Scene Graphs & Modeling Transformations

COS 426, Spring 2015

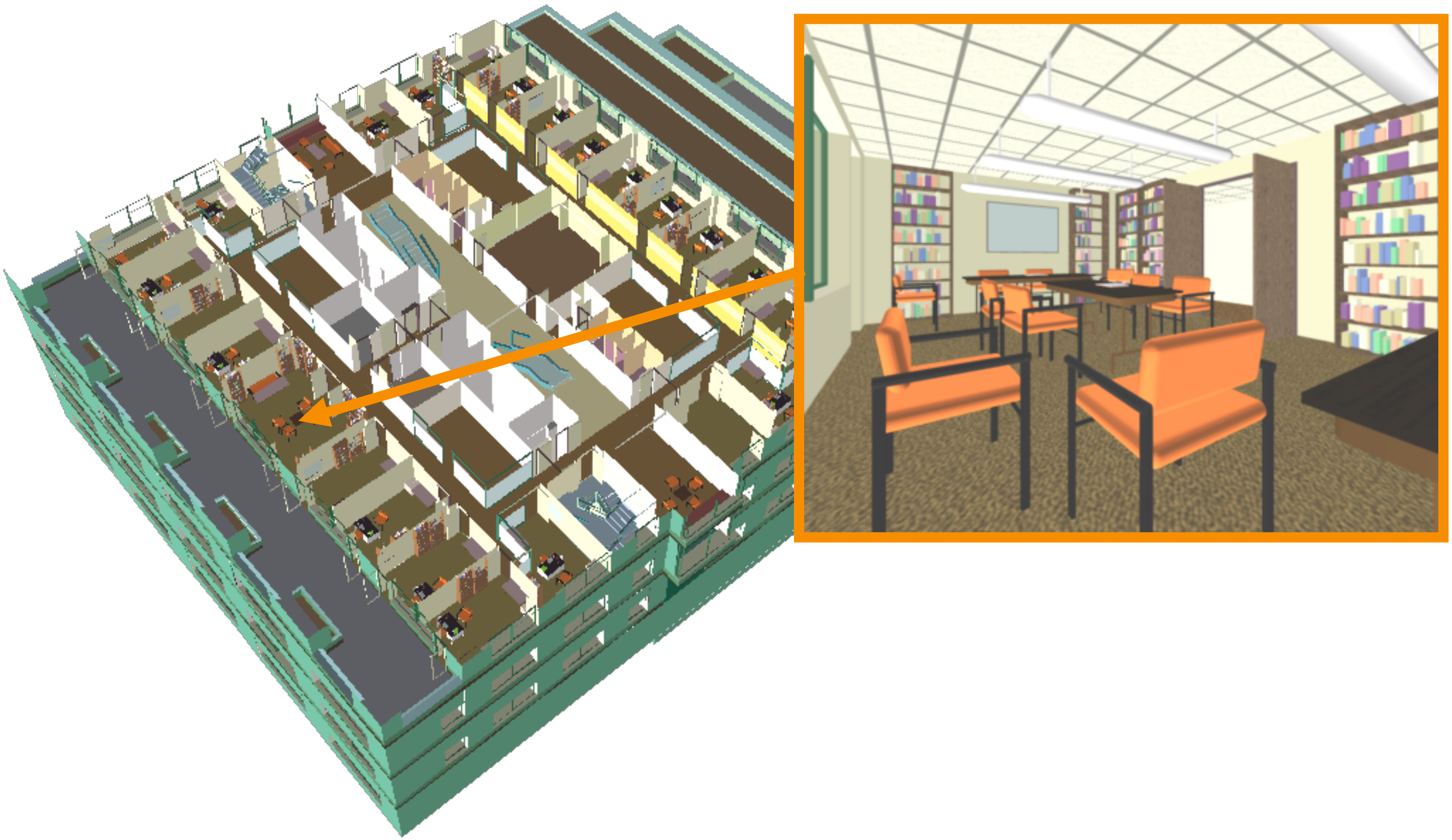
Princeton University

3D Object Representations

- Points
  - Range image
  - Point cloud
- Surfaces
  - Polygonal mesh
  - Subdivision
  - Parametric
  - Implicit
- Solids
  - Voxels
  - BSP tree
  - CSG
  - Sweep
- High-level structures
  - Scene graph
  - Application specific

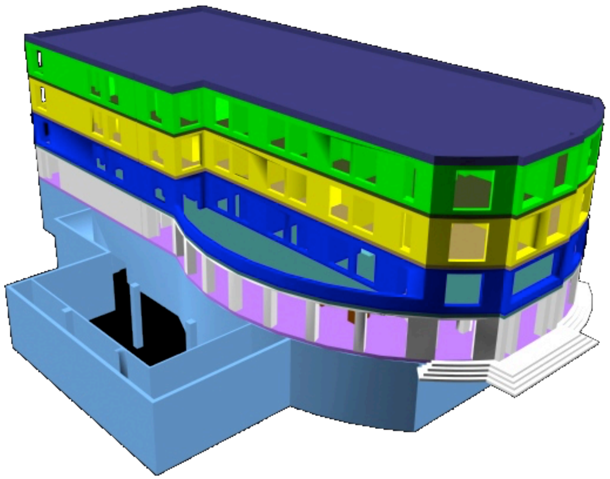
3D Object Representations

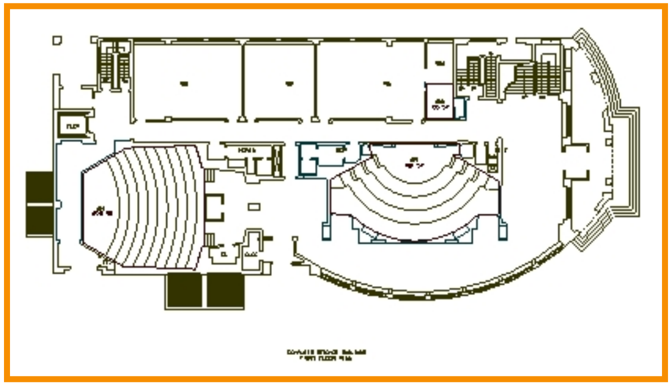
- What object representation is best for this?



3D Object Representations

- Desirable properties of an object representation
  - Easy to acquire
  - Accurate
  - Concise
  - Intuitive editing
  - Efficient editing
  - Efficient display
  - Efficient intersections
  - Guaranteed validity
  - Guaranteed smoothness
  - etc.





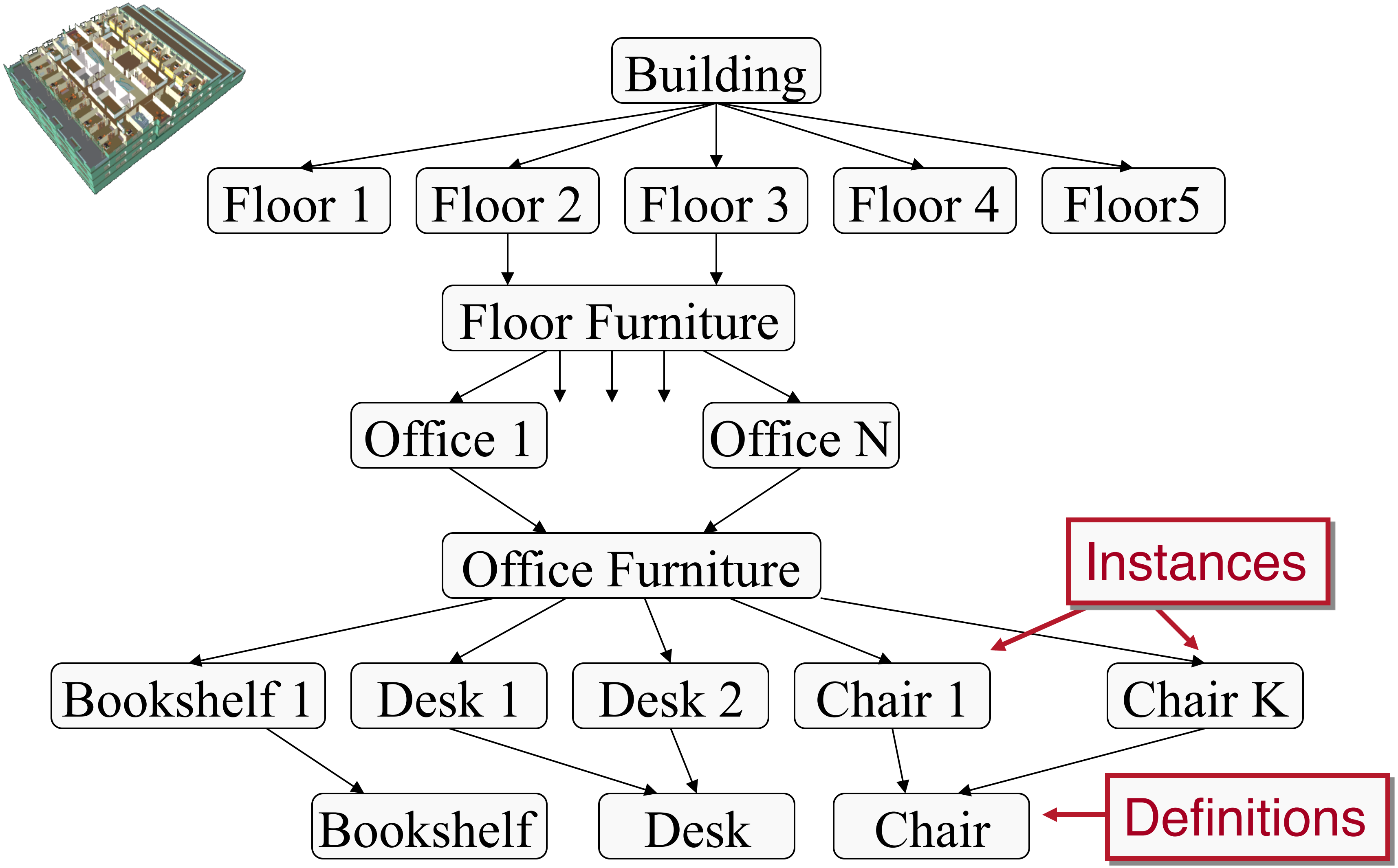
(CS Building, Princeton University)

Overview

- Scene graphs
  - Geometry & attributes
  - Transformations
  - Bounding volumes

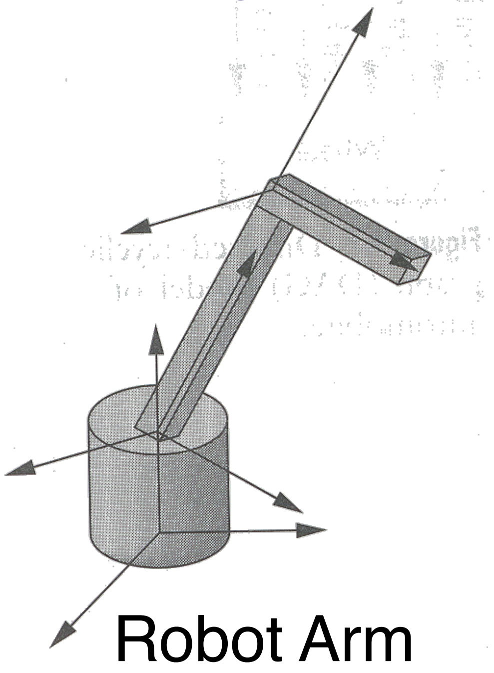
- Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
  - 3D transformations

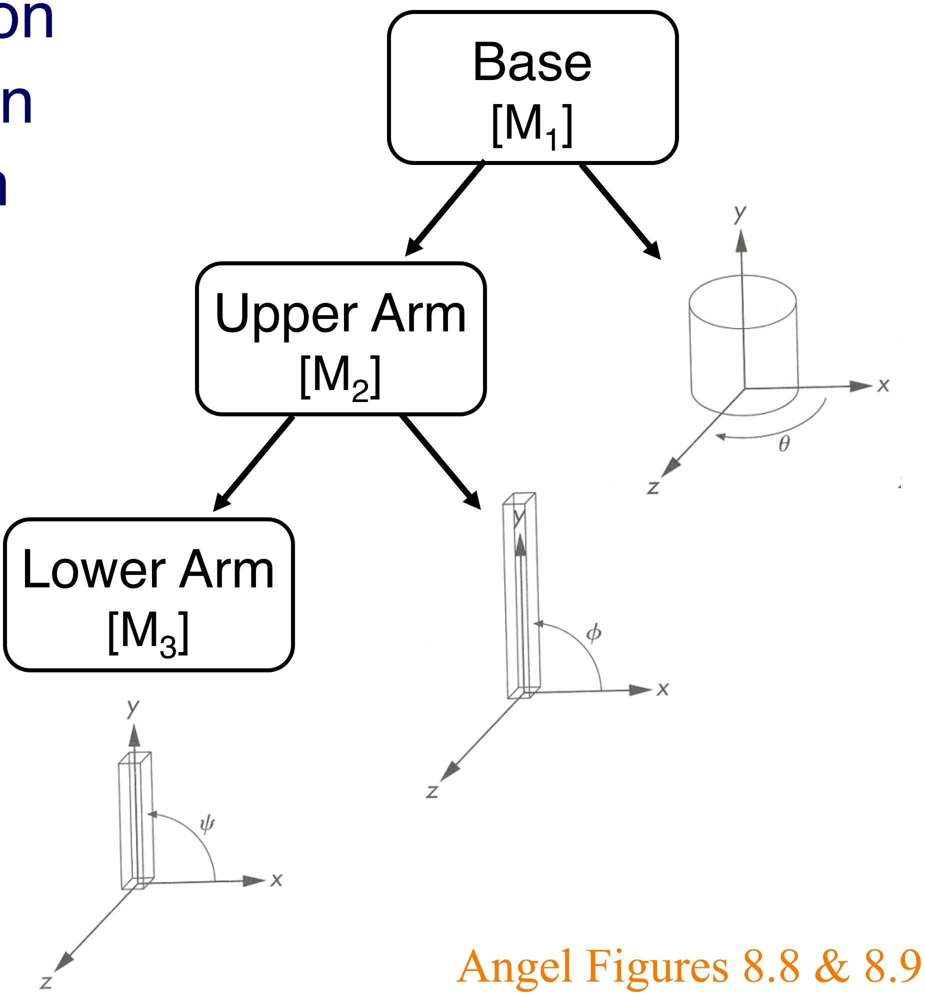
Scene Graphs



Scene Graphs

- Hierarchy (DAG) of nodes, where each may have:
  - Geometry representation
  - Modeling transformation
  - Parents and/or children
  - Bounding volume

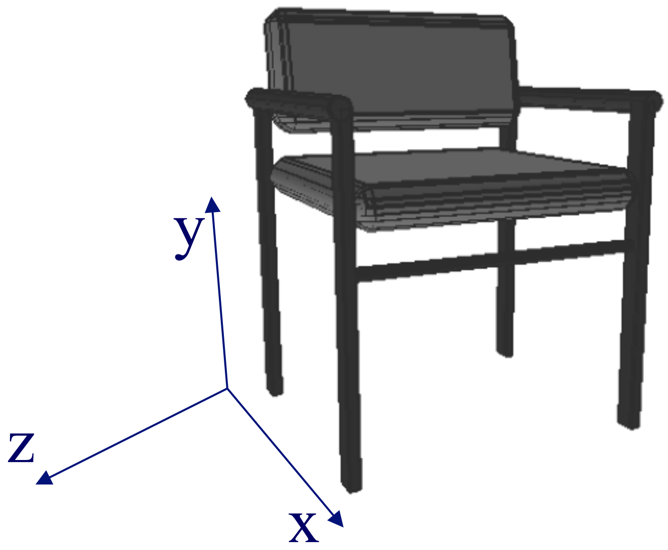




Angel Figures 8.8 & 8.9

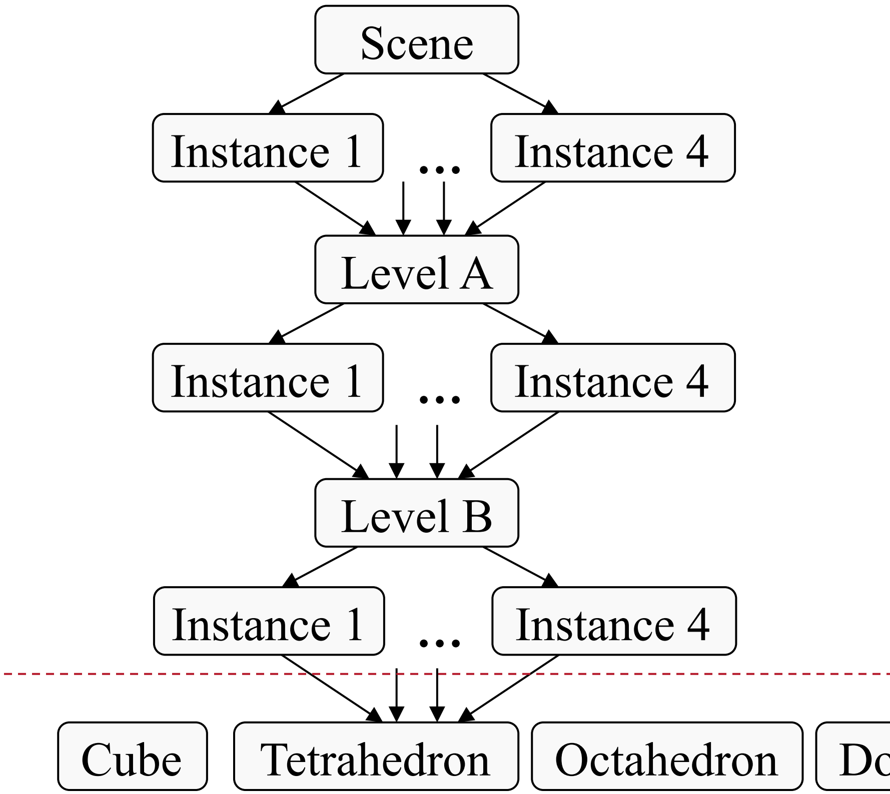
Scene Graphs

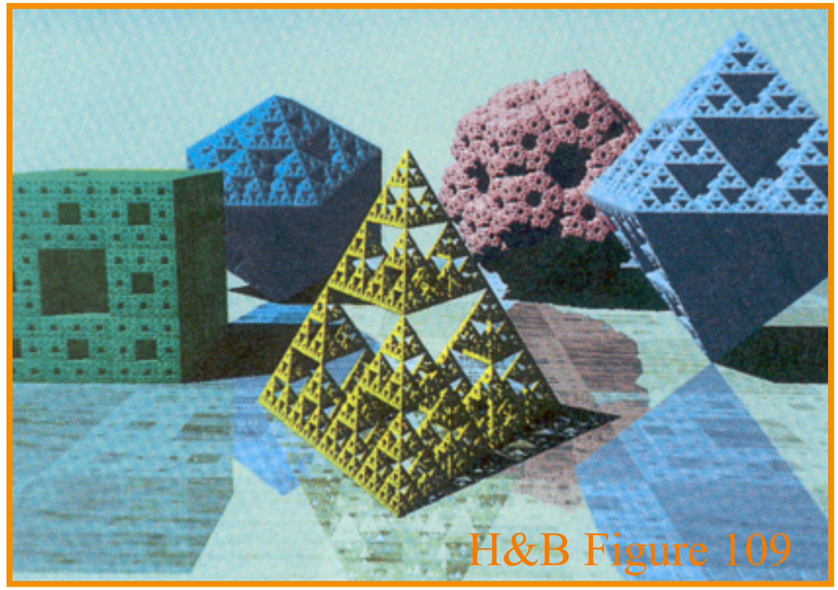
- Advantages
  - Allows definitions of objects in own coordinate systems



Scene Graphs

- Advantages
  - Allows definitions of objects in own coordinate systems
  - Allows use of object definition multiple times in a scene

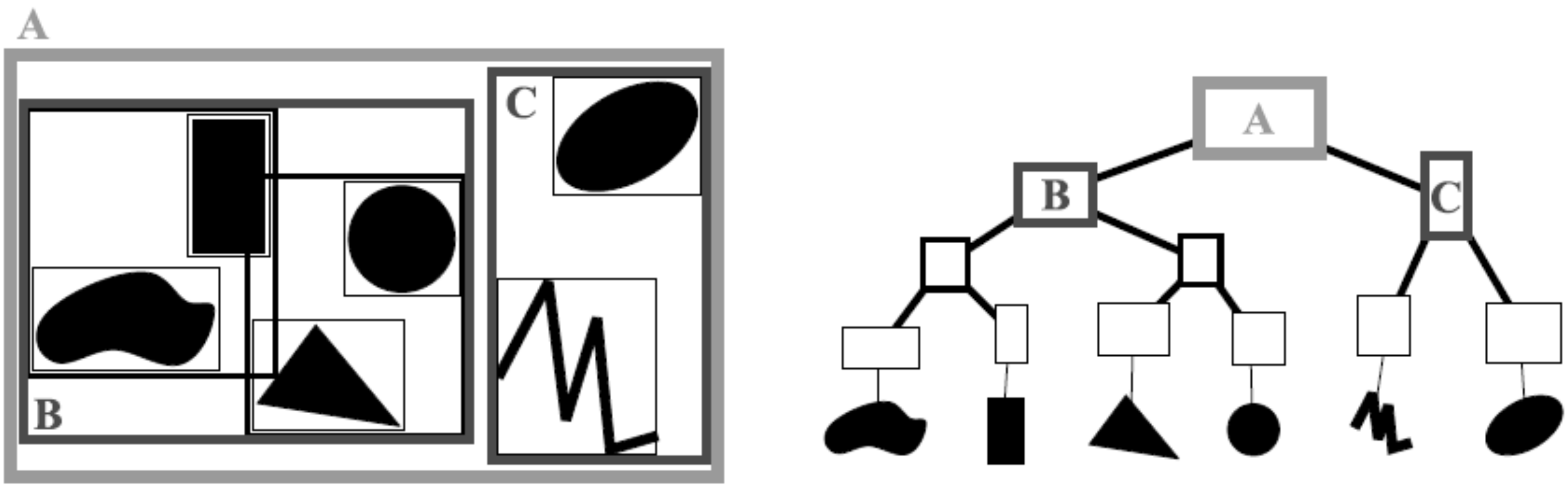




H&B Figure 109

Scene Graphs

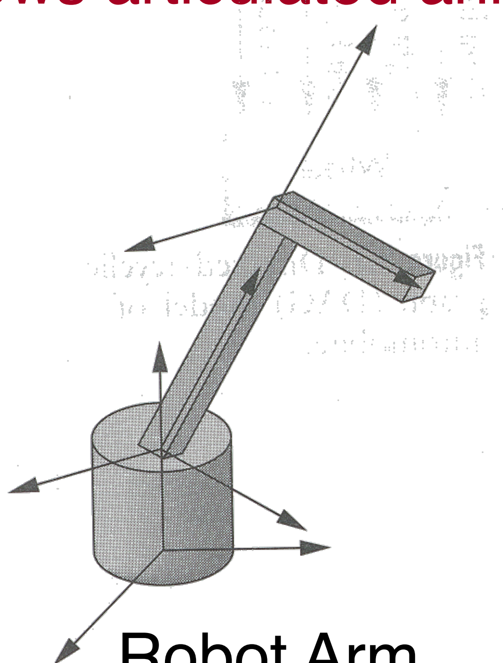
- Advantages
  - Allows definitions of objects in own coordinate systems
  - Allows use of object definition multiple times in a scene
  - Allows hierarchical processing (e.g., intersections)



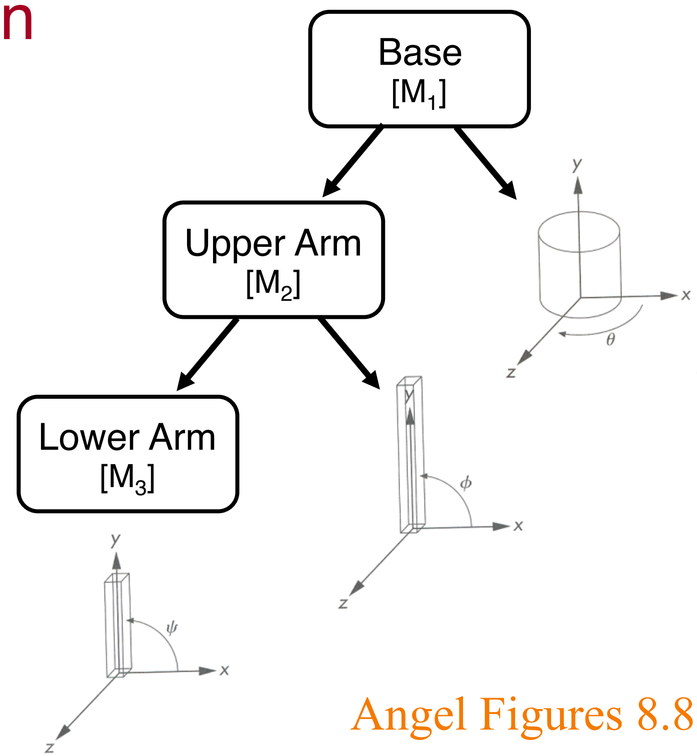
Haverkort

Scene Graphs

- Advantages
  - Allows definitions of objects in own coordinate systems
  - Allows use of object definition multiple times in a scene
  - Allows hierarchical processing (e.g., intersections)
  - Allows articulated animation

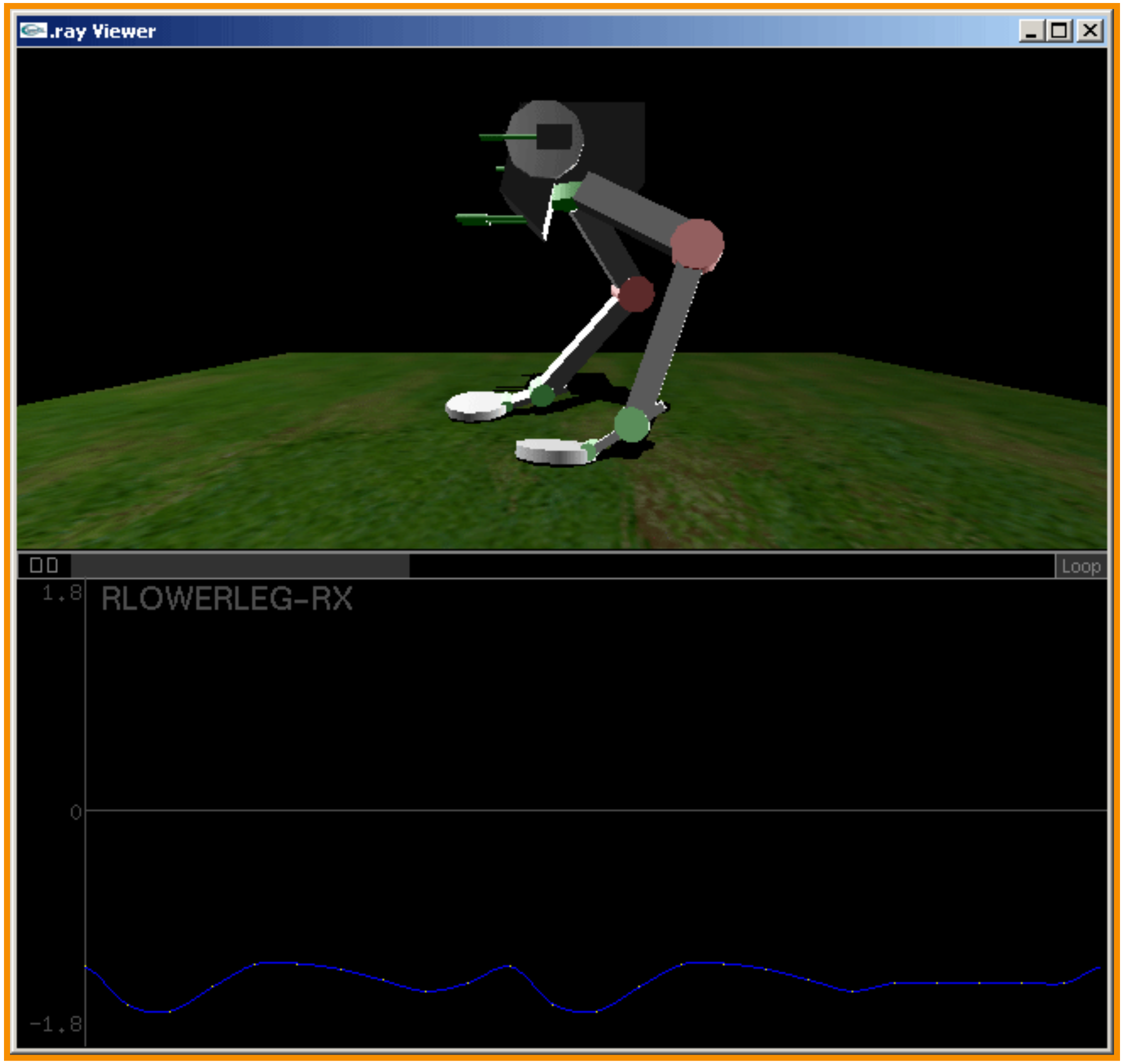


Robot Arm



Angel Figures 8.8 & 8.9

Transformations in Scene Graphs

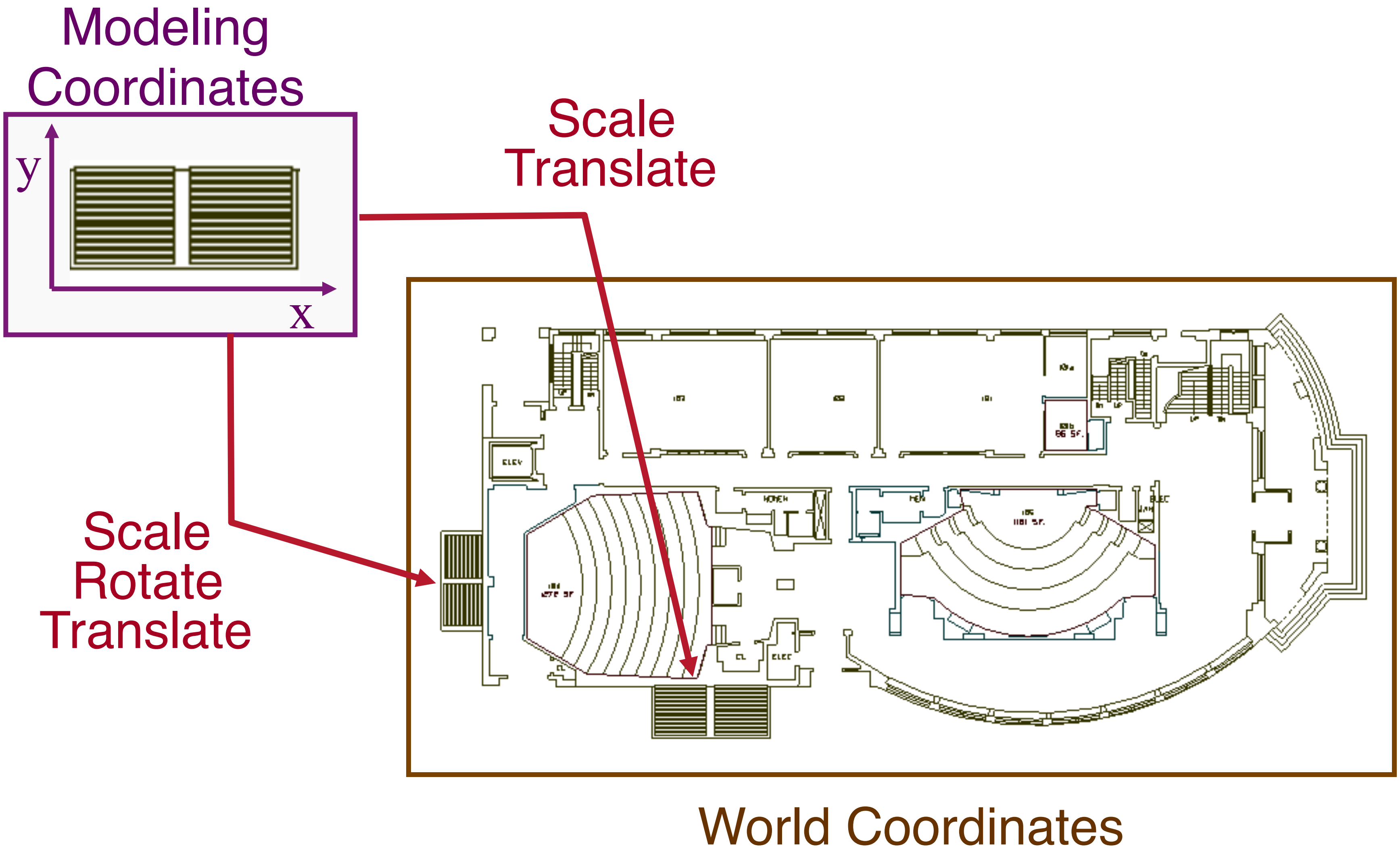


Mihai Parparita, COS 426

Overview

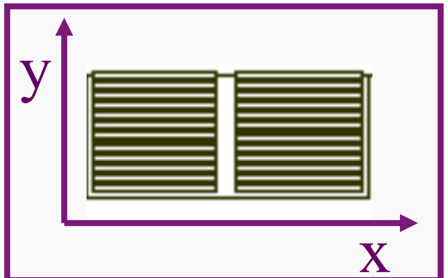
- Scene graphs
  - Geometry & attributes
  - Transformations
  - Bounding volumes
- Transformations
  - Basic 2D transformations
  - Matrix representation
  - Matrix composition
  - 3D transformations

2D Modeling Transformations

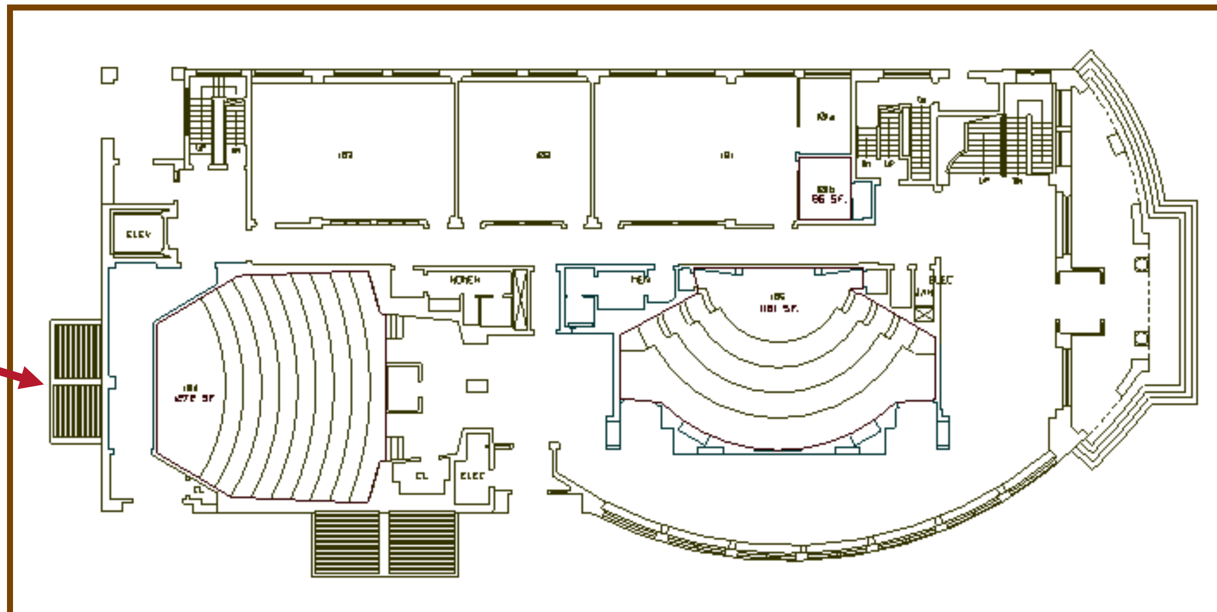


2D Modeling Transformations

Modeling Coordinates



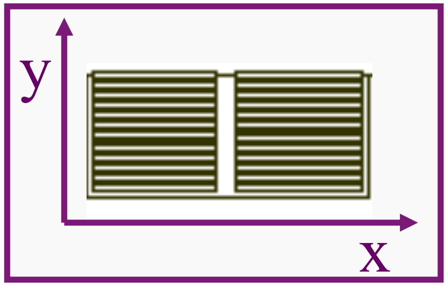
Let's look at this in detail…

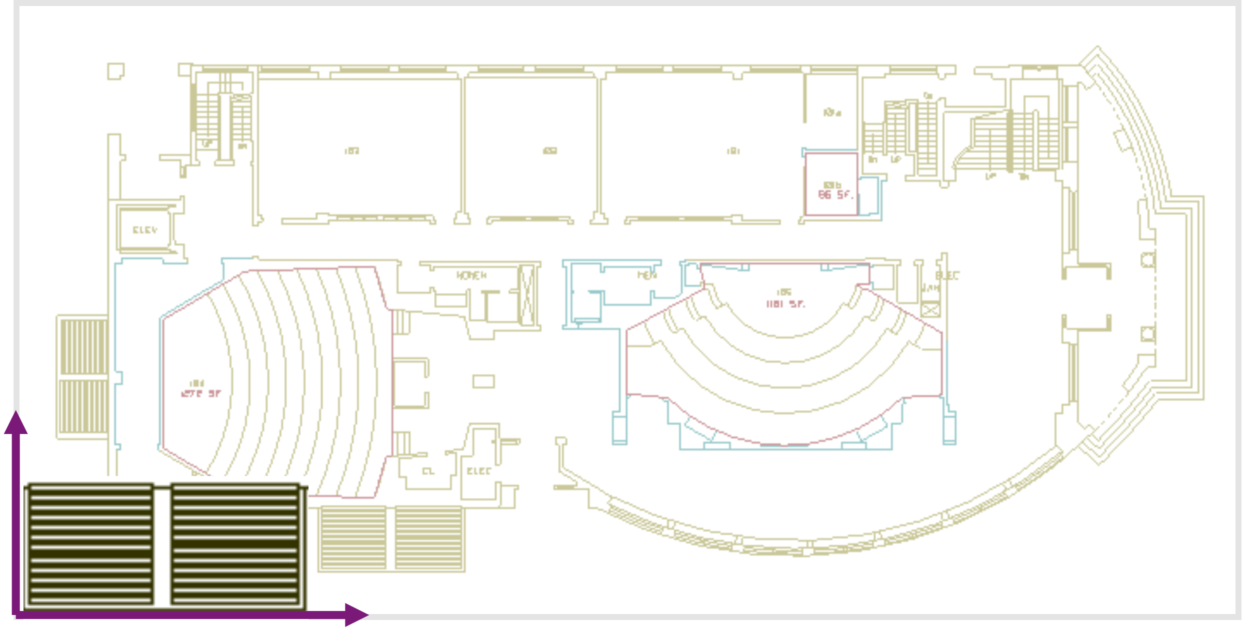


World Coordinates

2D Modeling Transformations

Modeling
Coordinates





2D Modeling Transformations

Modeling
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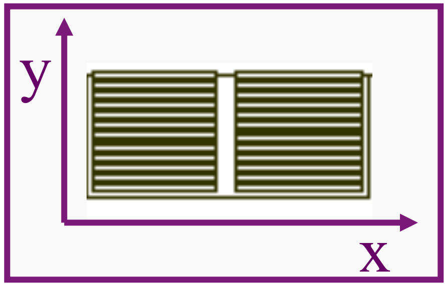
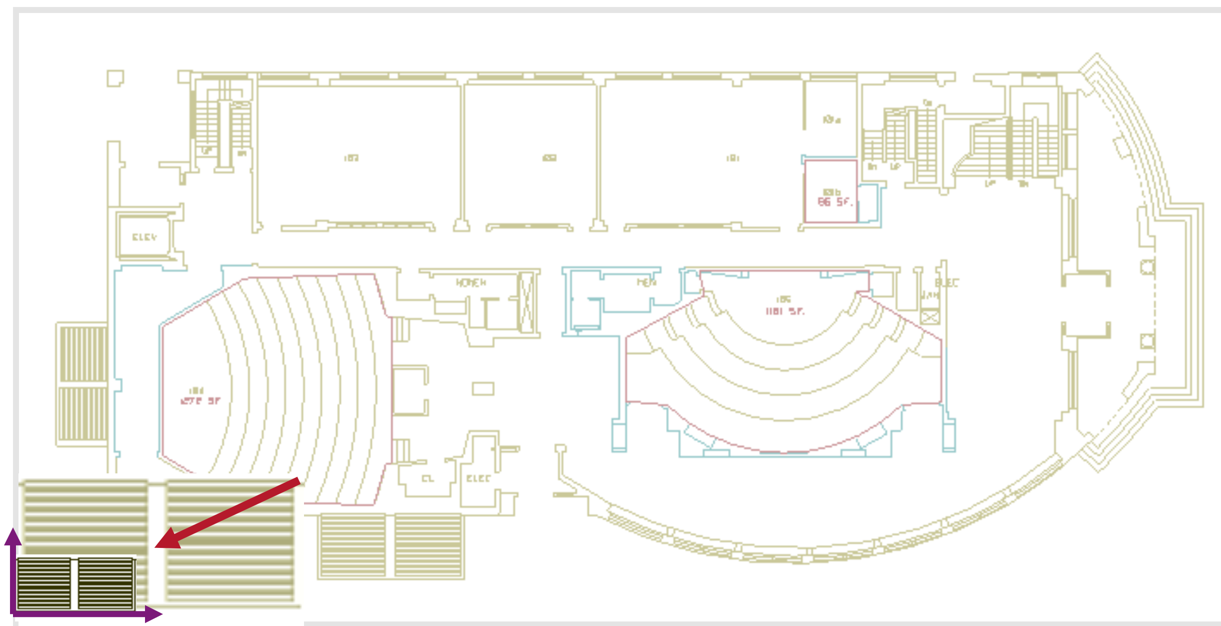
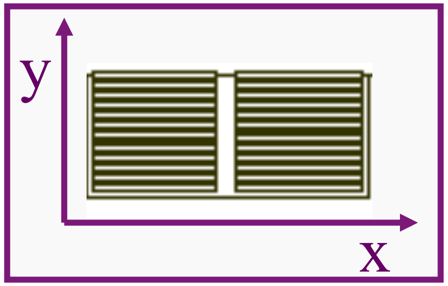


Scale .3, .3

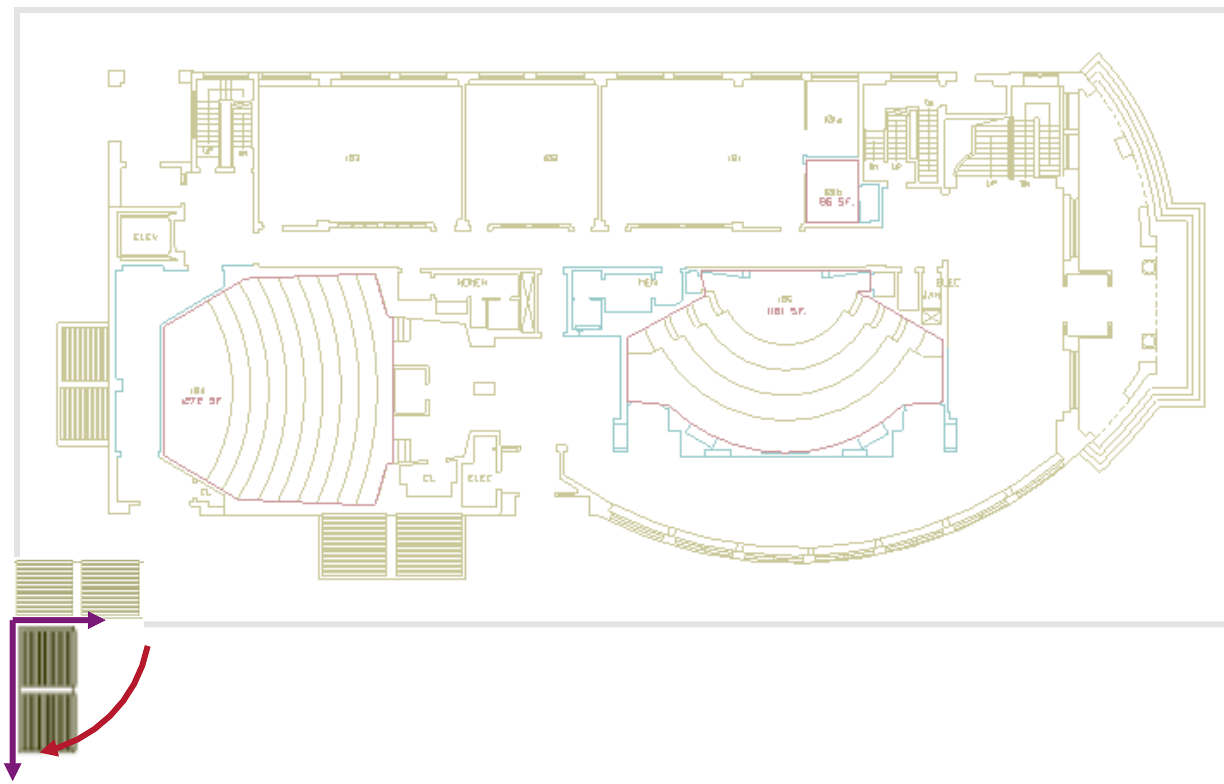


2D Modeling Transformations

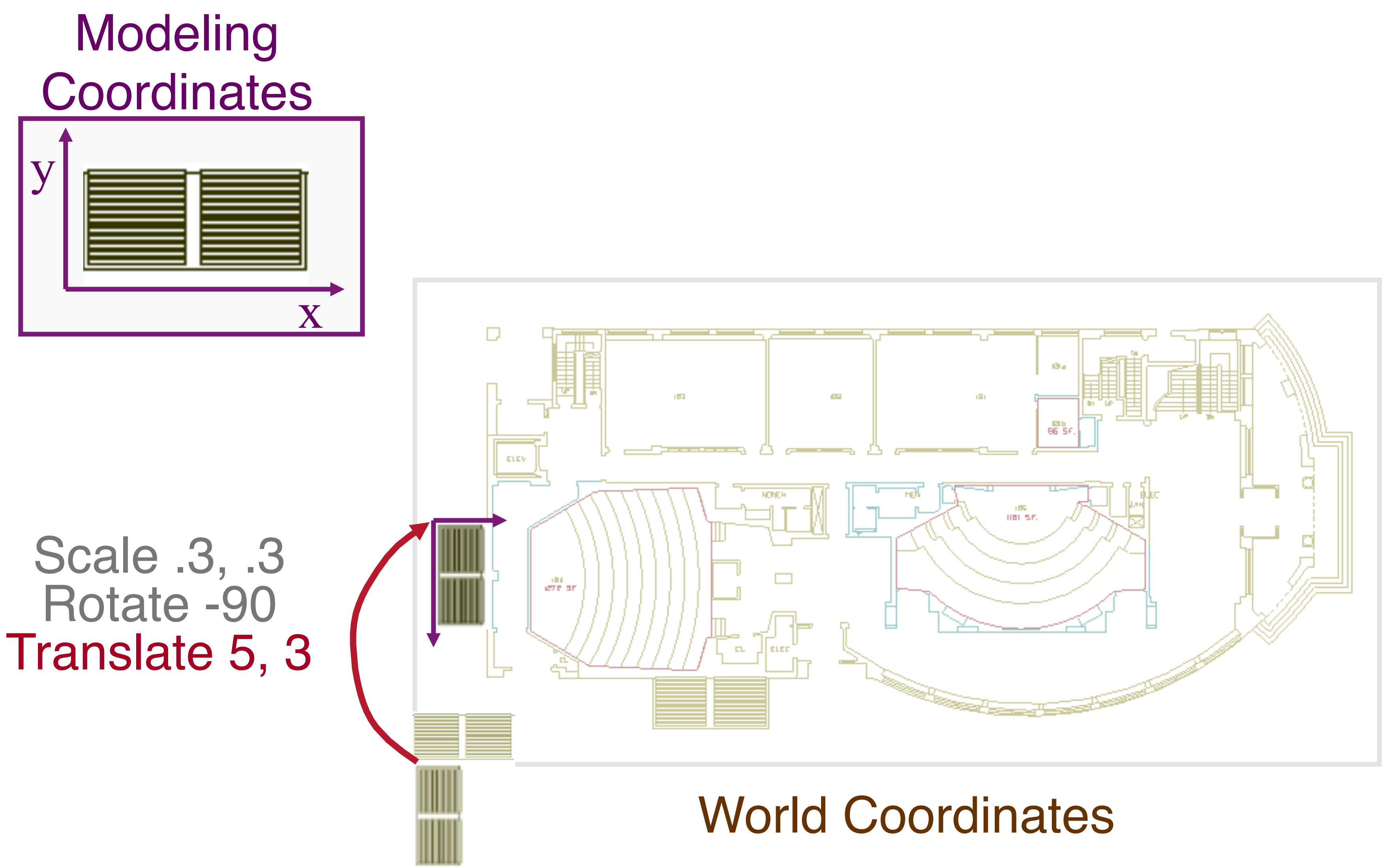
Modeling
Coordinates



Scale .3, .3
Rotate -90

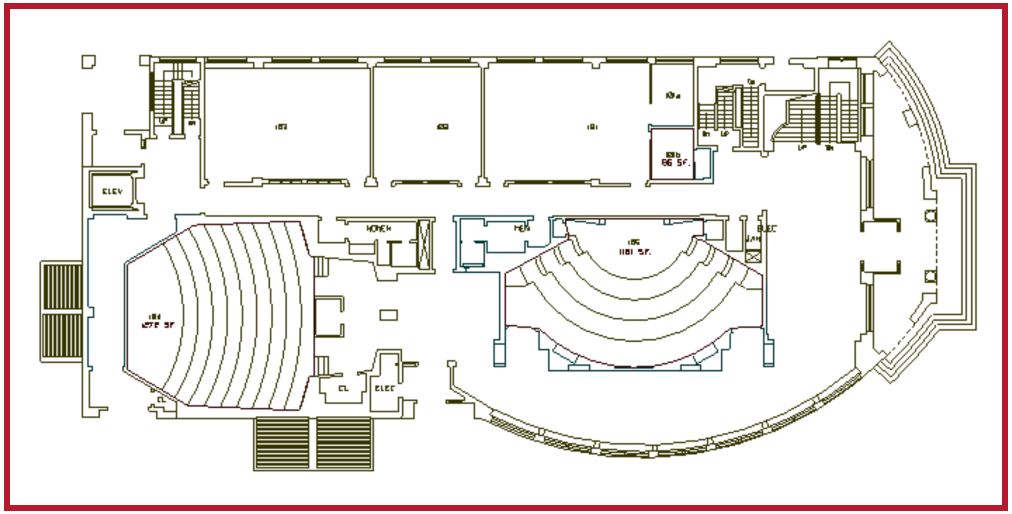


2D Modeling Transformations



Basic 2D Transformations

- Translation:
  - $x' = x + tx$
  - $y' = y + ty$
- Scale:
  - $x' = x * sx$
  - $y' = y * sy$
- Shear:
  - $x' = x + hx*y$
  - $y' = y + hy*x$
- Rotation:
  - $x' = x*\cos\Theta - y*\sin\Theta$
  - $y' = x*\sin\Theta + y*\cos\Theta$



Transformations can be combined (with simple algebra)

Basic 2D Transformations

- Translation:
  - $x' = x + tx$
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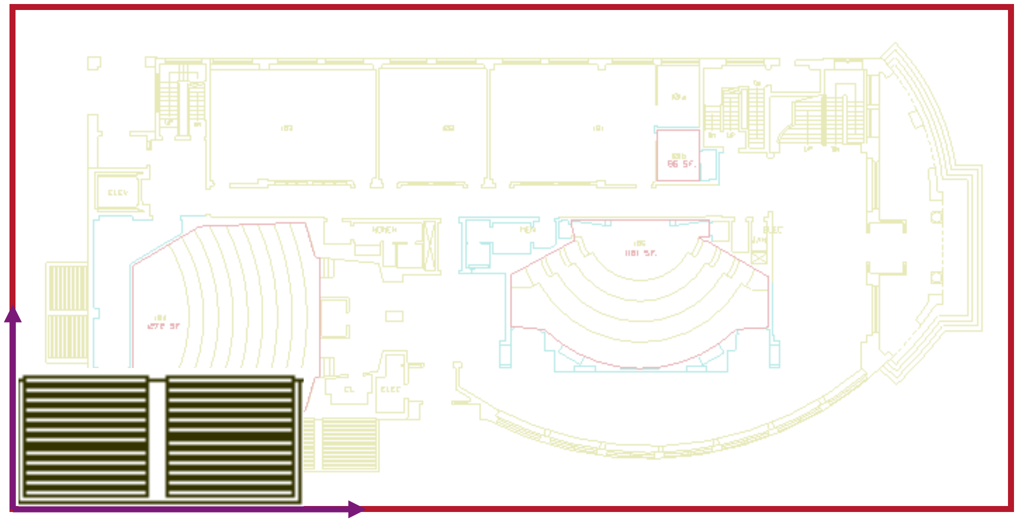

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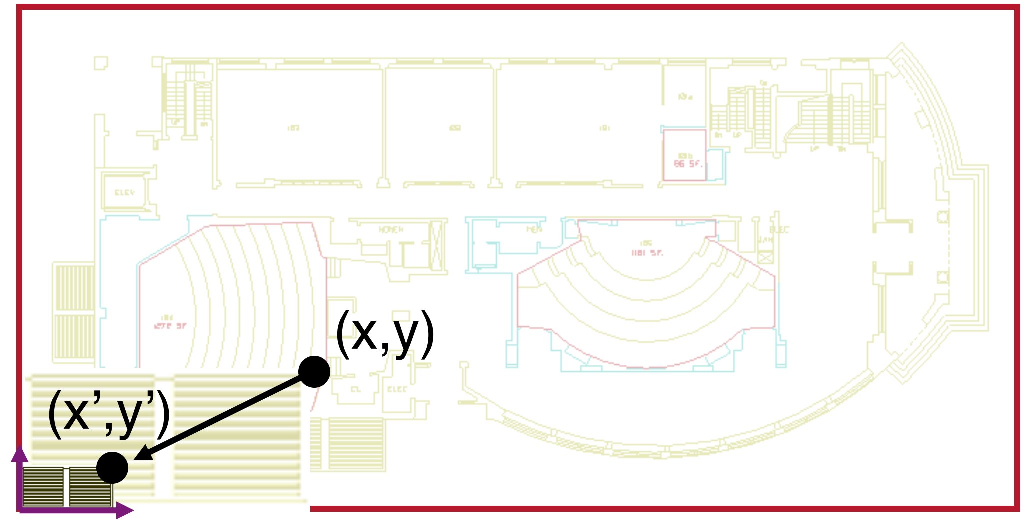
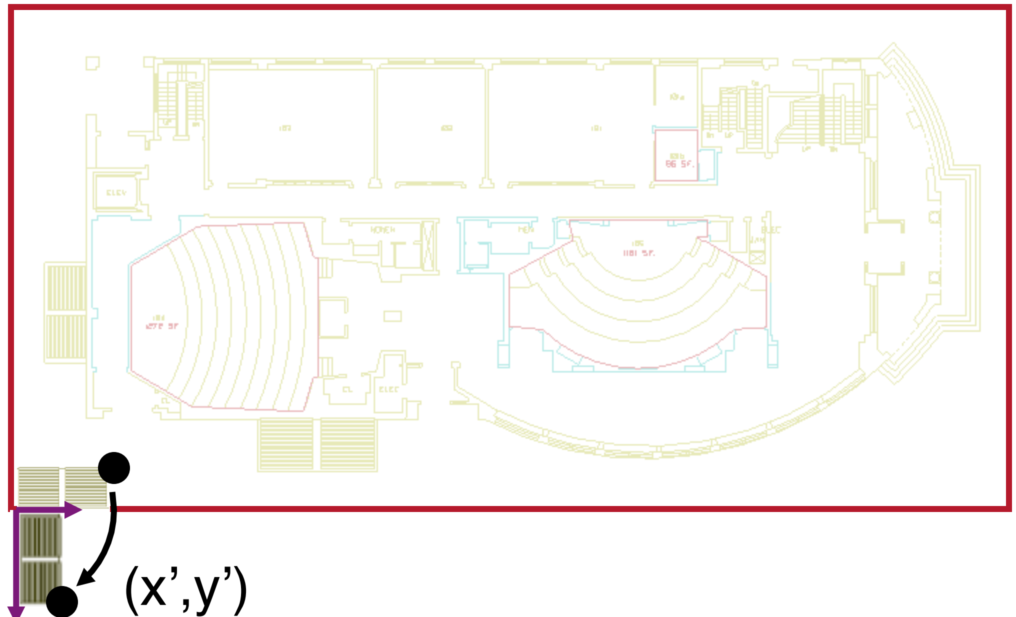


$x' = x*sx$
$y' = y*sy$

Basic 2D Transformations

- Translation:
  - x' = x + tx
  - y' = y + ty
- Scale:
  - x' = x * sx
  - y' = y * sy
- Shear:
  - x' = x + hx*y
  - y' = y + hy*x
- Rotation:
  - x' = x*cosΘ - y*sinΘ
  - y' = x*sinΘ + y*cosΘ



x' = (x*sx)*cosΘ − (y*sy)*sinΘ
y' = (x*sx)*sinΘ + (y*sy)*cosΘ

Basic 2D Transformations

- Translation:
  - $x' = x + tx$
  - $y' = y + ty$
- Scale:
  - $x' = x * sx$
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- Rotation:
  - $x' = x*\cos\Theta - y*\sin\Theta$
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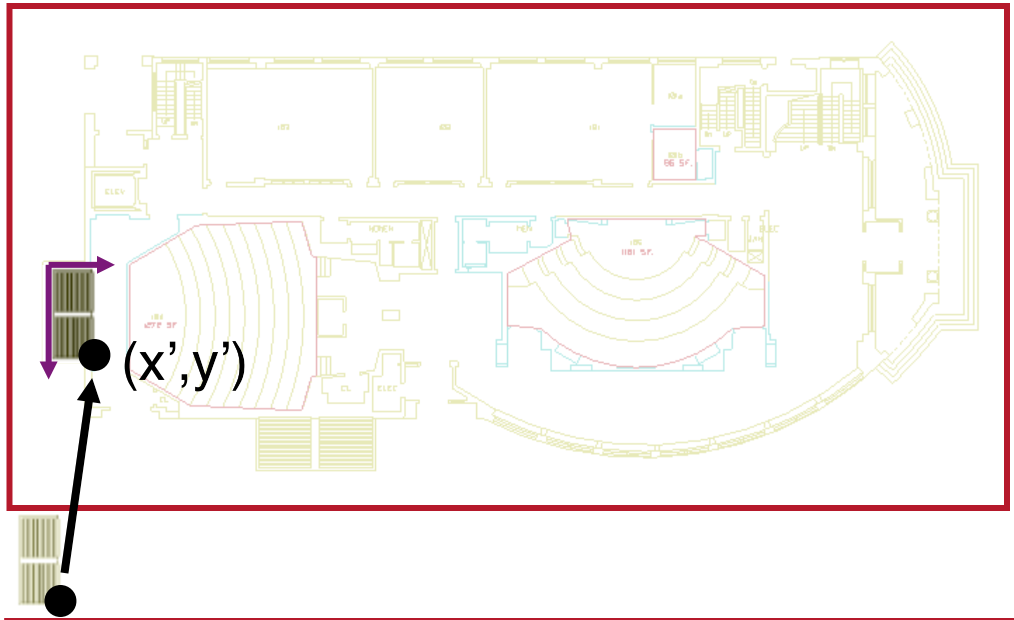
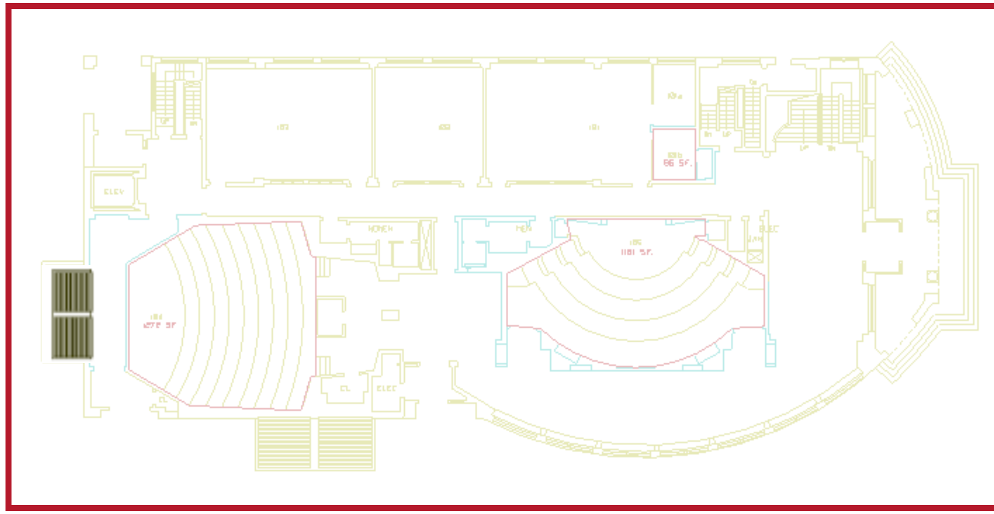


$$x' = ((x*sx)*\cos\Theta - (y*sy)*\sin\Theta) + tx$$
$$y' = ((x*sx)*\sin\Theta + (y*sy)*\cos\Theta) + ty$$

Basic 2D Transformations

- Translation:
  - $x' = x + tx$
  - $y' = y + ty$



- Scale:
  - $x' = x * sx$
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- Shear:
  - $x' = x + hx*y$
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$$x' = ((x*sx)*\cos\Theta - (y*sy)*\sin\Theta) + tx$$
$$y' = ((x*sx)*\sin\Theta + (y*sy)*\cos\Theta) + ty$$

- Rotation:
  - $x' = x*\cos\Theta - y*\sin\Theta$
  - $y' = x*\sin\Theta + y*\cos\Theta$

Overview

- Scene graphs
  - Geometry & attributes
  - Transformations
  - Bounding volumes
- Transformations
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  - Matrix representation
  - Matrix composition
  - 3D transformations

Matrix Representation

- Represent 2D transformation by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- Multiply matrix by column vector
  $\Leftrightarrow$ apply transformation to point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = ax + by$$
$$y' = cx + dy$$

Matrix Representation

- Transformations combined by multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Matrices are a convenient and efficient way to represent a sequence of transformations

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Identity?

$$x' = x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)?

$$x' = sx * x$$
$$y' = sy * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$x' = \cos\Theta * x - \sin\Theta * y$$
$$y' = \sin\Theta * x + \cos\Theta * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear?

$$x' = x + shx * y$$
$$y' = shy * x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & shx \\ shy & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Mirror over Y axis?

$$x' = -x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$$x' = -x$$
$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + tx$$
$$y' = y + ty$$

NO.

Only *linear* 2D transformations can be represented with a 2×2 matrix

Linear Transformations

- 2D linear transformations are combinations of …
  - Scale,
  - Rotation,
  - Shear, and
  - Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Properties of linear transformations:
  - Satisfies: $T(s_1\mathbf{p}_1 + s_2\mathbf{p}_2) = s_1 T(\mathbf{p}_1) + s_2 T(\mathbf{p}_2)$
  - Origin maps to origin
  - Points at infinity stay at infinity
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

2D Translation

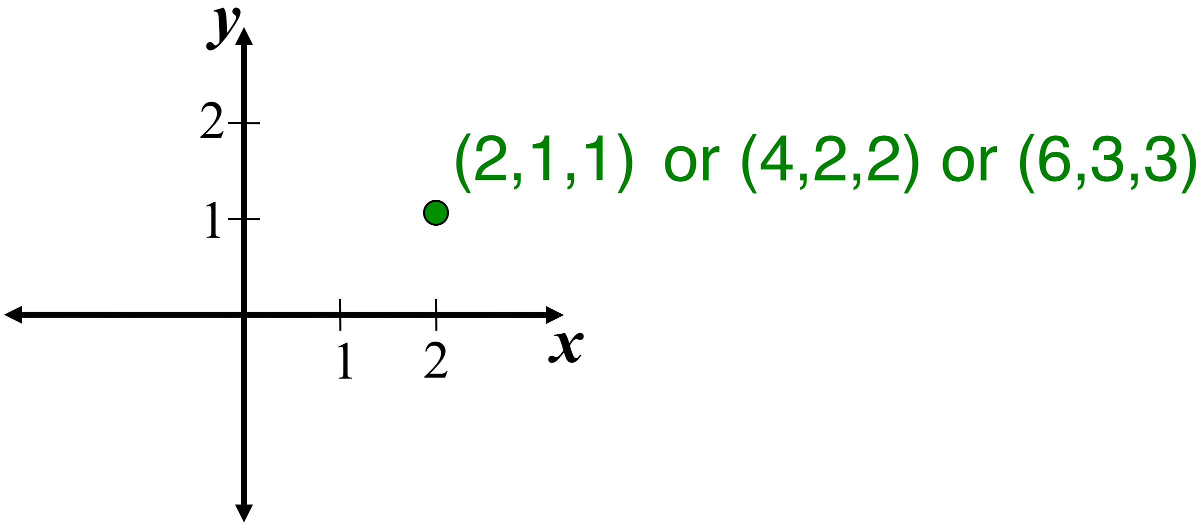
- 2D translation represented by a 3x3 matrix
  - Point represented with *homogeneous coordinates*

$$x' = x + tx$$
$$y' = y + ty$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Add a 3rd coordinate to every 2D point
  - (x, y, w) represents a point at location (x/w, y/w)
  - (x, y, 0) represents a point at infinity
  - (0, 0, 0) is not allowed



Convenient coordinate system to
represent many useful transformations

Basic 2D Transformations

- Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & shx & 0 \\ shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Affine Transformations

- Affine transformations are combinations of …
  - Linear transformations, and
  - Translations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of affine transformations:
  - Origin does not necessarily map to origin
  - Points at infinity remain at infinity
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

Projective Transformations

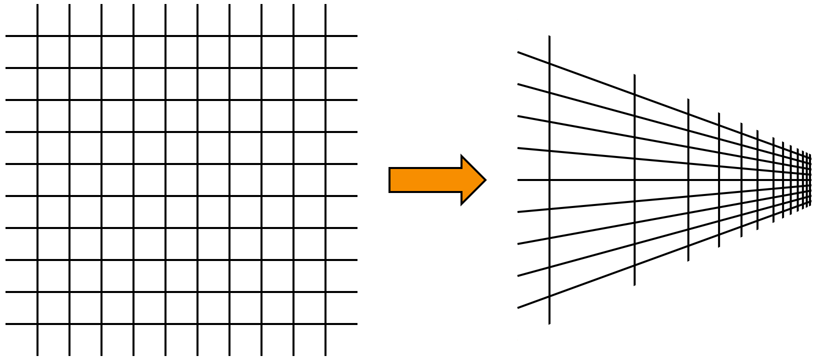
- Projective transformations (homographies):
  - Affine transformations, and
  - Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of projective transformations:
  - Origin does not necessarily map to origin
  - Point at infinity may map to finite point
  - Lines map to lines
  - Parallel lines do not necessarily remain parallel
  - Ratios are not preserved (but “cross-ratios” are)
  - Closed under composition

Projective Transformations

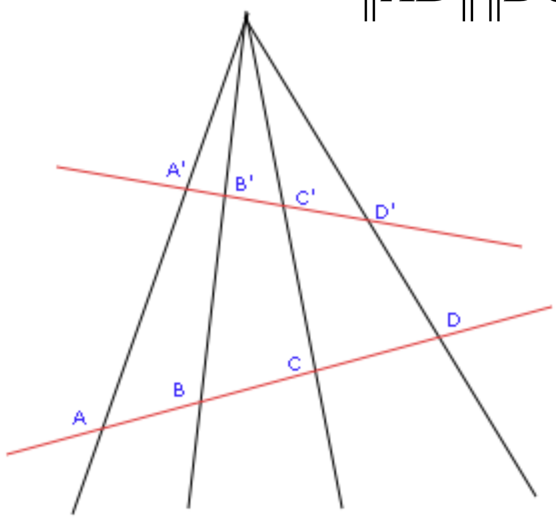
- Will be useful to model (pinhole) cameras: can represent camera projection in same framework as modeling transformations



Cross-Ratio

- Definition: for 4 collinear points A, B, C, D

$$(A,B;C,D) = \frac{\|AC\|\,\|BD\|}{\|AD\|\,\|BC\|}$$



- Projective Invariant: (A,B;C,D) = (A’,B’;C’,D’)

Wikipedia

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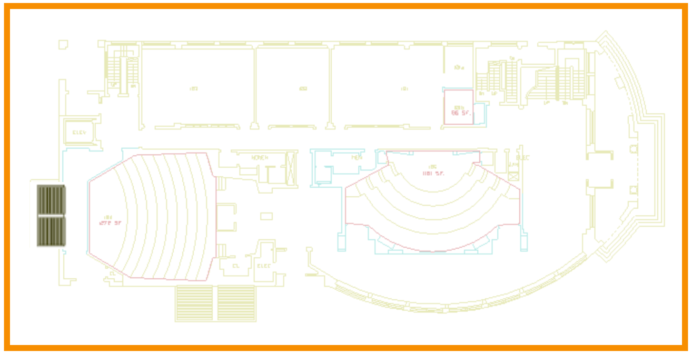
Matrix Composition

- Transformations can be combined by matrix multiplication

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

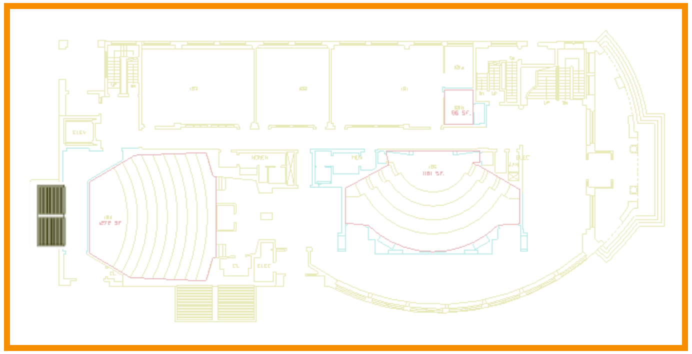
p' = T(tx,ty) R(Θ) S(sx,sy) **p**



Matrix Composition

- Matrices are a convenient and efficient way to represent a sequence of transformations
  - General purpose representation
  - Hardware matrix multiply
  - Efficiency with premultiplication
    - » Matrix multiplication is associative

$$\mathbf{p}' = (T * (R * (S*\mathbf{p})\ )\ )$$
$$\mathbf{p}' = (T*R*S) * \mathbf{p}$$

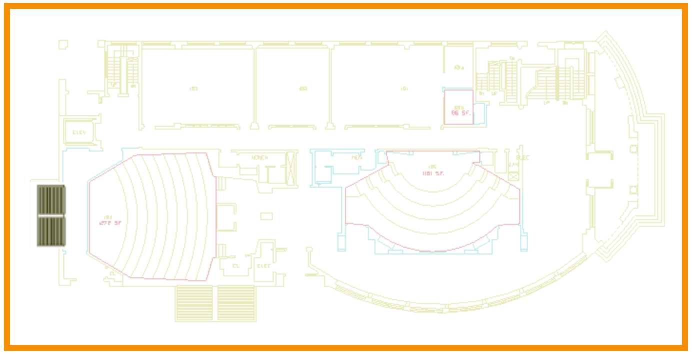


Matrix Composition

- Be aware: order of transformations matters
  - » Matrix multiplication is **not** commutative

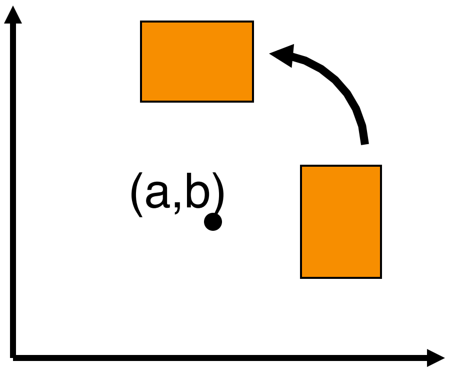
$$\mathbf{p}' = T * R * S * \mathbf{p}$$

"Global" "Local"



Matrix Composition

- Rotate by Θ around arbitrary point (a,b)
  - M=T(a,b) * R(Θ) * T(-a,-b)



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3D Transformations

- Same idea as 2D transformations
  - Homogeneous coordinates: (x,y,z,w)
  - 4x4 transformation matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Basic 3D Transformations

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Identity

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Mirror over X axis

Basic 3D Transformations

Rotate around Z axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

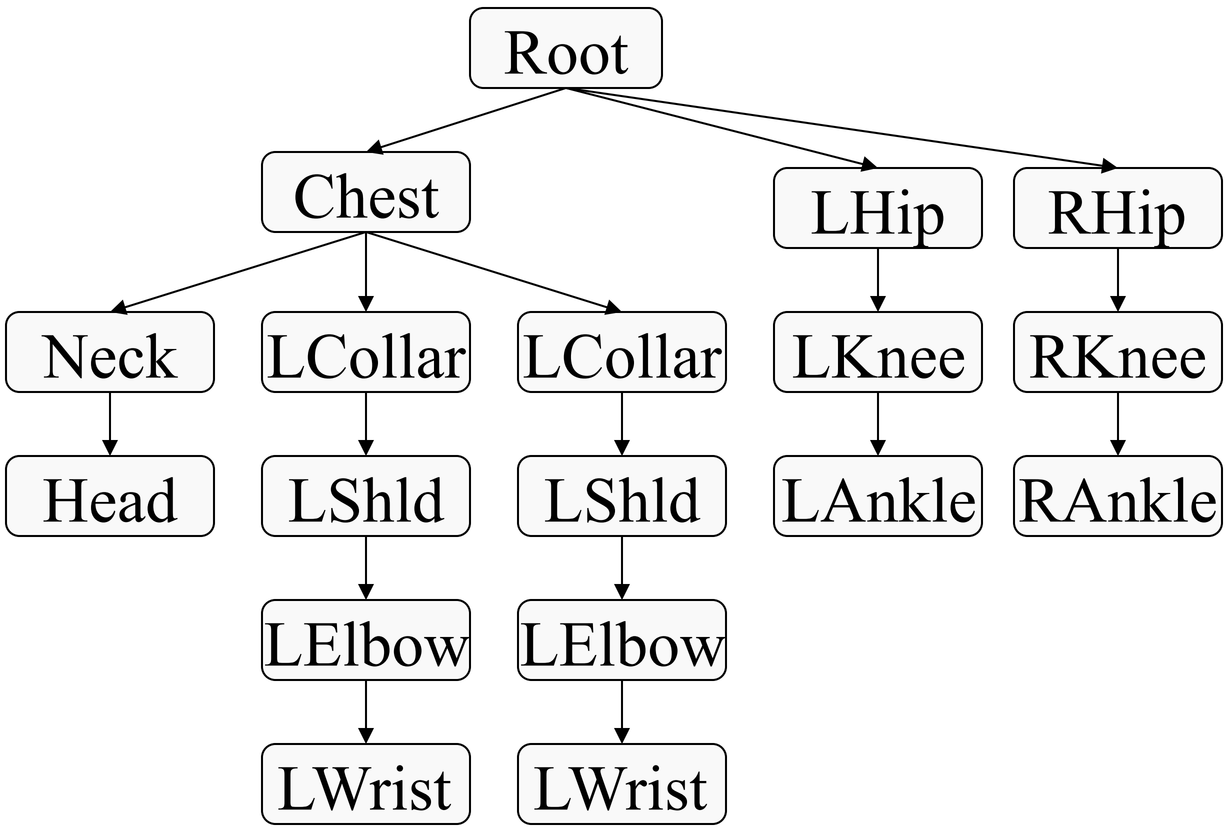
Rotate around Y axis:

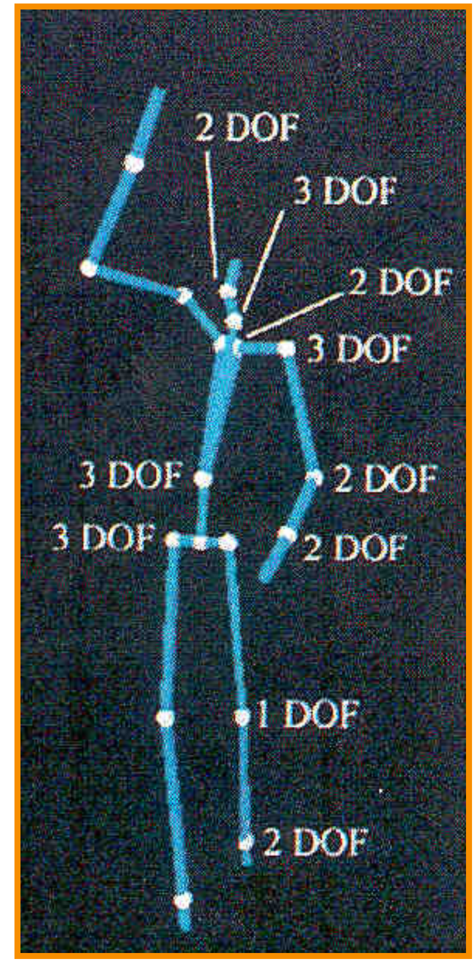
$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} \cos\Theta & 0 & -\sin\Theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\Theta & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Rotate around X axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta & -\sin\Theta & 0 \\ 0 & \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Transformations in Scene Graphs





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Scene Graph Example

Pixar

Summary

- Scene graphs
  - Hierarchical
  - Modeling transformations
  - Bounding volumes
- Coordinate systems
  - World coordinates
  - Modeling coordinates
- 3D modeling transformations
  - Represent most transformations by 4x4 matrices
  - Composite with matrix multiplication (order matters)