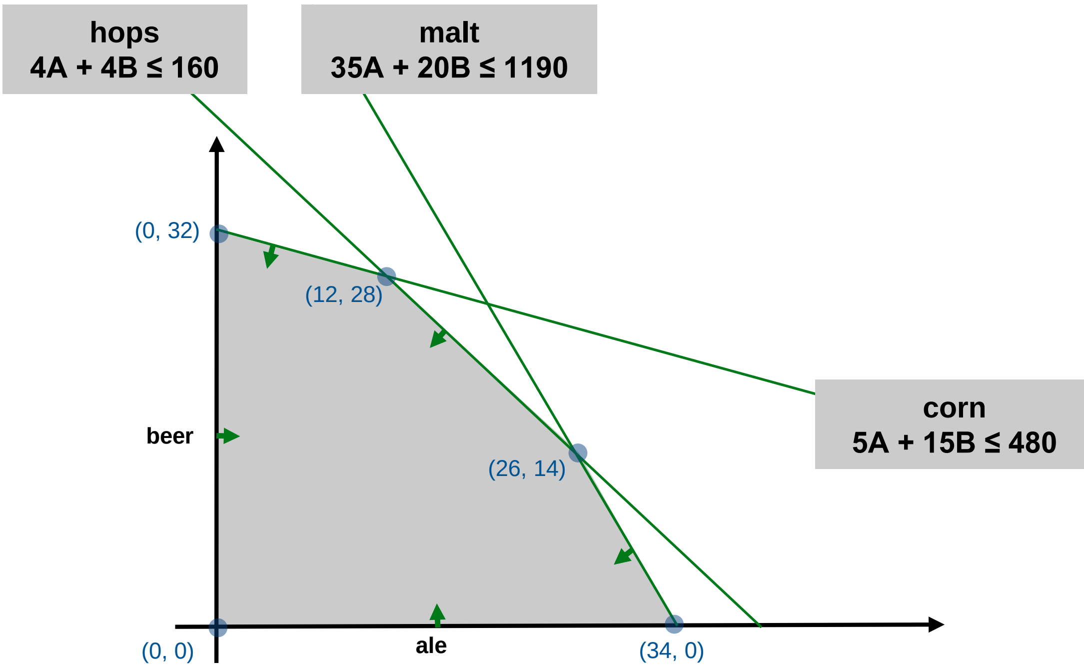
- Let A be the number of barrels of ale.
- Let B be the number of barrels of beer.

	ale		beer			
maximize	13A	+	23B			**profits**
subject	5A	+	15B	≤	480	**corn**
to the	4A	+	4B	≤	160	**hops**
constraints	35A	+	20B	≤	1190	**malt**
	A	,	B	≥	0	

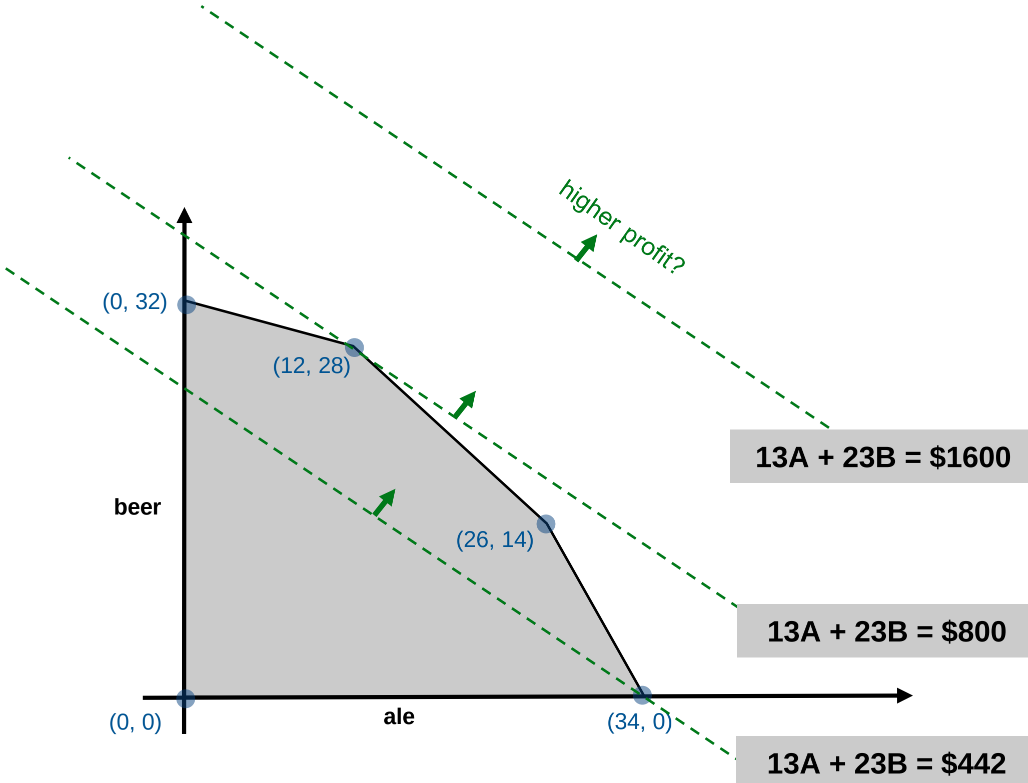


Brewer's problem: feasible region

Inequalities define halfplanes; feasible region is a convex polygon.



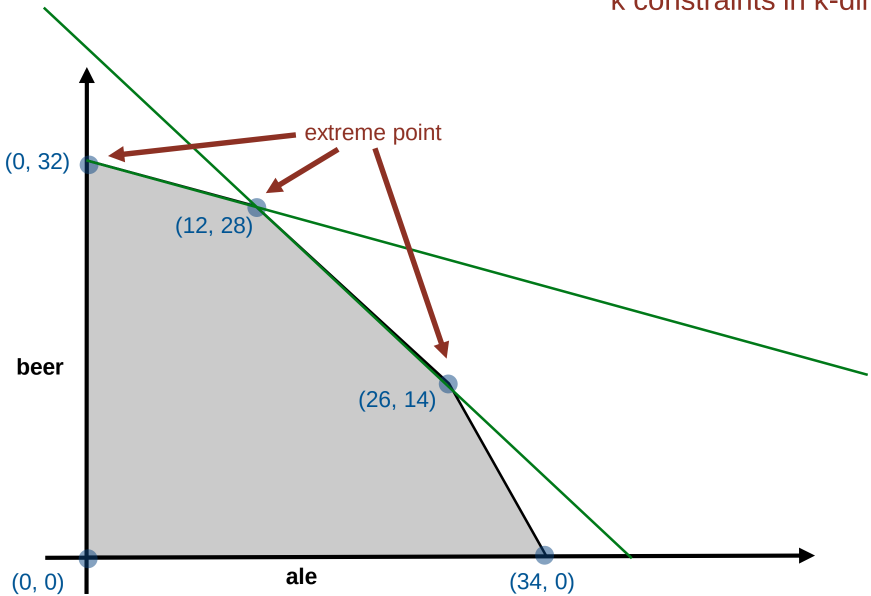
Brewer's problem: objective function



Brewer's problem: geometry

Optimal solution occurs at an extreme point.

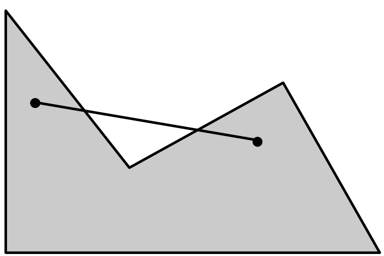
intersection of 2 constraints in 2d
k constraints in k-dims



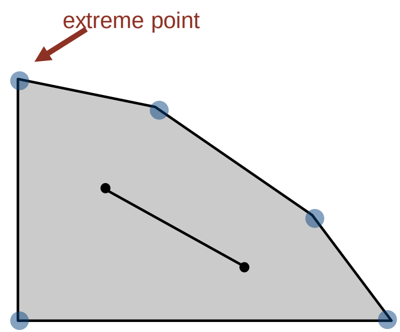
Inequalities define halfspaces; feasible region is a convex polyhedron.

A set is convex if for any two points a and b in the set, so is $\frac{1}{2}(a + b)$.

An extreme point of a set is a point in the set that can't be written as $\frac{1}{2}(a + b)$, where a and b are two distinct points in the set.

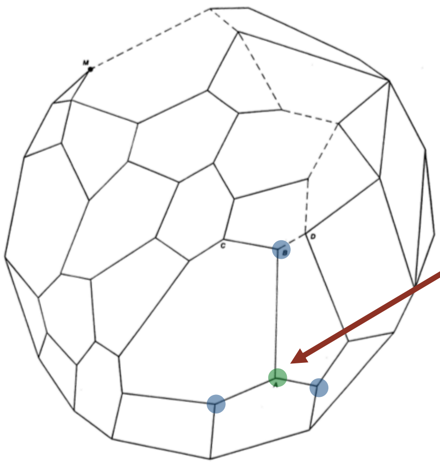


not convex



convex

Extreme point property. If there exists an optimal solution to (P),then there exists one that is an extreme point.
Good news? Bad news?



local optima are global optima
(follows because objective function
is linear
and feasible region is convex)

Simplex algorithm. [George Dantzig, 1947]

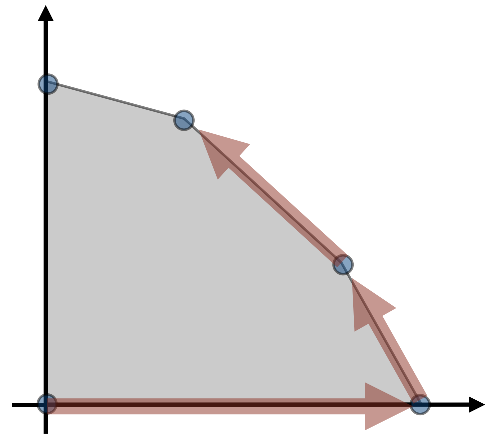
- Developed shortly after WWII in response to logistical problems, including Berlin airlift.
- Ranked as one of top 10 scientific algorithms of 20th century.

Generic algorithm.

never decreasing objective function

- Start at some extreme point.
- Pivot from one extreme point to an adjacent one.
- Repeat until optimal.

How to implement? Linear algebra.



Brewer's problem. Find optimal mix of beer and ale to maximize profits.

$$
\begin{array}{lrcrcrl}
\text{maximize} & 13A & + & 23B & & & \\
\text{subject} & 5A & + & 15B & \leq & 480 & \textbf{corn} \\
\text{to the} & 4A & + & 4B & \leq & 160 & \textbf{hops} \\
\text{constraints} & 35A & + & 20B & \leq & 1190 & \textbf{malt} \\
 & A & , & B & \geq & 0 &
\end{array}
$$

$A^* = 12$
$B^* = 28$
$OPT = 800$

Brewer to Analyst: "Can you prove to me that I can't make more than $800?"

Analyst:

Brewer's problem. Find optimal mix of beer and ale to maximize profits.

maximize	13A	+	23B			
subject to the constraints	5A	+	15B	≤	480	**corn**
	4A	+	4B	≤	160	**hops**
	35A	+	20B	≤	1190	**malt**
	A	,	B	≥	0	

$A^* = 12$
$B^* = 28$
$OPT = 800$

Brewer to Analyst: "Can you prove to me that I can't make more than $800?"

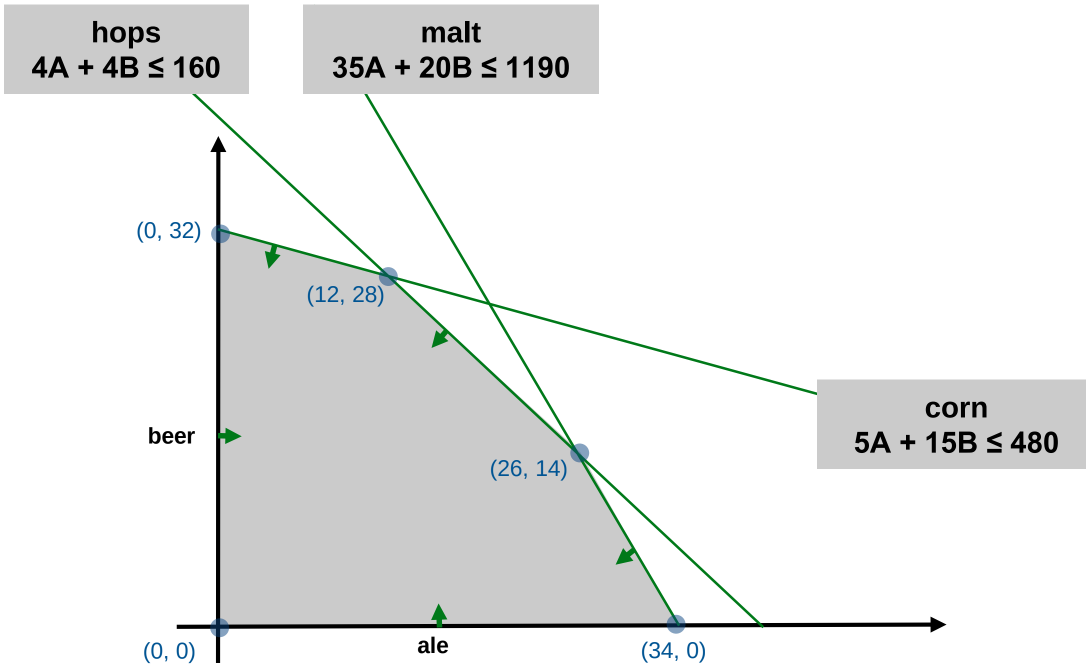
Analyst:

	13A	+	23B		
	5A	+	15B	≤	480
2 x	4A	+	4B	≤	160
	8 A	+	8 B	≤	320
1+3	13 A	+	23 B	≤	800

Brewer: "Amazing, how did you do it?"

Brewer's problem: feasible region

Inequalities define halfplanes; feasible region is a convex polygon.



Brewer's problem. Find optimal mix of beer and ale to maximize profits.

$$
\begin{array}{lrcrcrl}
\text{maximize} & 13A & + & 23B & & & \\
\text{subject} & 5A & + & 15B & \leq & 480 & \textbf{corn} \\
\text{to the} & 4A & + & 4B & \leq & 160 & \textbf{hops} \\
\text{constraints} & 35A & + & 20B & \leq & 1190 & \textbf{malt} \\
 & A & , & B & \geq & 0 &
\end{array}
$$

$A^* = 12$
$B^* = 28$
$OPT = 800$

Analyst: "Can I prove that Brewer can't make more than $800?"

Analyst:

Analyst's problem: give the best estimate on profits

Brewer's problem. Find optimal mix of beer and ale to maximize profits.

maximize	13A	+	23B				
subject	5A	+	15B	≤	480	**corn**	$A^* = 12$
to the	4A	+	4B	≤	160	**hops**	$B^* = 28$
constraints	35A	+	20B	≤	1190	**malt**	OPT = 800
	A	,	B	≥	0		

Analyst: "Can I prove that Brewer can't make more than $800?"

Analyst:

	13A	+	23B		
C x	5A	+	15B	≤	480
H x	4A	+	4B	≤	160
M x	35A	+	20B	≤	1190
	(5C+4H+35M) A	+	(15 C+ 4 H + 20 M) B	≤	480C + 160H + 1190 M

Analyst's problem

	13A	+	23B		
C x	5 A	+	15 B	≤	480
H x	4A	+	4B	≤	160
M x	35A	+	20B	≤	1190
	(5 C+4 H+35 M) A	+	(15 C+ 4 H + 20 M) B	≤	480C + 160H + 1190M

minimize	480 C	+	160 H	+	1190 M		
subject to the constraints	5 C	+	4 H	+	35 M	≥	13
	15 C	+	4 H	+	20 M	≥	23
	C	,	H	,	M	≥	0

C* = 1
H* = 2
M*=0
OPT = 800

- Self-certifying to be optimal!
- Not coincidental.

Strong LP duality theorem

Goal. Given a matrix A and vectors b and c, find vectors x and y that solve:

primal problem (P)		dual problem (D)	
maximize	$c^T x$	minimize	$b^T y$
subject to the constraints	$A x \leq b$	subject to the constraints	$A^T y \geq c$
	$x \geq 0$		$y \geq 0$

Proposition. If (P) and (D) have feasible solutions, then max = min.

LP duality: sensitivity analysis

Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?

Analyst's problem

	13A	+	23B		
C x	5 A	+	15 B	≤	480
H x	4A	+	4B	≤	160
M x	35A	+	20B	≤	1190
	(5 C+4 H+35 M) A	+	(15 C+ 4 H + 20 M) B	≤	480C + 160H + 1190M

minimize	480 C	+	160 H	+	1190 M			
subject to the constraints	5 C	+	4 H	+	35 M	≥	13	
	15 C	+	4 H	+	20 M	≥	23	
	C	,	H	,	M	≥	0	

C* = 1
H* = 2
M*=0
OPT = 800

- Self-certifying to be optimal!
- Not coincidental.

LP duality: sensitivity analysis

Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?

A. corn $1, hops $2, malt $0.

Q. Suppose a new product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?

Analyst's problem

	$13A$	$+$	$23B$		
$C\ \times$	$5A$	$+$	$15B$	$\le$	480
$H\ \times$	$4A$	$+$	$4B$	$\le$	160
$M\ \times$	$35A$	$+$	$20B$	$\le$	1190
	$(5C+4H+35M)\,A$	$+$	$(15C+4H+20M)\,B$	$\le$	$480C + 160H + 1190M$

$$
\begin{array}{lccccccccc}
\text{minimize} & 480\,C & + & 160\,H & + & 1190\,M & & \\
\text{subject to the constraints} & 5\,C & + & 4\,H & + & 35\,M & \ge & 13 \\
 & 15\,C & + & 4\,H & + & 20\,M & \ge & 23 \\
 & C & , & H & , & M & \ge & 0
\end{array}
$$

$C^* = 1$
$H^* = 2$
$M^* = 0$
$OPT = 800$

- Self-certifying to be optimal!
- Not coincidental.

LP duality: sensitivity analysis

Q. How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?

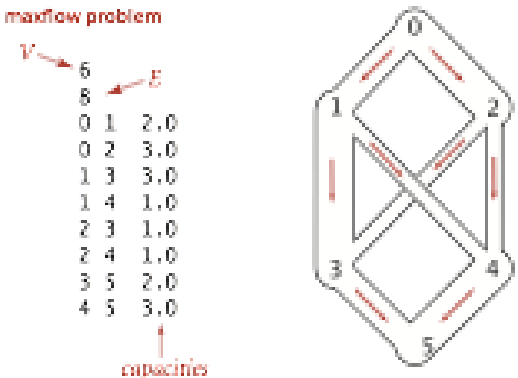
A. corn $1, hops $2, malt $0.

Q. Suppose a new product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?

A. At least 2 ($1) + 5 ($2) + 24 ($0) = $12 / barrel.

Modeling the maxflow problem as a linear program

Variables. x_{vw} = flow on edge $v \to w$.
Constraints. Capacity and flow conservation.
Objective function. Net flow into t.
Dual?



LP formulation

Maximize $x_{35} + x_{45}$
subject to the constraints

capacity constraints:

$$0 \le x_{01} \le 2$$
$$0 \le x_{02} \le 3$$
$$0 \le x_{13} \le 3$$
$$0 \le x_{14} \le 1$$
$$0 \le x_{23} \le 1$$
$$0 \le x_{24} \le 1$$
$$0 \le x_{35} \le 2$$
$$0 \le x_{45} \le 3$$

flow conservation constraints:

$$x_{01} = x_{13} + x_{14}$$
$$x_{02} = x_{23} + x_{24}$$
$$x_{13} + x_{23} = x_{35}$$
$$x_{14} + x_{24} = x_{45}$$

Shortest path as a linear program?

Maximize d(t) subject to

(1) $d(s) = 0$

(2) For each edge $u \to v$, $d(v) - d(u) \leq w(u \to v)$

Maximum cardinality bipartite matching problem

Input. Bipartite graph.

Goal. Find a matching of maximum cardinality.

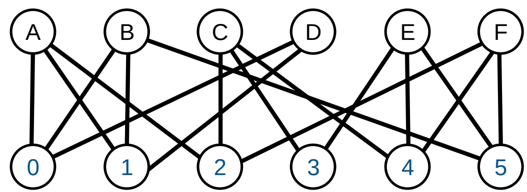
set of edges with no vertex appearing twice

Interpretation. Mutual preference constraints.

- People to jobs.
- Students to writing seminars.

Alice Adobe, Apple, Google	Adobe Alice, Bob, Dave
Bob Adobe, Apple, Yahoo	Apple Alice, Bob, Dave
Carol Google, IBM, Sun	Google Alice, Carol, Frank
Dave Adobe, Apple	IBM Carol, Eliza
Eliza IBM, Sun, Yahoo	Sun Carol, Eliza, Frank
Frank Google, Sun, Yahoo	Yahoo Bob, Eliza, Frank

Example: job offers



matching of cardinality 6:
A–1, B–5, C–2, D–0, E–3, F–4

Maximum cardinality bipartite matching problem

LP formulation. One variable per pair.

Interpretation. $x_{ij} = 1$ if person i assigned to job j.

maximize

$$x_{A0} + x_{A1} + x_{A2} + x_{B0} + x_{B1} + x_{B5} + x_{C2} + x_{C3} + x_{C4} + x_{D0} + x_{D1} + x_{E3} + x_{E4} + x_{E5} + x_{F2} + x_{F4} + x_{F5}$$

subject to the constraints

at most one job per person		at most one person per job	
$x_{A0} + x_{A1} + x_{A2}$	≤ 1	$x_{A0} + x_{B0} + x_{D0}$	≤ 1
$x_{B0} + x_{B1} + x_{B5}$	≤ 1	$x_{A1} + x_{B1} + x_{D1}$	≤ 1
$x_{C2} + x_{C3} + x_{C4}$	≤ 1	$x_{A2} + x_{C2} + x_{F2}$	≤ 1
$x_{D0} + x_{D1}$	≤ 1	$x_{C3} + x_{E3}$	≤ 1
$x_{E3} + x_{E4} + x_{E5}$	≤ 1	$x_{C4} + x_{E4} + x_{F4}$	≤ 1
$x_{F2} + x_{F4} + x_{F5}$	≤ 1	$x_{B5} + x_{E5} + x_{F5}$	≤ 1

all $x_{ij} \geq 0$

Theorem. [Birkhoff 1946, von Neumann 1953]
All extreme points of the above polyhedron have integer (0 or 1) coordinates.

Corollary. Can solve matching problem by solving LP. ← not usually so lucky!

Connection to game theory

- Zero-sum games: games where total payoff is zero.
- Typically, best strategies are *mixed:* probability distributions.
- Knowing the opponent's strategy gives you an advantage.

			
	0 / 0	+1/-1	-1/+1
	-1/+1	0 / 0	+1/-1
	+1/-1	-1/+1	0 / 0

Connection to game theory

- Zero-sum games: games where total payoff is zero.
- Typically, best strategies are *mixed:* probability distributions.
- Knowing the opponent’s strategy gives you an advantage.
- Or does it?

Expected payoff		 1/2	 1/4	 1/4
0		0 / 0	+1/-1	-1/+1
-1/4		-1/+1	0 / 0	+1/-1
+1/4		+1/-1	-1/+1	0 / 0

Looking for the best strategy

- Variables R, S, P ≥ 0
- Constraint: R+S+P=1
- Minimize: max(S-P,P-R,R-S)

- Not an LP?
- Easily becomes one!
- Minimize V
- Additional constraints:
- V $\geq$ S-P
- V $\geq$ P-R
- V $\geq$ R-S

Solution: R*=S*=P*=$\frac{1}{3}$
V*=0

	R 	S 	P 
S-P 	0 / 0	+1/-1	-1/+1
P-R 	-1/+1	0 / 0	+1/-1
R-S 	+1/-1	-1/+1	0 / 0

Von Neumann's MiniMax

- Knowing opponent's distribution is not helpful in zero-sum games!
- If for each distribution of opponent's moves I have a strategy that achieves a payoff V, then there is a distribution p* of my strategies that achieves payoff at least V *for all* moves of the opponent.
- I can guarantee myself a payoff of V.
- Let V* be the highest payoff I can guarantee myself.

Proof via LP duality!

- **Claim: opponent can guarantee itself a payoff of –V*.**
- If not, then for each strategy of the opponent, I have a strategy that pays opponent less than –V* (and therefore pays me more than V*)
- Then I must have one strategy that pays me more than V* by the Minimax Theorem.
- There is a pair of strategies that guarantees the best possible (V*,-V*) payoff – called an *equilibrium.* **V* in this case is the *value of the game.***

LP wrap-up

- Powerful generic tool for obtaining efficient algorithms.
- Typically not as good as purpose-designed algorithms (though this might change).
- Duality is an important general concept.
- Extended to *convex optimization:* optimize over a convex domain that is not necessarily a polytope
- In practice: a testing ground for *non-convex* optimization which is important in Machine Learning and other areas
- Major limitation: only works over continuous variables!
- Integer Programming (LP with integer variables) is NP-complete.
- Sometimes: lucky, as with matchings
- More often: solve LP then round solution – a major area of research in approximation algorithms