Algorithms — Robert Sedgewick | Kevin Wayne

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2.4 Priority Queues

- API and elementary implementations
- binary heaps
- heapsort
- ~~event-driven simulation~~

See video/book/booksite

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2.4 Priority Queues

- API and elementary implementations
- binary heaps
- heapsort
- event-driven simulation

Collections

A collection is a data type that stores a group of items.

data type	core operations	data structure
stack	Push, Pop	*linked list, resizing array*
queue	Enqueue, Dequeue	*linked list, resizing array*
priority queue	insert, Delete-Max	*binary heap* ←
symbol table	Put, Get, Delete	*binary search tree, hash table*
set	Add, Contains, Delete	*binary search tree, hash table*

Priority queue

Collections. Insert and delete items. Which item to delete?

Stack. Remove the item most recently added.
Queue. Remove the item least recently added.
Randomized queue. Remove a random item.

Priority queue. Remove the largest (or smallest) item.
Generalizes: stack, queue, randomized queue.

operation	*argument*	*return value*
insert	P	
insert	Q	
insert	E	
remove max		Q
insert	X	
insert	A	
insert	M	
remove max		X
insert	P	
insert	L	
insert	E	
remove max		P

Priority queue API

Requirement. Items are generic; they must also be `Comparable`.

Key must be Comparable (bounded type parameter)

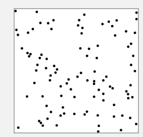
`public class MaxPQ<Key extends Comparable<Key>>`

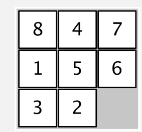
	`MaxPQ()`	*create an empty priority queue*
	`MaxPQ(Key[] a)`	*create a priority queue with given keys*
`void`	`insert(Key v)`	*insert a key into the priority queue*
`Key`	`delMax()`	*return and remove a largest key*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`Key`	`max()`	*return a largest key*
`int`	`size()`	*number of entries in the priority queue*

Note. Duplicate keys allowed; `delMax()` picks any maximum key.

Priority queue: applications

- Event-driven simulation. [customers in a line, colliding particles]
- Numerical computation. [reducing roundoff error]
- Discrete optimization. [bin packing, scheduling]
- Artificial intelligence. [A* search]
- Computer networks. [web cache]
- Operating systems. [load balancing, interrupt handling]
- Data compression. [Huffman codes]
- Graph searching. [Dijkstra's algorithm, Prim's algorithm]
- Number theory. [sum of powers]
- Spam filtering. [Bayesian spam filter]
- Statistics. [online median in data stream]





Priority queue: client example

Challenge. Find the largest M items in a stream of N items.

- Fraud detection: isolate $$ transactions.
- NSA monitoring: flag most suspicious documents.

N huge, M large

Constraint. Not enough memory to store N items.

Q. Would you use a MaxPQ or a MinPQ?

Transaction data type is Comparable (ordered by $$)

```
MinPQ<Transaction> pq = new MinPQ<Transaction>();

while (StdIn.hasNextLine())
{
   String line = StdIn.readLine();
   Transaction transaction = new Transaction(line);
   pq.insert(transaction);
   if (pq.size() > M)
      pq.delMin();
}
```

pq now contains largest M items

Priority queue: unordered and ordered array implementation

operation	*argument*	*return value*	*size*	*contents (unordered)*	*contents (ordered)*
insert	P		1	P	P
insert	Q		2	P Q	P Q
insert	E		3	P Q E	E P Q
remove max		Q	2	P E	E P
insert	X		3	P E X	E P X
insert	A		4	P E X A	A E P X
insert	M		5	P E X A M	A E M P X
remove max		X	4	P E M A	A E M P
insert	P		5	P E M A P	A E M P P
insert	L		6	P E M A P L	A E L M P P
insert	E		7	P E M A P L E	A E E L M P P
remove max		P	6	E M A P L E	A E E L M P

A sequence of operations on a priority queue

Priority queue: implementations cost summary

Challenge. Implement all operations efficiently.

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
goal	$\log N$	$\log N$	$\log N$

order of growth of running time for priority queue with N items

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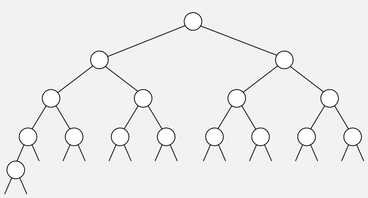
2.4 Priority Queues

- API and elementary implementations
- *binary heaps*
- heapsort
- event-driven simulation

Complete binary tree

Binary tree. Empty or node with links to left and right binary trees.

Complete tree. Perfectly balanced, except for bottom level.



complete binary tree with N = 16 nodes (height = 4)

Property. Height of complete binary tree with N nodes is $\lfloor \lg N \rfloor$.

A complete binary tree in nature

Binary heap: representation

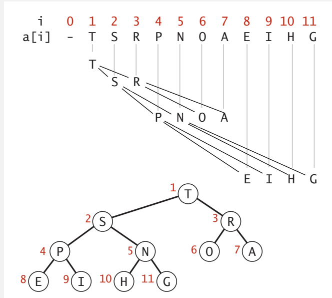
Binary heap. Array representation of a heap-ordered complete binary tree.

Heap-ordered binary tree.

- Keys in nodes.
- Parent's key no smaller than children's keys.

Array representation.

- Indices start at 1.
- Take nodes in level order.
- No explicit links needed!



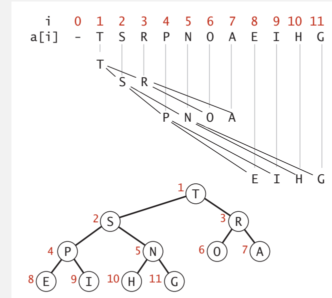
Heap representations

Binary heap: properties

Proposition. Largest key is `a[1]`, which is root of binary tree.

Proposition. Can use array indices to move through tree.

- Parent of node at `k` is at `k/2`.
- Children of node at `k` are at `2k` and `2k+1`.

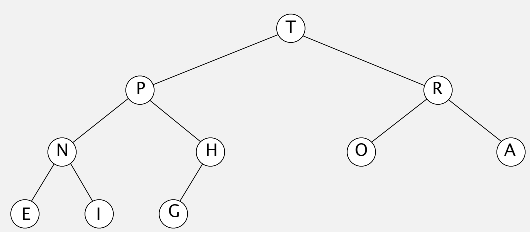


Heap representations

Binary heap demo

Insert. Add node at end, then swim it up.
Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



T	P	R	N	H	O	A	E	I	G	

Binary heap demo

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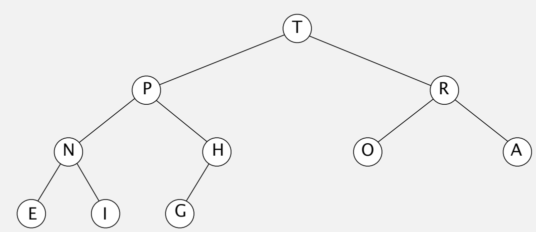

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Binary heap demo

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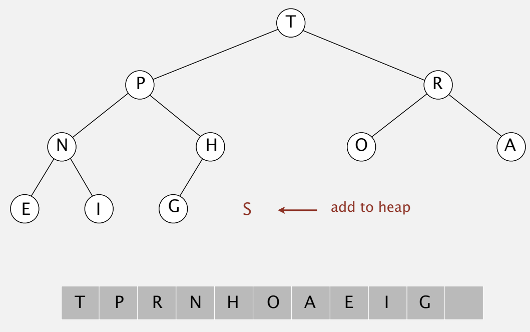
insert S



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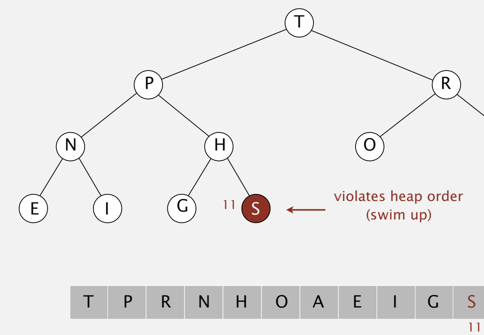

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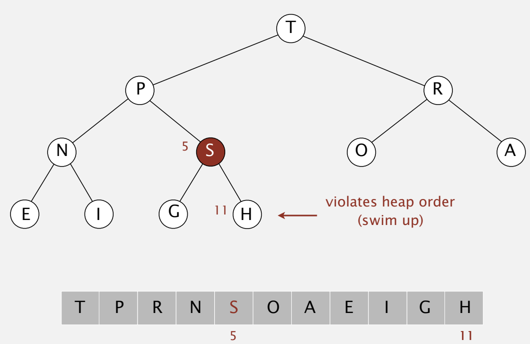

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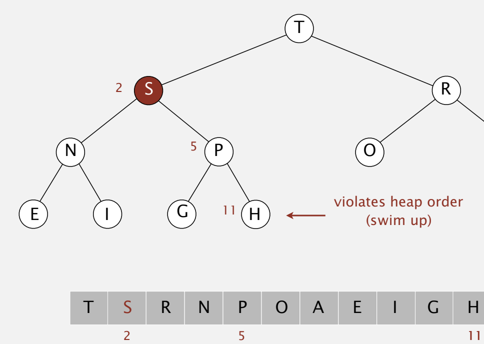

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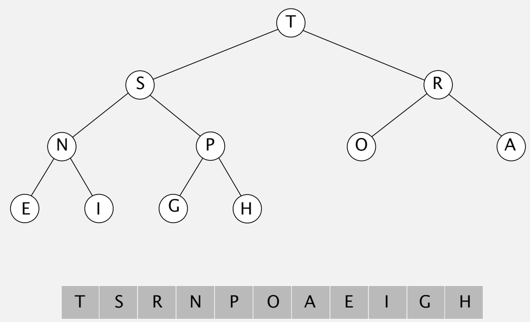
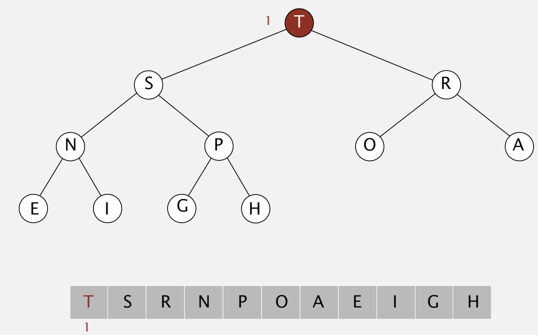


Binary heap demo

Insert. Add node at end, then swim it up.
Remove the maximum. Exchange root with node at end, then sink it down.

remove the maximum



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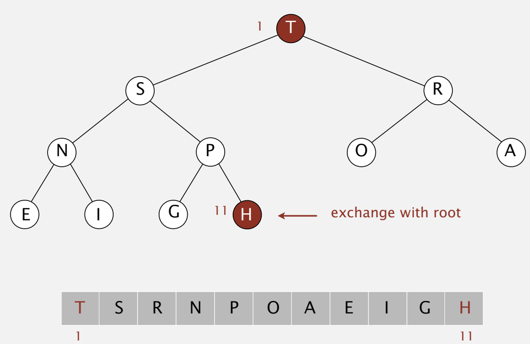

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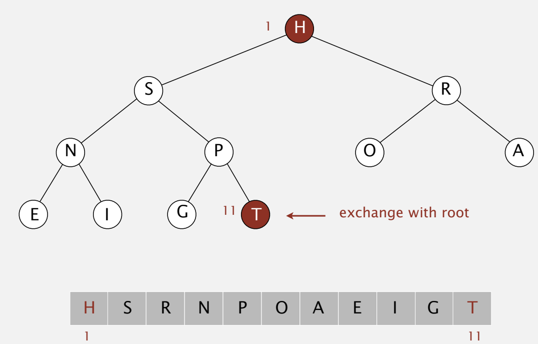

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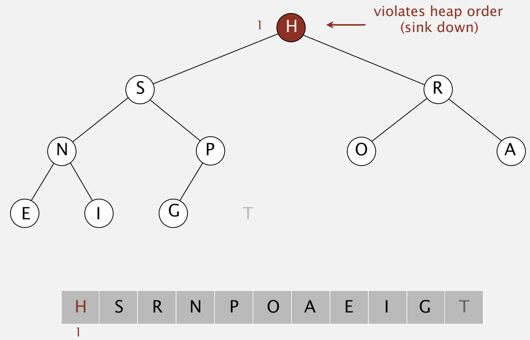

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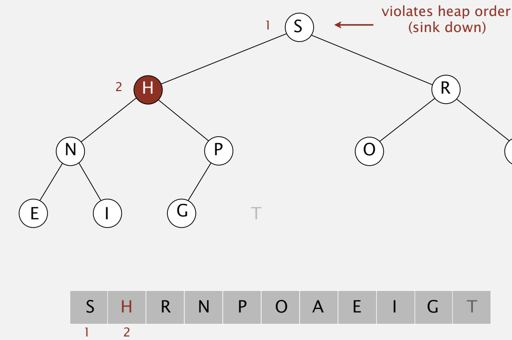

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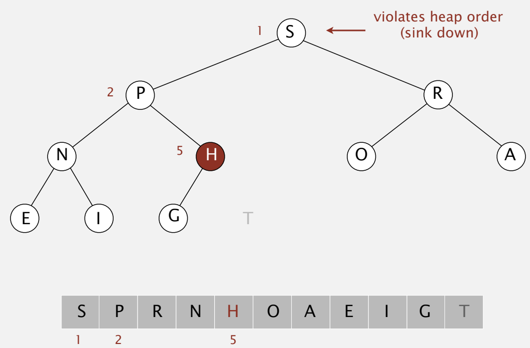
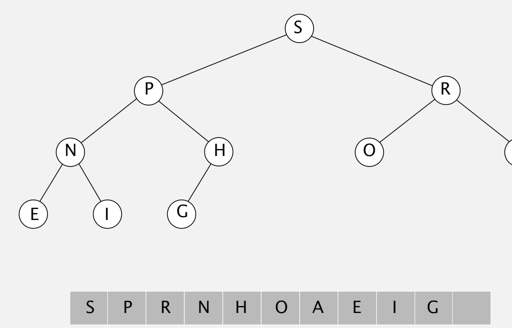


Binary heap demo

Insert. Add node at end, then swim it up.
Remove the maximum. Exchange root with node at end, then sink it down.

heap ordered



Binary heap: promotion

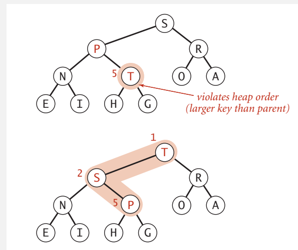
Scenario. A key becomes larger than its parent's key.

To eliminate the violation:

- Exchange key in child with key in parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

parent of node at k is at k/2

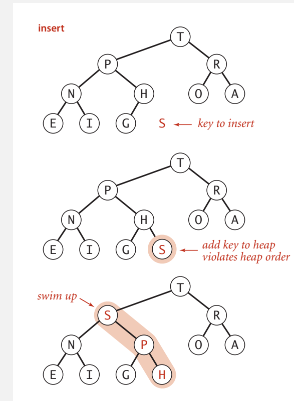


Peter principle. Node promoted to level of incompetence.

Binary heap: insertion

Insert. Add node at end, then swim it up.
Cost. At most $1 + \lg N$ compares.

```
public void insert(Key x)
{
   pq[++N] = x;
   swim(N);
}
```



Binary heap: demotion

Scenario. A key becomes smaller than one (or both) of its children's.

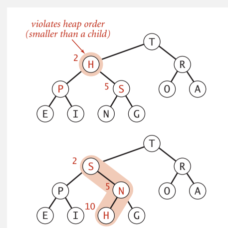
To eliminate the violation:

why not smaller child?

- Exchange key in parent with key in larger child.
- Repeat until heap order restored.

```
private void sink(int k)
{
   while (2*k <= N)
   {
      int j = 2*k;
      if (j < N && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

children of node at k are 2*k and 2*k+1



Top-down reheapify (sink)

Power struggle. Better subordinate promoted.

Binary heap: demotion

Q. Write a recursive version of sink

```
private void sink(int k)
{
   while (2*k <= N)
   {
      int j = 2*k;
      if (j < N && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

```
private void sink(int k)
{
   if (2*k > N)
     return;
   int j = 2*k;
   if (j < N && less(j, j+1)) j++;
   if (!less(k, j)) return;
   exch(k, j);
   sink(j);
}
```

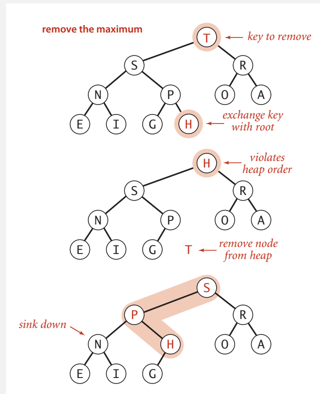
This is just an exercise. No particular reason to implement this recursively.

In fact, many compilers will *automatically* convert the recursive version to the iterative one. This is called tail-call elimination or tail-call optimization.

Binary heap: delete the maximum

Delete max. Exchange root with node at end, then sink it down.
Cost. At most $2 \lg N$ compares.

```
public Key delMax()
{
   Key max = pq[1];
   exch(1, N--);
   sink(1);
   pq[N+1] = null;      ← prevent loitering
   return max;
}
```



Binary heap: Java implementation

```
public class MaxPQ<Key extends Comparable<Key>>
{
   private Key[] pq;
   private int N;

   public MaxPQ(int capacity)                                  ← fixed capacity (for simplicity)
   {  pq = (Key[]) new Comparable[capacity+1];  }

   public boolean isEmpty()                                    ← PQ ops
   {  return N == 0;  }
   public void insert(Key key)  // see previous code
   public Key delMax()          // see previous code

   private void swim(int k)     // see previous code          ← heap helper functions
   private void sink(int k)     // see previous code

   private boolean less(int i, int j)                          ← array helper functions
   {  return pq[i].compareTo(pq[j]) < 0;  }
   private void exch(int i, int j)
   {  Key t = pq[i]; pq[i] = pq[j]; pq[j] = t;  }
}
```

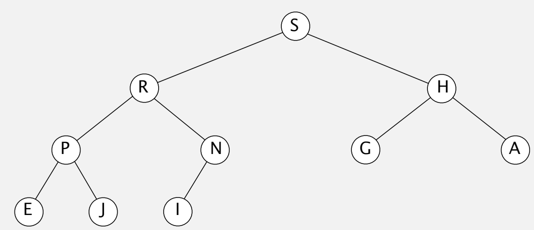

Priority queue: implementations cost summary

implementation	insert	del max	max
unordered array	1	N	N
ordered array	N	1	1
binary heap	$\log N$	$\log N$	1

order-of-growth of running time for priority queue with N items

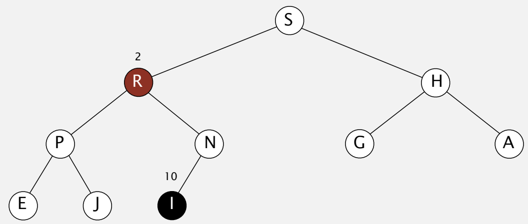
Delete-random from a binary heap

Problem. Delete a random key from a binary heap in logarithmic time.



Delete-random from a binary heap

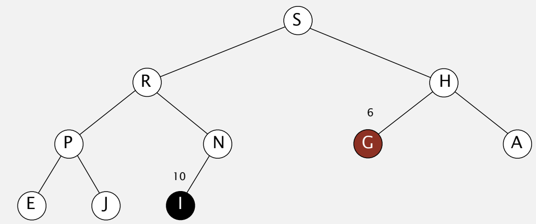
Problem. Delete a random key from a binary heap in logarithmic time.



Solution.
- Pick a random index `r` between `1` and `N`.
- Perform `exch(r, N--)`.
- Perform either `sink(r)` or `swim(r)`.

Delete-random from a binary heap

Problem. Delete a random key from a binary heap in logarithmic time.

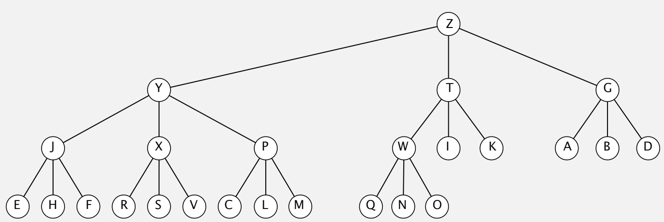


Solution.
- Pick a random index `r` between `1` and `N`.
- Perform `exch(r, N--)`.
- Perform either `sink(r)` or `swim(r)`.

Binary heap: practical improvements

Multiway heaps.
- Complete d-way tree.
- Parent's key no smaller than its children's keys.

Fact. Height of complete d-way tree on N nodes is $\sim \log_d N$.



3-way heap

Priority queue: implementation cost summary

implementation	insert	del max	max	
unordered array	1	N	N	
ordered array	N	1	1	
binary heap	$\log N$	$\log N$	1	
d-ary heap	$\log_d N$	$d \log_d N$	1	⟵ sweet spot: d = 4
Fibonacci	1	$\log N$ †	1	
Brodal queue	1	$\log N$	1	
impossible	1	1	1	⟵ why impossible?

† amortized

order-of-growth of running time for priority queue with N items

Binary heap: considerations

Underflow and overflow.

- Underflow: throw exception if deleting from empty PQ.
- Overflow: use resizing array.

Minimum-oriented priority queue.

- Replace `less()` with `greater()`.
- Implement `greater()`.

Other operations.

- Remove an arbitrary item.
- Change the priority of an item.

can implement efficiently with `sink()` and `swim()`

Immutability of keys.

- Assumption: client does not change keys while they're on the PQ.
- Best practice: use immutable keys.

Immutability: implementing in Java

Immutable data type. Can't change the data type value once created.

Examples: `String`, `Integer`, `Double`, `Color`, `Vector`, `Transaction`, `Point2D`.
Mutable: `StringBuilder`, `Stack`, `Counter`, Java array.

To create your own immutable data types:

- Make defensive copy of client-provided mutable variables in constructor
- Don't change instance variables in instance methods

Immutability: properties

Data type. Set of values and operations on those values.
Immutable data type. Can't change the data type value once created.

Advantages.

- Simplifies debugging.
- Simplifies concurrent programming.
- More secure in presence of hostile code.
- Safe to use as key in priority queue or symbol table.

Disadvantage. Must create new object for each data type value.

> *" Classes should be immutable unless there's a very good reason to make them mutable.... If a class cannot be made immutable, you should still limit its mutability as much as possible. "*
> *— Joshua Bloch (Java architect)*

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2.4 Priority Queues

- API and elementary implementations
- binary heaps
- *heapsort*
- event-driven simulation

Priority queues: quiz 4

Verify that this is a correct sorting algorithm. What are its properties?

```
public void sort(String[] a)
{
    int N = a.length;
    MaxPQ<String> pq = new MaxPQ<String>();
    for (int i = 0; i < N; i++)
        pq.insert(a[i]);
    for (int i = N-1; i >= 0; i--)
        a[i] = pq.delMax();
}
```

A. $N \log N$ compares in the worst case.

B. In-place.

C. Stable.

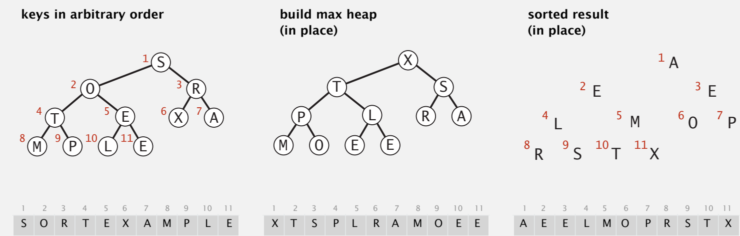
D. *All of the above.*

E. *I don't know.*

Heapsort

Basic plan for in-place sort.

- View input array as a complete binary tree.
- Heap construction: build a max-heap with all N keys.
- Sortdown: repeatedly remove the maximum key.

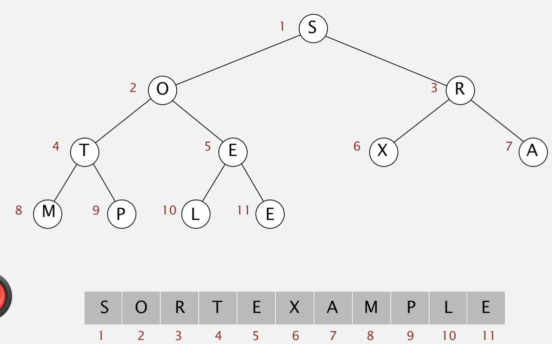


Heapsort demo

Heap construction. Build max heap using bottom-up method.

assume array entries are indexed 1 to N

array in arbitrary order



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Heap construction. Build max heap using bottom-up method.

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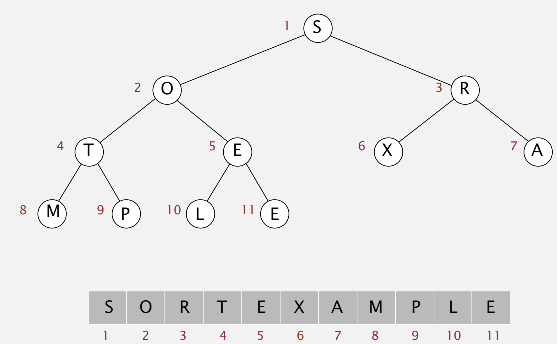

Heapsort demo

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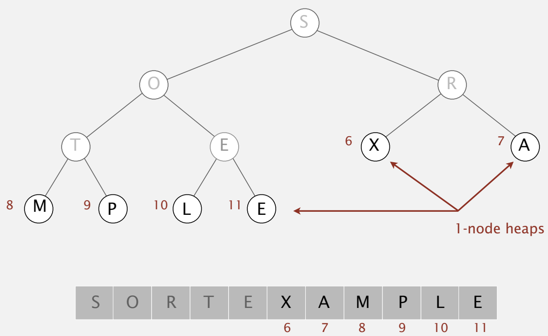
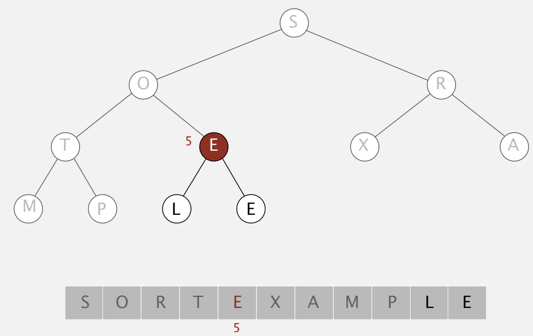


Heapsort demo

Heap construction. Build max heap using bottom-up method.

sink 5



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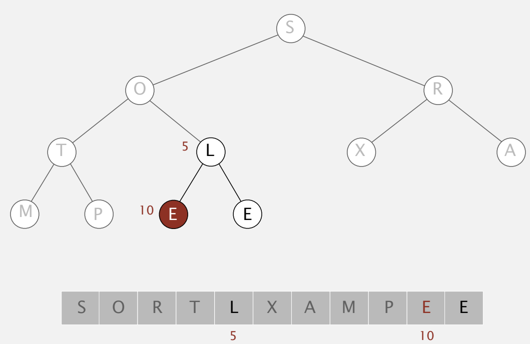
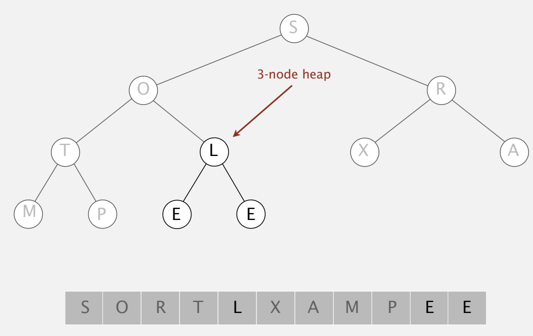


Heapsort demo

Heap construction. Build max heap using bottom-up method.

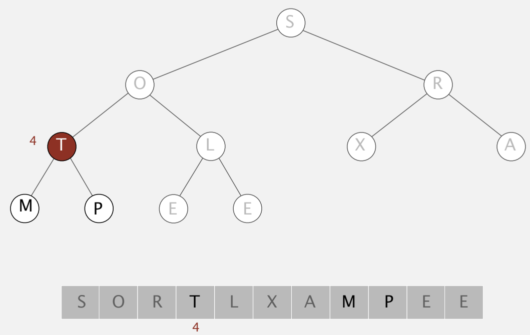
sink 5



Heapsort demo

Heap construction. Build max heap using bottom-up method.

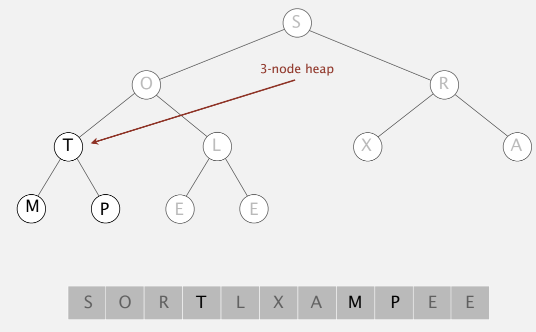
sink 4



Heapsort demo

Heap construction. Build max heap using bottom-up method.

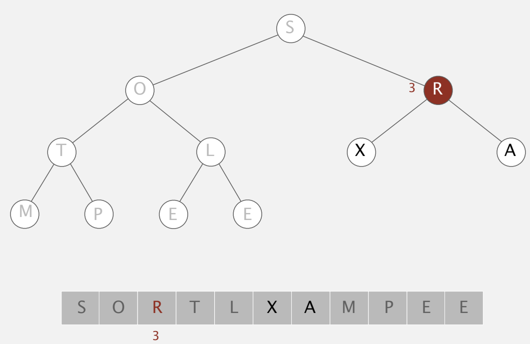
sink 4



Heapsort demo

Heap construction. Build max heap using bottom-up method.

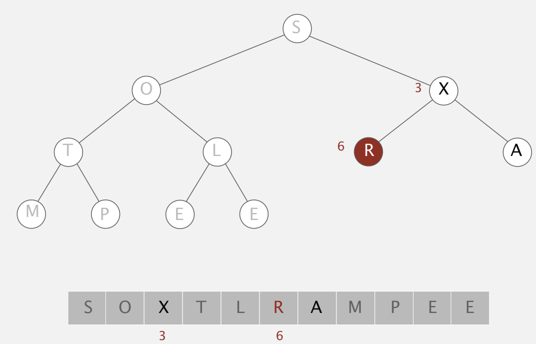
sink 3



Heapsort demo

Heap construction. Build max heap using bottom-up method.

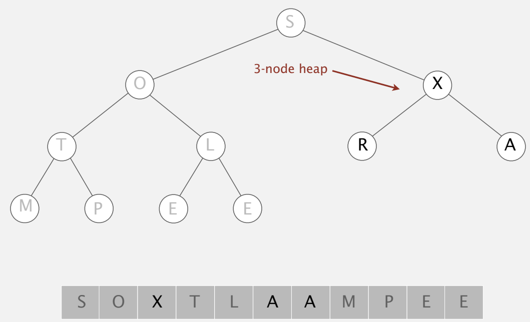
sink 3



Heapsort demo

Heap construction. Build max heap using bottom-up method.

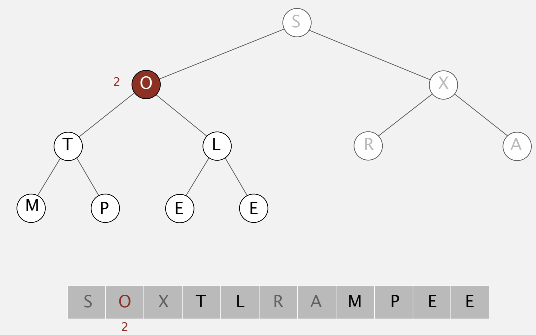
sink 3



Heapsort demo

Heap construction. Build max heap using bottom-up method.

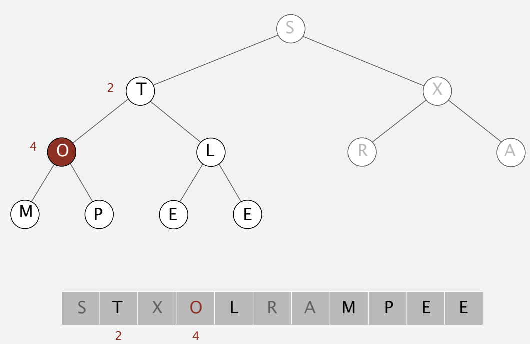
sink 2



Heapsort demo

Heap construction. Build max heap using bottom-up method.

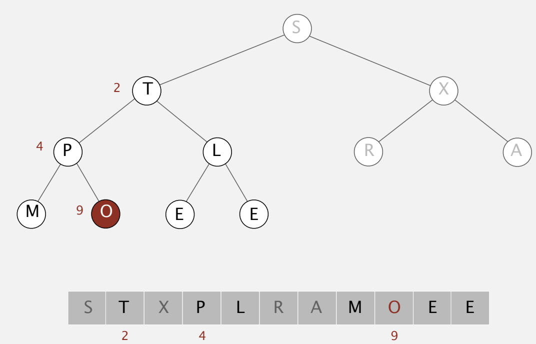
sink 2



Heapsort demo

Heap construction. Build max heap using bottom-up method.

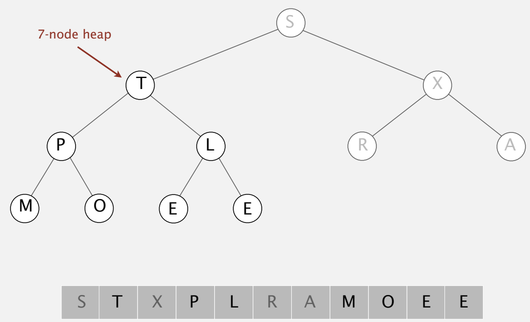
sink 2



Heapsort demo

Heap construction. Build max heap using bottom-up method.

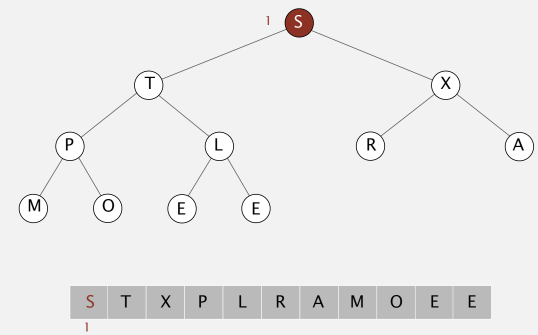
sink 2



Heapsort demo

Heap construction. Build max heap using bottom-up method.

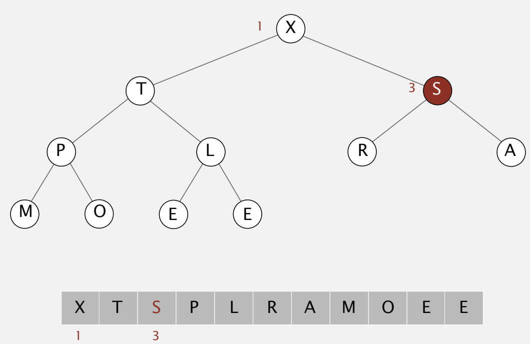
sink 1



Heapsort demo

Heap construction. Build max heap using bottom-up method.

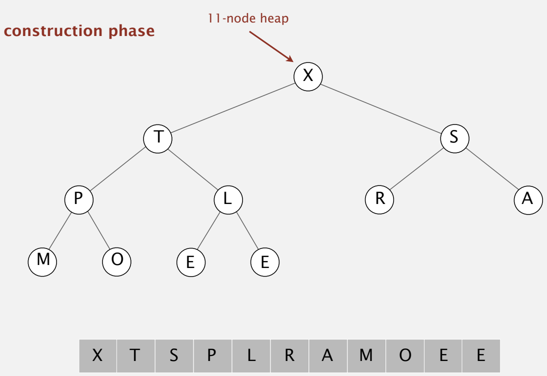
sink 1



Heapsort demo

Heap construction. Build max heap using bottom-up method.

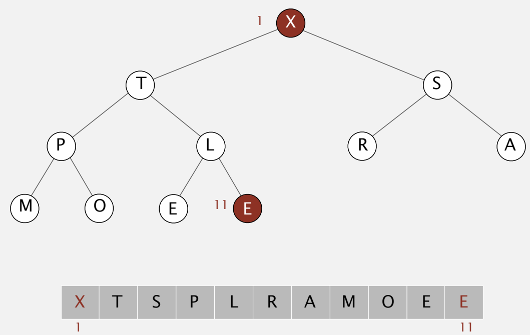
end of construction phase



Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

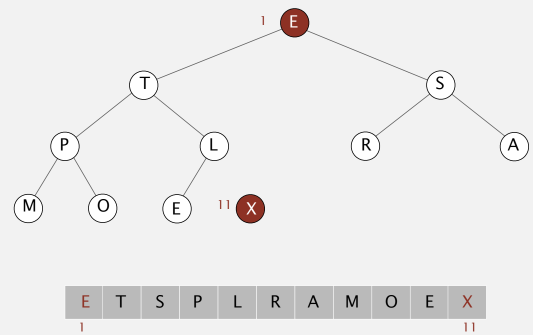
exchange 1 and 11



Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

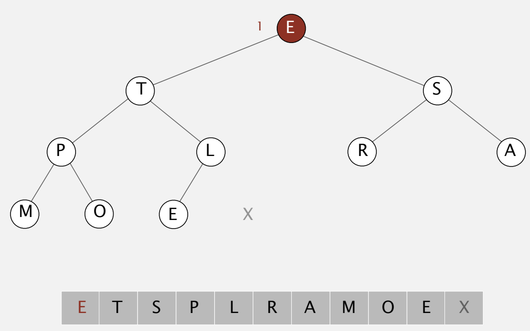
exchange 1 and 11



Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

sink 1



Heapsort demo

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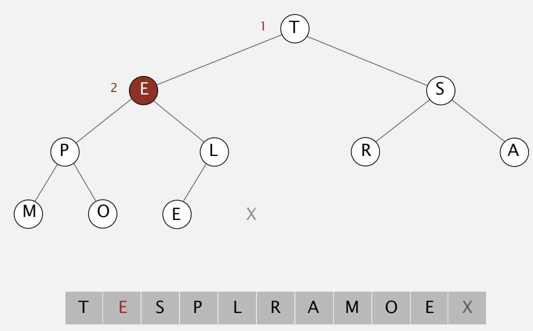
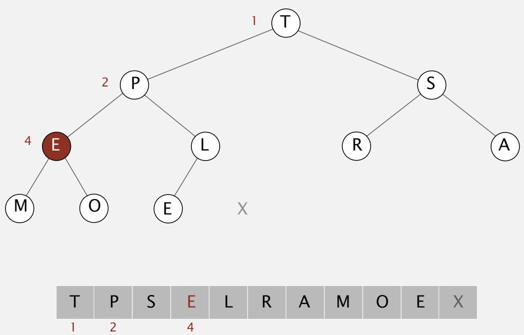


Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

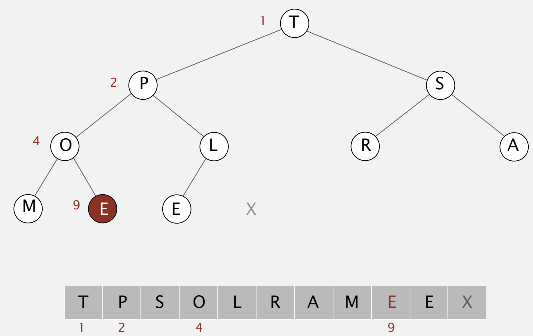
sink 1



Heapsort demo

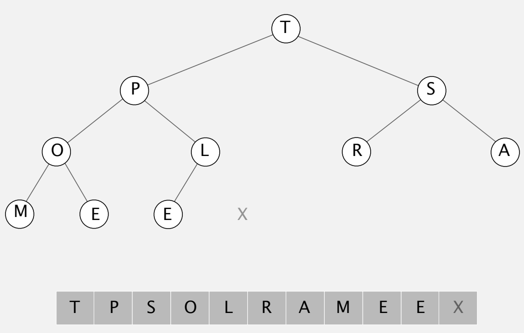
Sortdown. Repeatedly delete the largest remaining item.

sink 1



Heapsort demo

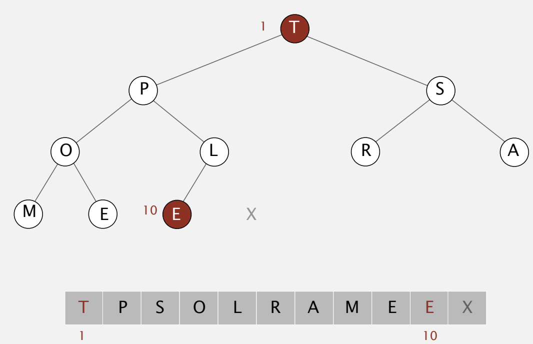
Sortdown. Repeatedly delete the largest remaining item.



Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

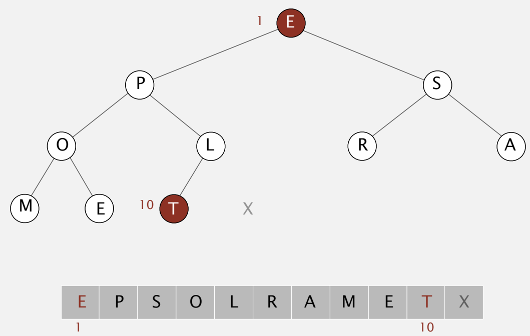
exchange 1 and 10



Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

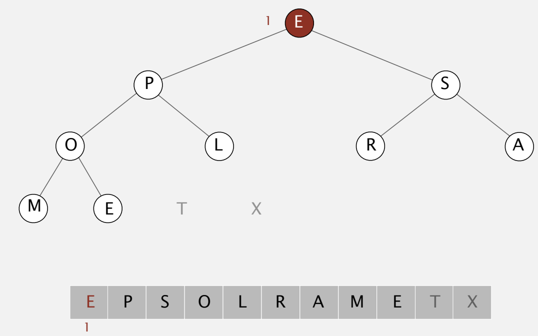
exchange 1 and 10



Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

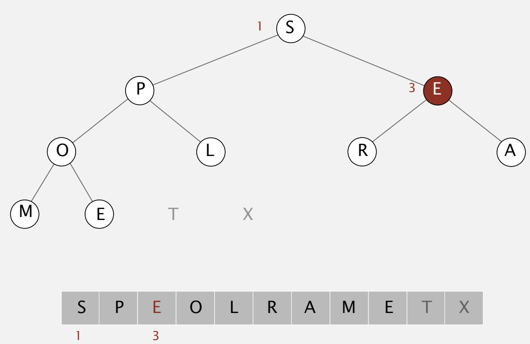
sink 1



Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

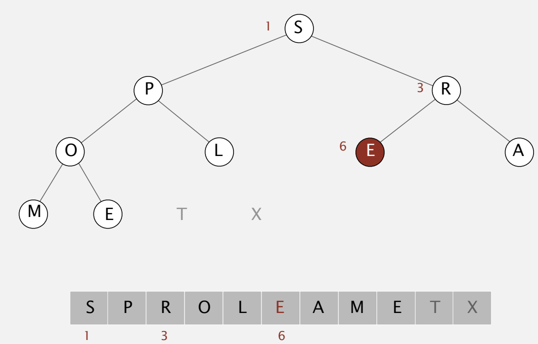
sink 1



Heapsort demo

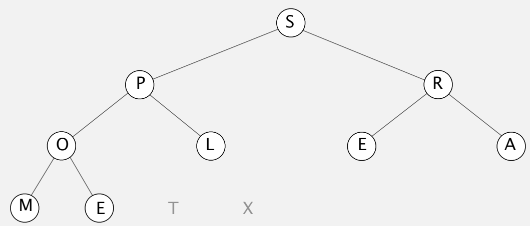
Sortdown. Repeatedly delete the largest remaining item.

sink 1



Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

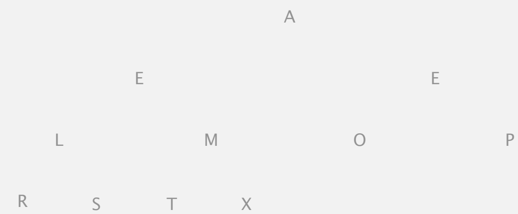


S	P	R	O	L	E	A	M	E	T	X

Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

array in sorted order

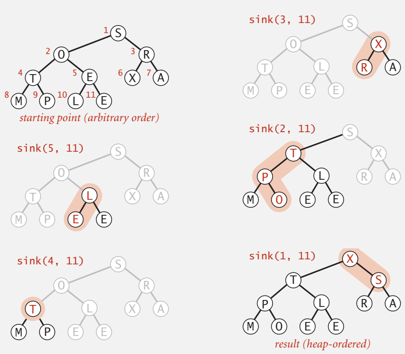


A	E	E	L	M	O	P	R	S	T	X
1	2	3	4	5	6	7	8	9	10	11

Heapsort: heap construction

First pass. Build heap using bottom-up method.

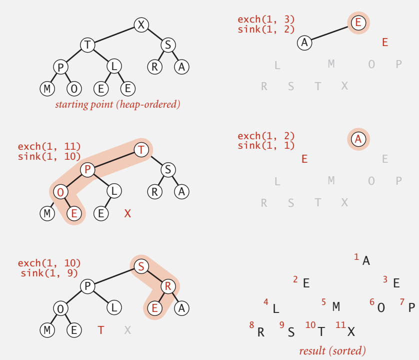
```
for (int k = N/2; k >= 1; k--)
   sink(a, k, N);
```



Heapsort: sortdown

Second pass.
- Remove the maximum, one at a time.
- Leave in array, instead of nulling out.

```
while (N > 1)
{
   exch(a, 1, N--);
   sink(a, 1, N);
}
```



Heapsort: Java implementation

```
public class Heap
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int k = N/2; k >= 1; k--)
         sink(a, k, N);
      while (N > 1)
      {
         exch(a, 1, N);
         sink(a, 1, --N);
      }
   }

   private static void sink(Comparable[] a, int k, int N)
   {  /* as before */  }

   private static boolean less(Comparable[] a, int i, int j)
   {  /* as before */  }

   private static void exch(Object[] a, int i, int j)
   {  /* as before */  }
}
```

but make static (and pass arguments)

but convert from 1-based indexing to 0-base indexing

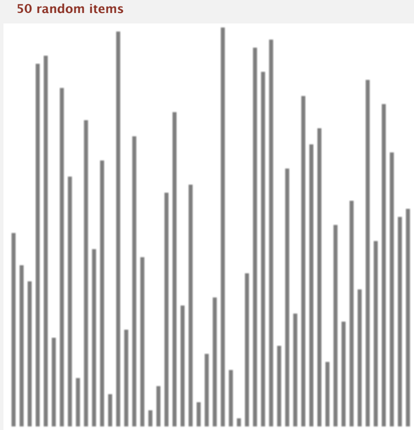
Heapsort: trace

N	k	0	1	2	3	4	5	6	7	8	9	10	11
initial values			S	O	R	T	E	X	A	M	P	L	E
11	5		S	O	R	T	L	X	A	M	P	E	E
11	4		S	O	R	T	L	X	A	M	P	E	E
11	3		S	O	X	T	L	R	A	M	P	E	E
11	2		S	T	X	P	L	R	A	M	O	E	E
11	1		X	T	S	P	L	R	A	M	O	E	E
heap-ordered			X	T	S	P	L	R	A	M	O	E	E
10	1		T	P	S	O	L	R	A	M	E	E	X
9	1		S	P	R	O	L	E	A	M	E	T	X
8	1		R	P	E	O	L	E	A	M	S	T	X
7	1		P	O	E	M	L	E	A	R	S	T	X
6	1		O	M	E	A	L	E	P	R	S	T	X
5	1		M	L	E	A	E	O	P	R	S	T	X
4	1		L	E	E	A	M	O	P	R	S	T	X
3	1		E	A	E	L	M	O	P	R	S	T	X
2	1		E	A	E	L	M	O	P	R	S	T	X
1	1		A	E	E	L	M	O	P	R	S	T	X
sorted result			A	E	E	L	M	O	P	R	S	T	X

Heapsort trace (array contents just after each sink)

Heapsort animation

50 random items



▲ algorithm position
in order
not in order

http://www.sorting-algorithms.com/heap-sort

Heapsort: mathematical analysis

Proposition. Heap construction uses $\leq 2N$ compares and $\leq N$ exchanges.

Proposition. Heapsort uses $\leq 2N \lg N$ compares and exchanges.

algorithm can be improved to ~ 1 N lg N (but no such variant is known to be practical)

Significance. Deterministic in-place $N \log N$ sorting algorithm.

- Mergesort: no, linear extra space. ← in-place merge possible, not practical
- Quicksort: no, randomized. ← deterministic N log N quicksort possible, not practical
- Heapsort: yes!

Bottom line. Heapsort is optimal for both time and space, but:

- Inner loop longer than quicksort's.
- Makes poor use of cache.
- Not stable.

can be improved using advanced caching tricks

Sorting algorithms: summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	N exchanges
insertion	✔	✔	N	$\frac{1}{4} N^2$	$\frac{1}{2} N^2$	use for small N or partially ordered
merge		✔	$\frac{1}{2} N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee; stable
timsort		✔	N	$N \lg N$	$N \lg N$	improves mergesort when preexisting order
quick	✔		$N \lg N$	$2 N \ln N$ (expected)	$\frac{1}{2} N^2$	$N \log N$ probabilistic guarantee; fastest in practice
3-way quick	✔		N	$2 N \ln N$ (expected)	$\frac{1}{2} N^2$	improves quicksort when duplicate keys
heap	✔		$3 N$	$2 N \lg N$	$2 N \lg N$	$N \log N$ guarantee; in-place
?	✔	✔	N	$N \lg N$	$N \lg N$	holy sorting grail

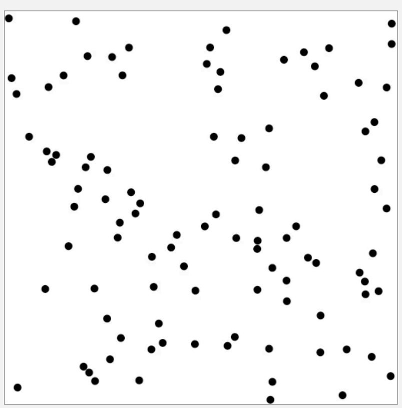
http://algs4.cs.princeton.edu

2.4 Priority Queues

- *API and elementary implementations*
- *binary heaps*
- *heapsort*
- *event-driven simulation*

Molecular dynamics simulation of hard discs

Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.



Molecular dynamics simulation of hard discs

Goal. Simulate the motion of N moving particles that behave according to the laws of elastic collision.

Hard disc model.

- Moving particles interact via elastic collisions with each other and walls.
- Each particle is a disc with known position, velocity, mass, and radius.
- No other forces.

temperature, pressure, diffusion constant

motion of individual atoms and molecules

Significance. Relates macroscopic observables to microscopic dynamics.

- Maxwell-Boltzmann: distribution of speeds as a function of temperature.
- Einstein: explain Brownian motion of pollen grains.

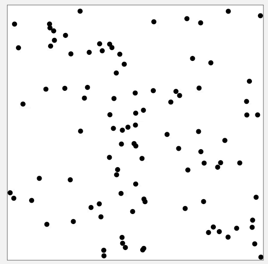
Warmup: bouncing balls

Time-driven simulation. N bouncing balls in the unit square.

```
public class BouncingBalls
{
   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      Ball[] balls = new Ball[N];
      for (int i = 0; i < N; i++)
         balls[i] = new Ball();
      while(true)
      {
         StdDraw.clear();
         for (int i = 0; i < N; i++)
         {
            balls[i].move(0.5);
            balls[i].draw();
         }
         StdDraw.show(50);
      }
   }
}
```

main simulation loop

```
% java BouncingBalls 100
```



Warmup: bouncing balls

```
public class Ball
{
    private double rx, ry;         // position
    private double vx, vy;         // velocity
    private final double radius;   // radius
    public Ball(...)
    {  /* initialize position and velocity */  }

    public void move(double dt)
    {
        if ((rx + vx*dt < radius) || (rx + vx*dt > 1.0 - radius)) { vx = -vx; }
        if ((ry + vy*dt < radius) || (ry + vy*dt > 1.0 - radius)) { vy = -vy; }
        rx = rx + vx*dt;
        ry = ry + vy*dt;
    }
    public void draw()
    {  StdDraw.filledCircle(rx, ry, radius);  }
}
```

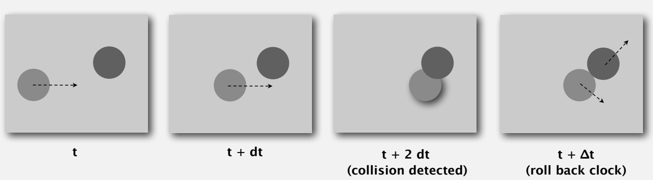
check for collision with walls

Missing. Check for balls colliding with each other.

- Physics problems: when? what effect?
- CS problems: which object does the check? too many checks?

Time-driven simulation

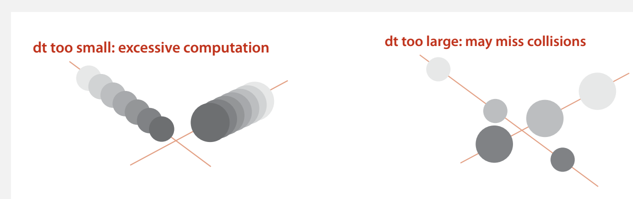
- Discretize time in quanta of size dt.
- Update the position of each particle after every dt units of time, and check for overlaps.
- If overlap, roll back the clock to the time of the collision, update the velocities of the colliding particles, and continue the simulation.



Time-driven simulation

Main drawbacks.

- ~ $N^2 / 2$ overlap checks per time quantum.
- Simulation is too slow if dt is very small.
- May miss collisions if dt is too large.
  (if colliding particles fail to overlap when we are looking)



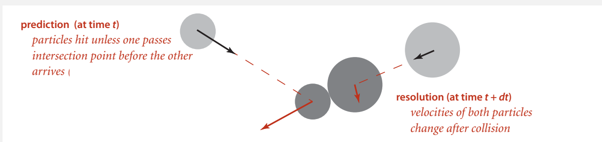
Event-driven simulation

Change state only when something interesting happens.

- Between collisions, particles move in straight-line trajectories.
- Focus only on times when collisions occur.
- Maintain PQ of collision events, prioritized by time.
- Delete min = get next collision.

Collision prediction. Given position, velocity, and radius of a particle, when will it collide next with a wall or another particle?

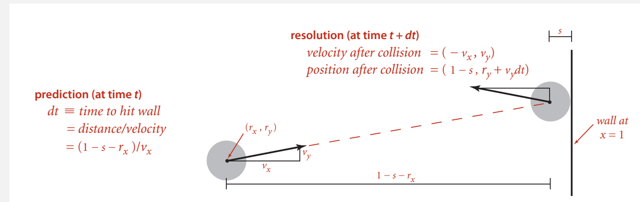
Collision resolution. If collision occurs, update colliding particle(s) according to laws of elastic collisions.



Particle-wall collision

Collision prediction and resolution.

- Particle of radius s at position (rx, ry).
- Particle moving in unit box with velocity (vx, vy).
- Will it collide with a vertical wall? If so, when?

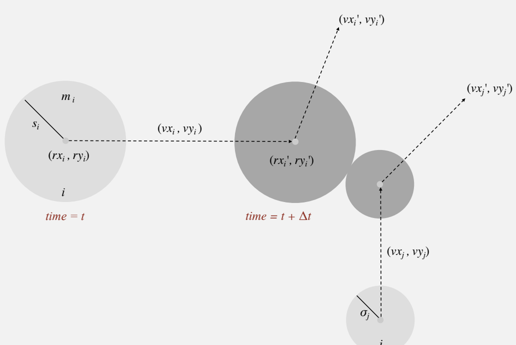


Predicting and resolving a particle-wall collision

Particle-particle collision prediction

Collision prediction.

- Particle i: radius s_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius s_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?



Particle-particle collision prediction

Collision prediction.

- Particle i: radius s_i, position (rx_i, ry_i), velocity (vx_i, vy_i).
- Particle j: radius s_j, position (rx_j, ry_j), velocity (vx_j, vy_j).
- Will particles i and j collide? If so, when?

$$\Delta t = \begin{cases} \infty & \text{if } \Delta v \cdot \Delta r \geq 0, \\ \infty & \text{if } d < 0, \\ -\dfrac{\Delta v \cdot \Delta r \; + \; \sqrt{d}}{\Delta v \cdot \Delta v} & \text{otherwise} \end{cases}$$

$$d \;=\; (\Delta v \cdot \Delta r)^2 \;-\; (\Delta v \cdot \Delta v)\,(\Delta r \cdot \Delta r \;-\; s^2), \quad s = s_i + s_j$$

$$\Delta v = (\Delta vx,\ \Delta vy) \;=\; (vx_i - vx_j,\ vy_i - vy_j)$$
$$\Delta r = (\Delta rx,\ \Delta ry) \;=\; (rx_i - rx_j,\ ry_i - ry_j)$$

$$\Delta v \cdot \Delta v = (\Delta vx)^2 + (\Delta vy)^2$$
$$\Delta r \cdot \Delta r = (\Delta rx)^2 + (\Delta ry)^2$$
$$\Delta v \cdot \Delta r = (\Delta vx)(\Delta rx) + (\Delta vy)(\Delta ry)$$

Important note: This is physics, so we won't be testing you on it!

Particle-particle collision resolution

Collision resolution. When two particles collide, how does velocity change?

$$vx_i' = vx_i + Jx / m_i$$
$$vy_i' = vy_i + Jy / m_i$$
$$vx_j' = vx_j - Jx / m_j$$
$$vy_i' = vy_j - Jy / m_j$$

Newton's second law (momentum form)

$$Jx = \frac{J\,\Delta rx}{s}, \quad Jy = \frac{J\,\Delta ry}{s}, \quad J = \frac{2\, m_i\, m_j\, (\Delta v \cdot \Delta r)}{s\,(m_i + m_j)}$$

impulse due to normal force
(conservation of energy, conservation of momentum)

Important note: This is physics, so we won't be testing you on it!

Particle data type skeleton

```
public class Particle
{
    private double rx, ry;       // position
    private double vx, vy;       // velocity
    private final double radius; // radius
    private final double mass;   // mass
    private int count;           // number of collisions

    public Particle( ... )      { ... }

    public void move(double dt) { ... }
    public void draw()          { ... }

    public double timeToHit(Particle that)  { }
    public double timeToHitVerticalWall()   { }
    public double timeToHitHorizontalWall() { }

    public void bounceOff(Particle that)    { }
    public void bounceOffVerticalWall()     { }
    public void bounceOffHorizontalWall()   { }

}
```

predict collision with particle or wall

resolve collision with particle or wall

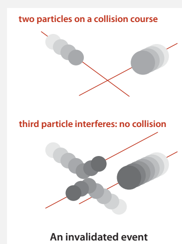
http://algs4.cs.princeton.edu/61event/Particle.java.html

Collision system: event-driven simulation main loop

Initialization.

- Fill PQ with all potential particle-wall collisions.
- Fill PQ with all potential particle-particle collisions.

"potential" since collision is invalidated if some other collision intervenes



Main loop.

- Delete the impending event from PQ (min priority = t).
- If the event has been invalidated, ignore it.
- Advance all particles to time t, on a straight-line trajectory.
- Update the velocities of the colliding particle(s).
- Predict future particle-wall and particle-particle collisions involving the colliding particle(s) and insert events onto PQ.

Event data type

Conventions.

- Neither particle `null` ⇒ particle-particle collision.
- One particle `null` ⇒ particle-wall collision.
- Both particles `null` ⇒ redraw event.

```
private static class Event implements Comparable<Event>
{
    private final double time;         // time of event
    private final Particle a, b;       // particles involved in event
    private final int countA, countB;  // collision counts of a and b

    public Event(double t, Particle a, Particle b)
    { ... }

    public int compareTo(Event that)
    {  return this.time - that.time;   }

    public boolean isValid()
    { ... }
}
```

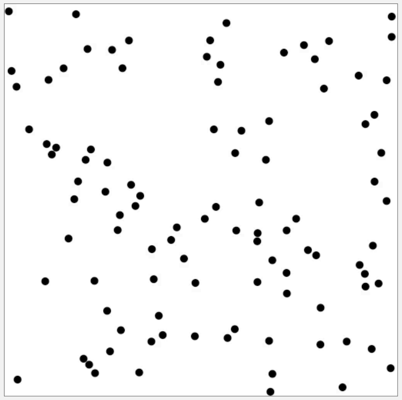
create event

ordered by time

valid if no intervening collisions (compare collision counts)

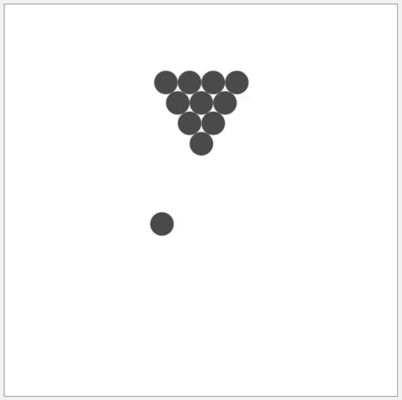
Particle collision simulation: example 1

```
% java CollisionSystem 100
```



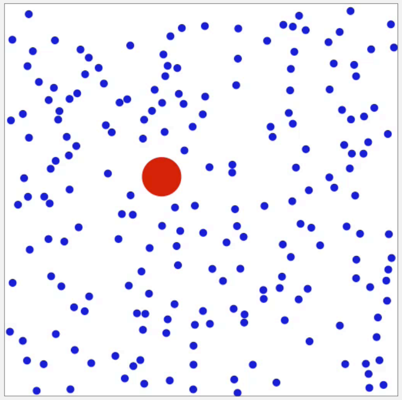
Particle collision simulation: example 2

```
% java CollisionSystem < billiards.txt
```



Particle collision simulation: example 3

```
% java CollisionSystem < brownian.txt
```



Particle collision simulation: example 4

```
% java CollisionSystem < diffusion.txt
```