SIMPLY TYPED LAMBDA CALCULUS

SYNTAX

$$e ::= x \mid \lambda x:\tau.e \mid e_1 e_2 \mid \text{true} \mid \text{false} \mid \text{if } e_1 \text{ then } e_2 \text{ else } e_3$$

$$\tau ::= \text{Bool} \mid \tau_1 \to \tau_2$$

Values

$$v ::= \lambda x:\tau.e \mid \text{true} \mid \text{false}$$

Substitution

$$x[e/x] = e$$

$$y[e/x] = y \qquad y \neq x$$

$$(\lambda x:\tau.e_1)[e/x] = \lambda x:\tau.e_1$$

$$(\lambda y:\tau.e_1)[e/x] = \lambda y:\tau.(e_1[e/x]) \qquad y \neq x \text{ and } y \notin fv(e)$$

$$(\lambda y:\tau.e_1)[e/x] = \lambda z:\tau.(e_1[z/y][e/x] \qquad y \neq x \text{ and } z \notin fv(e, e_1)$$

$$\text{true}[e/x] = \text{true}$$

$$\text{false}[e/x] = \text{false}$$

$$\text{if } e_1 \text{ then } e_2 \text{ else } e_3\ [e/x] = \text{if } e_1[e/x] \text{ then } e_2[e/x] \text{ else } e_3[e/x]$$

SMALL STEP OPERATIONAL SEMANTICS

$$\frac{\text{value } v_2}{(\lambda x:\tau.e_1)\, v_2 \mapsto e_1[v_2/x]} \qquad \frac{e_1 \mapsto e_1'}{e_1 e_2 \mapsto e_1' e_2} \qquad \frac{\text{value } v_1 \quad e_2 \mapsto e_2'}{v_1 e_2 \mapsto v_1 e_2'}$$

$$\frac{}{\text{if true then } e_2 \text{ else } e_3 \mapsto e_2} \qquad \frac{e_1 \mapsto e_1'}{\text{if } e_1 \text{ then } e_2 \text{ else } e_3 \mapsto \text{if } e_1' \text{ then } e_2 \text{ else } e_3}$$

$$\frac{}{\text{if false then } e_2 \text{ else } e_3 \mapsto e_3}$$

TYPE SYSTEM

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x:\tau} \qquad \frac{\Gamma, x:\tau_1 \vdash \tau_2}{\Gamma \vdash (\lambda x:\tau_1.\, e):\tau_2} \qquad \frac{\Gamma \vdash e_1:\tau_1 \to \tau_2 \quad \Gamma \vdash e_2:\tau_1}{\Gamma \vdash e_1 e_2:\tau_2}$$

$$\frac{}{\Gamma \vdash \text{true}:\text{Bool}} \qquad \frac{}{\Gamma \vdash \text{false}:\text{Bool}} \qquad \frac{\Gamma \vdash e_1:\text{Bool} \quad \Gamma \vdash e_2:\tau \quad \Gamma \vdash e_3:\tau}{\Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : \tau}$$

FREE VARIABLES

$$\frac{}{x \in fv(x)} \qquad \frac{x \in fv(e_1)}{x \in fv(e_1 e_2)} \qquad \frac{x \in fv(e_2)}{x \in fv(e_1 e_2)} \qquad \frac{x \in fv(e) \quad x \neq y}{x \in fv(\lambda y:\tau.e)} \qquad \frac{x \in fv(e_1)}{x \in fv(\text{if } e_1 \text{ then} \ldots)}$$

etc.

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CANONICAL FORMS

$\vdash e : \mathrm{Bool} \to \mathrm{value}\ e \to e \equiv \mathrm{true} \vee e \equiv \mathrm{false}$

$\vdash e : \tau_1 \to \tau_2 \to \mathrm{value}\ e \to \exists x, e'.\ e \equiv \lambda x{:}\tau_1 .\, e'$

PROGRESS

$\vdash e : \tau \to \mathrm{value}\ e \vee \exists e'.\ e \mapsto e'$

by induction on $\vdash e : \tau$

CLOSED

e closed $:= fv(e) = \{\}$ (that is, $\exists x.\ x \in fv(e)$)

FREE IN CONTEXT:

$x \in fv(e) \to \Gamma \vdash e : \tau \to \exists \tau'.\ \Gamma(x) = \tau'$

by induction on $x \in fv(e)$

CONTEXT INVARIANCE:

$\Gamma \vdash e : \tau \to \left(\forall x.\ x \in fv(e) \to \Gamma(x) = \Gamma'(x)\right) \to \Gamma' \vdash e : \tau$

by induction on $\Gamma \vdash e : \tau$

SUBSTITUTION PRESERVES TYPING

$\Gamma, x : \tau' \vdash e : \tau \to \vdash v : \tau' \to \Gamma \vdash e[v/x] : \tau$

by induction on e

PRESERVATION

$\vdash e : \tau \to e \mapsto e' \to \vdash e' : \tau$

by induction on $\vdash e : \tau$

SAFETY

$\mathrm{safe}\ e := \forall e'.\ e \mapsto^* e' \to \mathrm{value}\ e' \vee \exists e''.\ e' \mapsto e''$

SOUNDNESS (of type system)

$\vdash e : \tau \to \mathrm{safe}\ e$