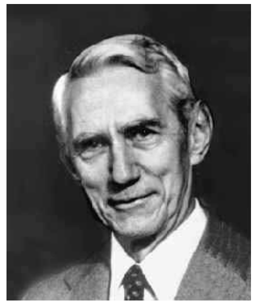
6.1 Combinational Circuits

George Boole (1815 – 1864)

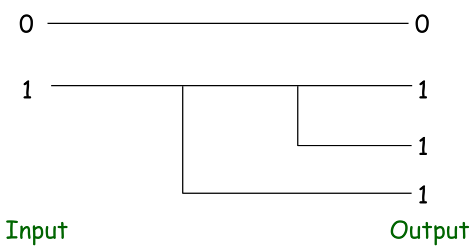


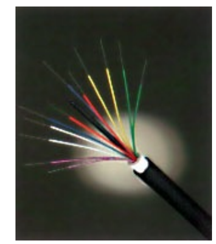
Claude Shannon (1916 – 2001)

Signals and Wires

Digital signals
- Binary (or "logical") values: 1 or 0, on or off, high or low voltage

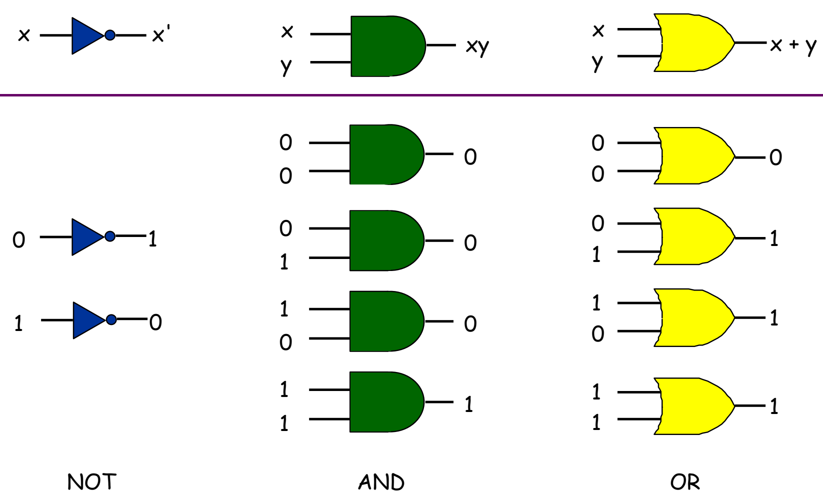
Wires.
- Propagate logical values from place to place.
- Signals "flow" from left to right.
  - A drawing convention, sometimes violated
  - Actually: flow from producer to consumer(s) of signal





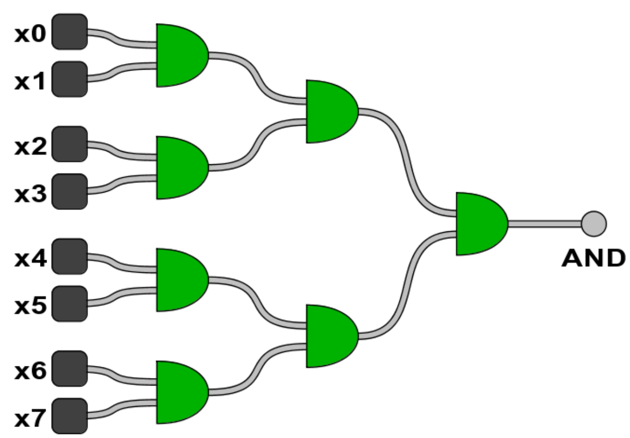
Logic Gates

Logical gates.
- Fundamental building blocks.



Multiway AND Gates

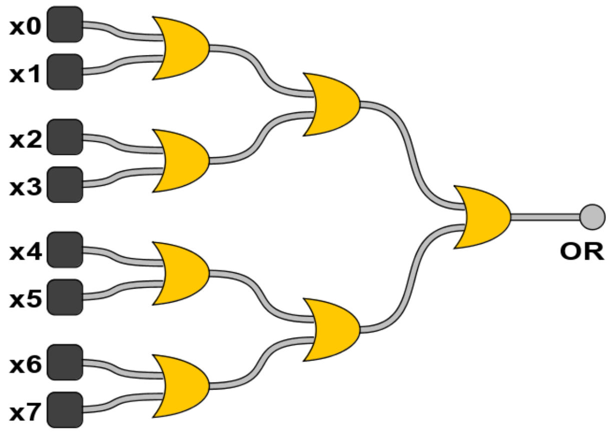
AND(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7).
- 1 if all inputs are 1.
- 0 otherwise.



Multiway OR Gates

$OR(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$.

- 1 if at least one input is 1.
- 0 otherwise.



Boolean Algebra

History.

- Developed by Boole to solve mathematical logic problems (1847).
- Shannon master's thesis applied it to digital circuits (1937).

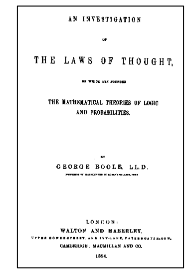
"possibly the most important, and also the most famous, master's thesis of the [20th] century" --Howard Gardner

Basics.

- Boolean variable: value is 0 or 1.
- Boolean function: function whose inputs and outputs are 0, 1.

Relationship to circuits.

- Boolean variables: signals.
- Boolean functions: circuits.

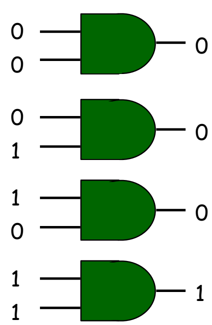


Truth Table

Truth table.

- Systematic method to describe Boolean function.
- One row for each possible input combination.
- N inputs $\Rightarrow$ 2^N rows.

AND Truth Table		
x	y	AND(x, y)
0	0	0
0	1	0
1	0	0
1	1	1



AND

Truth Table for Functions of 2 Variables

Truth table.

- 16 Boolean functions of 2 variables.
  - every 4-bit value represents one

Truth Table for All Boolean Functions of 2 Variables									
x	y	ZERO	AND		x		y	XOR	OR
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Truth Table for All Boolean Functions of 2 Variables									
x	y	NOR	EQ	y'		x'		NAND	ONE
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

Truth Table for Functions of 3 Variables

Truth table.
- 16 Boolean functions of 2 variables.
  - every 4-bit value represents one
- 256 Boolean functions of 3 variables.
  - every 8-bit value represents one
- 2^(2^N) Boolean functions of N variables!

Some Functions of 3 Variables						
x	y	z	AND	OR	MAJ	ODD
0	0	0	0	0	0	0
0	0	1	0	1	0	1
0	1	0	0	1	0	1
0	1	1	0	1	1	0
1	0	0	0	1	0	1
1	0	1	0	1	1	0
1	1	0	0	1	1	0
1	1	1	1	1	1	1

Universality of AND, OR, NOT

Any Boolean function can be expressed using AND, OR, NOT.
- "Universal."
- XOR(x,y) = xy' + x'y

Notation	Meaning
x'	NOT x
x y	x AND y
x + y	x OR y

Expressing XOR Using AND, OR, NOT							
x	y	x'	y'	x'y	xy'	x'y + xy'	XOR
0	0	1	1	0	0	0	0
0	1	1	0	1	0	1	1
1	0	0	1	0	1	1	1
1	1	0	0	0	0	0	0

Exercise. Show {AND, NOT}, {OR, NOT}, {NAND}, {AND, XOR} are universal.
Hint. Use DeMorgan's Law: (xy)' = (x' + y') and (x + y)' = (x'y')

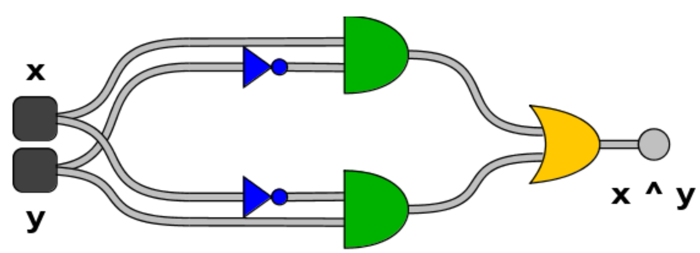
Sum-of-Products

Any Boolean function can be expressed using AND, OR, NOT.
- Sum-of-products is systematic procedure.
  - form AND term for each 1 in truth table of Boolean function
  - OR terms together

Expressing MAJ Using Sum-of-Products								
x	y	z	MAJ	x'yz	xy'z	xyz'	xyz	x'yz + xy'z + xyz' + xyz
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	1
1	0	0	0	0	0	0	0	0
1	0	1	1	0	1	0	0	1
1	1	0	1	0	0	1	0	1
1	1	1	1	0	0	0	1	1

Translate Boolean Formula to Boolean Circuit

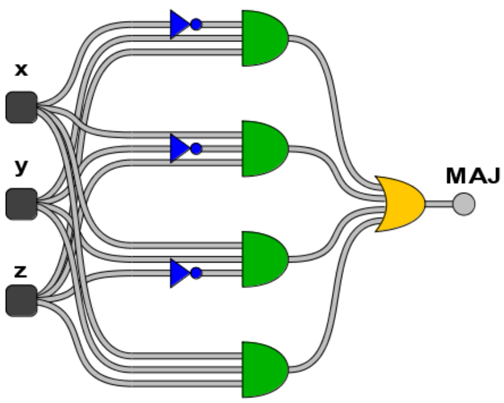
Use sum-of-products form.
- XOR(x, y) = xy' + x'y.



Translate Boolean Formula to Boolean Circuit

Use sum-of-products form.

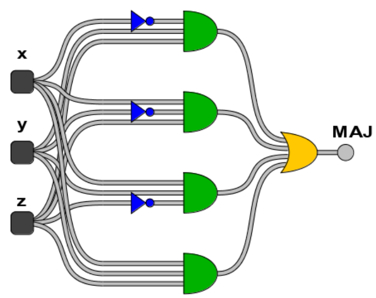
- MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz.



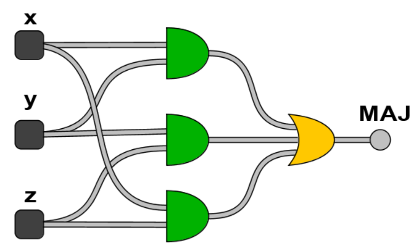
Simplification Using Boolean Algebra

Many possible circuits for each Boolean function.

- Sum-of-products not necessarily optimal in:
  - number of gates (space)
  - depth of circuit (time)
- MAJ(x, y, z) = x'yz + xy'z + xyz' + xyz = xy + yz + xz.



size = 8, depth = 3



size = 4, depth = 2

Expressing a Boolean Function Using AND, OR, NOT

Ingredients.

- AND gates.
- OR gates.
- NOT gates.
- Wire.

Instructions.

- Step 1: represent input and output signals with Boolean variables.
- Step 2: construct truth table to carry out computation.
- Step 3: derive (simplified) Boolean expression using sum-of products.
- Step 4: transform Boolean expression into circuit.

ODD Parity Circuit

ODD(x, y, z).

- 1 if odd number of inputs are 1.
- 0 otherwise.

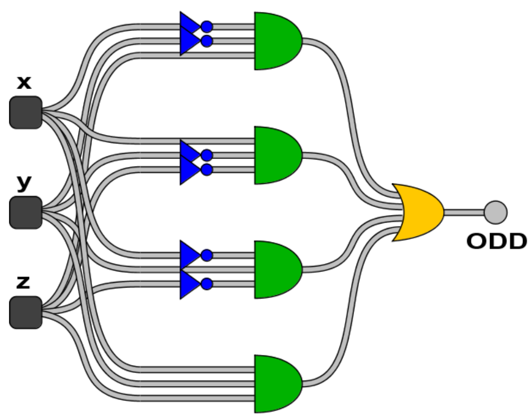
Expressing ODD Using Sum-of-Products

x	y	z	ODD	x'y'z	x'yz'	xy'z'	xyz	x'y'z + x'yz' + xy'z' + xyz
0	0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0	1
0	1	0	1	0	1	0	0	1
0	1	1	0	0	0	0	0	0
1	0	0	1	0	0	1	0	1
1	0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	1	1	1	0	0	0	1	1

ODD Parity Circuit

ODD(x, y, z).

- 1 if odd number of inputs are 1.
- 0 otherwise.



Let's Make an Adder Circuit

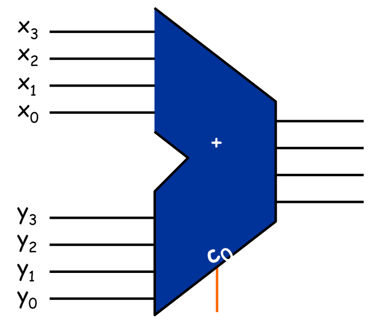
Goal: x + y = z for 4-bit integers.

- We build 4-bit adder: 9 inputs, 4 outputs.
- Same idea scales to 128-bit adder.
- Key computer component.

	1	1	1	0
	2	4	8	7
+	3	5	7	9
	6	0	6	6

Step 1.

- Represent input and output in binary.



	1	1	0	0
	0	0	1	0
+	0	1	1	1
	1	0	0	1

	c_3	c_2	c_1	c_0
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Let's Make an Adder Circuit

Goal: x + y = z for 4-bit integers.

				c_0
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Step 2. (first attempt)

- Build truth table.
- Why is this a bad idea?
  - 128-bit adder: 2^{256+1} rows > # electrons in universe!

4-Bit Adder Truth Table

c_0	x_3	x_2	x_1	x_0	y_3	y_2	y_1	y_0	z_3	z_2	z_1	z_0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	1	1	0	0	1	1
0	0	0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	0	1	0	1	0	1	0	1
.	.	.	.	.	.	.	.	.	.	.	.	.
1	1	1	1	1	1	1	1	1	1	1	1	1

2^{8+1} = 512 rows!

Let's Make an Adder Circuit

Goal: x + y = z for 4-bit integers.

	c_3	c_2	c_1	$c_0 = 0$
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Step 2. (do one bit at a time)

- Build truth table for carry bit.
- Build truth table for summand bit.

Carry Bit

x_i	y_i	c_i	c_{i+1}
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Summand Bit

x_i	y_i	c_i	z_i
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Let's Make an Adder Circuit

Goal: $x + y = z$ for 4-bit integers.

Step 3.

- Derive (simplified) Boolean expression.

	c_3	c_2	c_1	$c_0 = 0$
	x_3	x_2	x_1	x_0
+	y_3	y_2	y_1	y_0
	z_3	z_2	z_1	z_0

Carry Bit				
x_i	y_i	c_i	c_{i+1}	MAJ
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

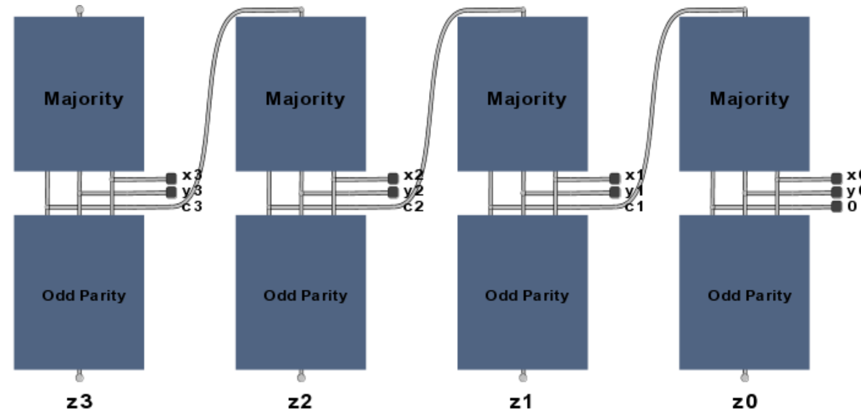
Summand Bit				
x_i	y_i	c_i	z_i	ODD
0	0	0	0	0
0	0	1	1	1
0	1	0	1	1
0	1	1	0	0
1	0	0	1	1
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

Let's Make an Adder Circuit

Goal: $x + y = z$ for 4-bit integers.

Step 4.

- Transform Boolean expression into circuit.
- Chain together 1-bit adders.



Let's Make an Adder Circuit

Goal: $x + y = z$ for 4-bit integers.

Step 4.

- Transform Boolean expression into circuit.
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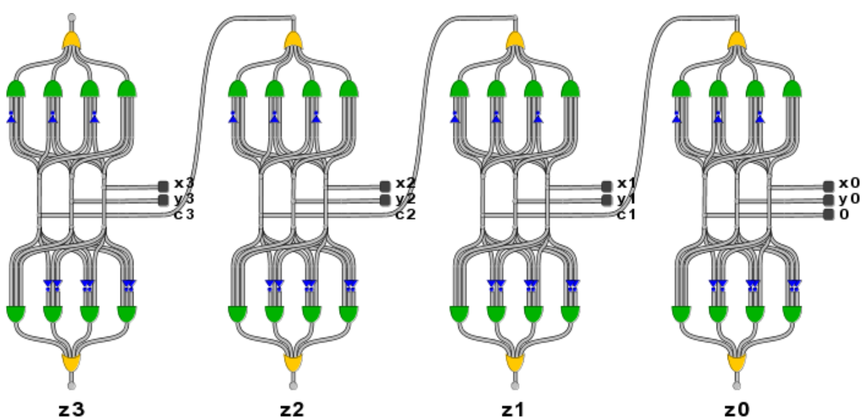
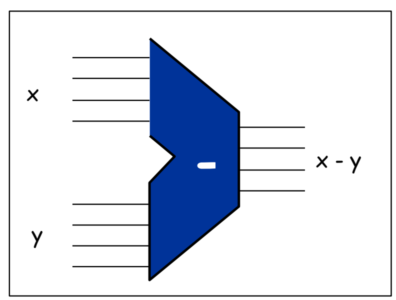


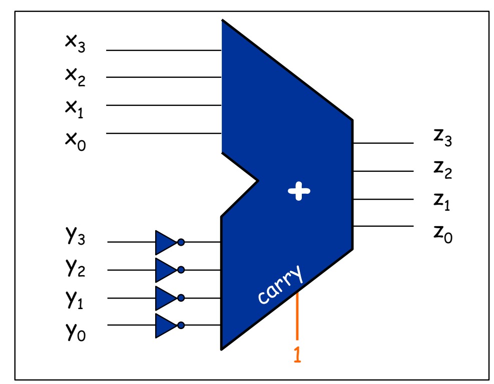
Subtractor

Subtractor circuit: $z = x - y$.

- One approach: new design, like adder circuit.
- Better idea: reuse adder circuit.
  - 2's complement: to negate an integer, flip bits, then add 1



4-Bit Subtractor Interface



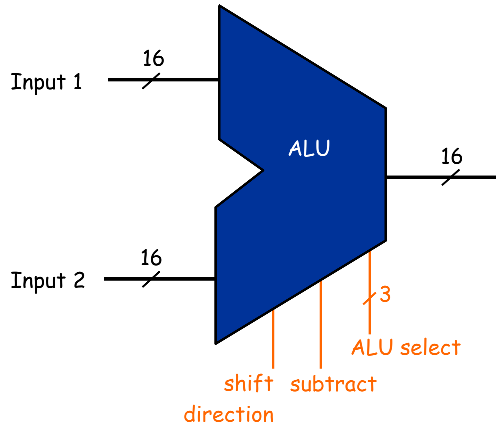
4-Bit Subtractor Implementation

TOY Arithmetic Logic Unit: Interface

ALU Interface.

- Add, subtract, bitwise and, bitwise xor, shift left, shift right, copy.
- Associate 3-bit integer with 5 primary ALU operations.
  - ALU performs operations in parallel
  - control wires select which result ALU outputs

op	2	1	0
+, -	0	0	0
&	0	0	1
^	0	1	0
<<, >>	0	1	1
input 2	1	0	0

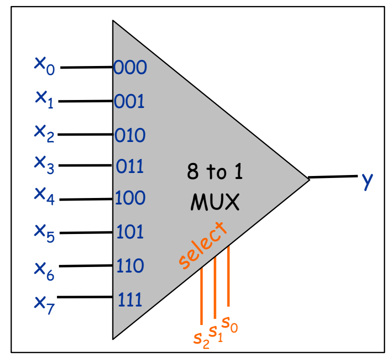


2^n-to-1 Multiplexer

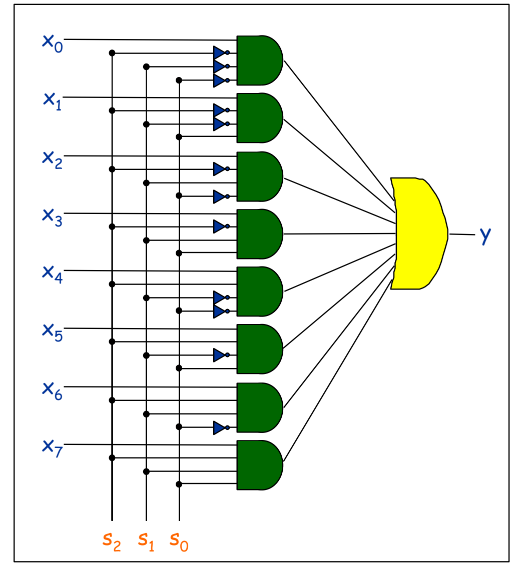
n = 8 for main memory

2^n-to-1 multiplexer.

- n select inputs, 2^n data inputs, 1 output.
- Copies "selected" data input bit to output.

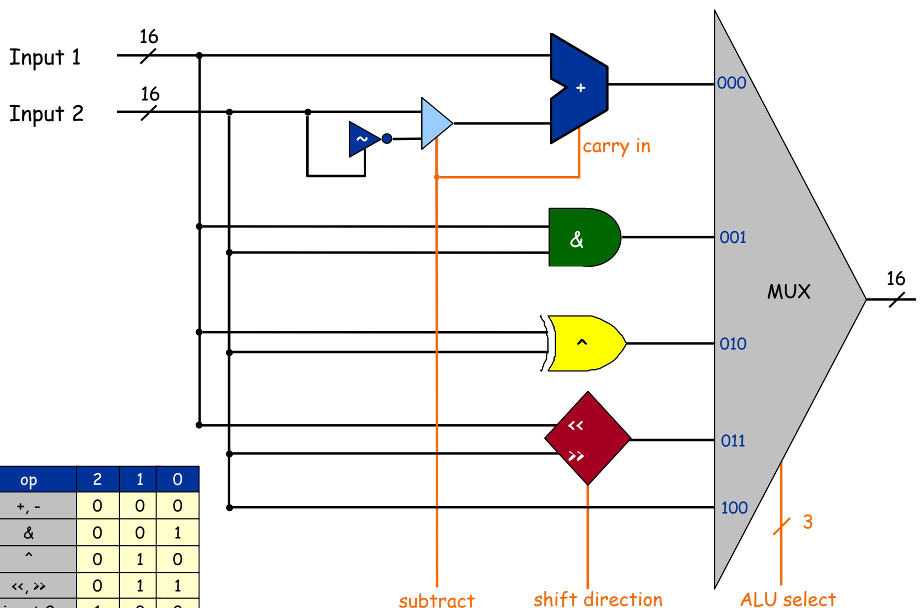


8-to-1 Mux Interface



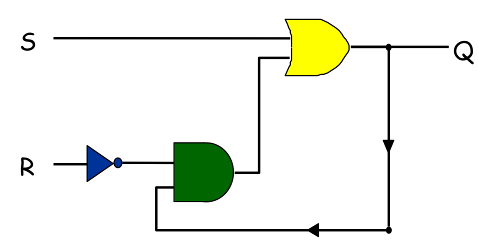
8-to-1 Mux Implementation

TOY Arithmetic Logic Unit: Implementation



op	2	1	0
+, -	0	0	0
&	0	0	1
^	0	1	0
<<, >>	0	1	1
input 2	1	0	0

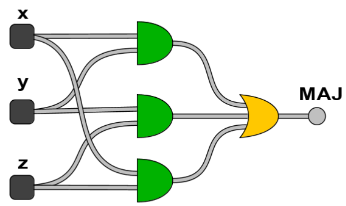
6.2: Sequential Circuits



Sequential vs. Combinational Circuits

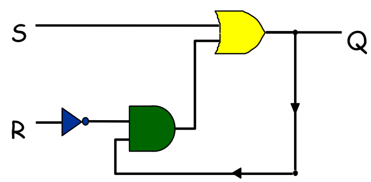
Combinational circuits.

- Output determined solely by inputs.
- Can draw solely with left-to-right signal paths.



Sequential circuits.

- Output determined by inputs AND previous outputs.
- Feedback loop.



Flip-Flop

Flip-flop.

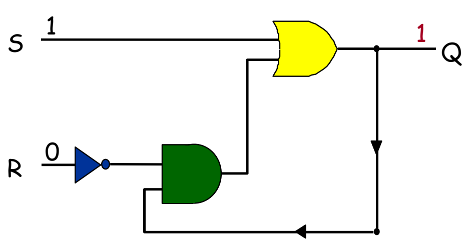
- A small and useful sequential circuit.
- Abstraction that "remembers" one bit.
- Basis of important computer components:
  - memory
  - counter

We will consider several flavors.

SR Flip-Flop

What is the value of Q if:

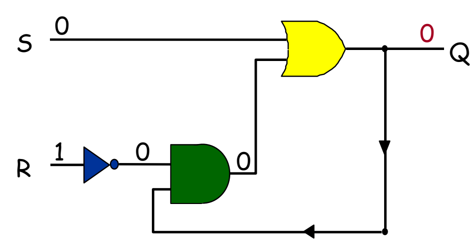
- S = 1 and R = 0 ? ⇒ Q is surely 1



SR Flip-Flop

What is the value of Q if:

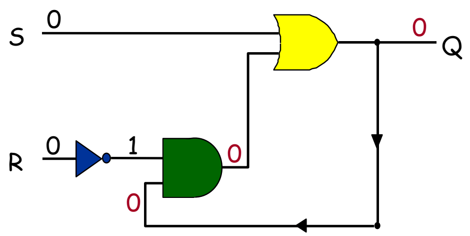
- S = 1 and R = 0 ? ⇒ Q is surely 1
- S = 0 and R = 1 ? ⇒ Q is surely 0



SR Flip-Flop

What is the value of Q if:

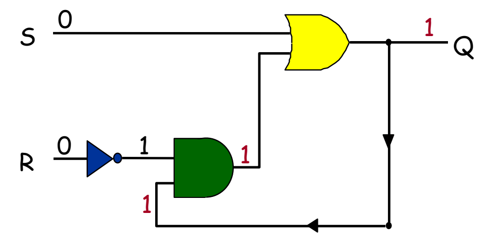
- S = 1 and R = 0 ? ⇒ Q is surely 1
- S = 0 and R = 1 ? ⇒ Q is surely 0
- S = 0 and R = 0 ? ⇒ Q is possibly 0



SR Flip-Flop

What is the value of Q if:

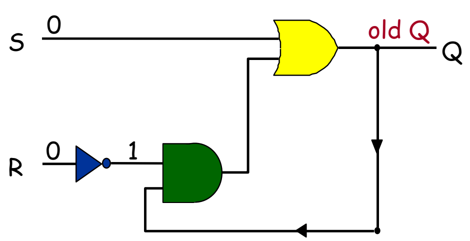
- S = 1 and R = 0 ? ⇒ Q is surely 1
- S = 0 and R = 1 ? ⇒ Q is surely 0
- S = 0 and R = 0 ? ⇒ Q is possibly 0 . . . or possibly 1 !



SR Flip-Flop

What is the value of Q if:

- S = 1 and R = 0 ? ⇒ Q is surely 1
- S = 0 and R = 1 ? ⇒ Q is surely 0
- S = 0 and R = 0 ? ⇒ Q is possibly 0 . . . or possibly 1 !

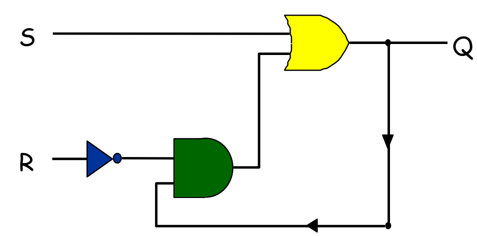


While S = R = 0 , Q remembers what it was the last time S or R was 1.

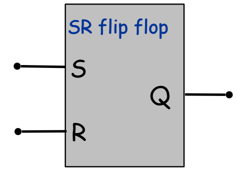
SR Flip-Flop

SR Flip-Flop.

- S = 1, R = 0 (Set) ⇒ "Flips" bit on.
- S = 0, R = 1 (Reset) ⇒ "Flops" bit off.
- S = R = 0 ⇒ Status quo.
- S = R = 1 ⇒ Not allowed.



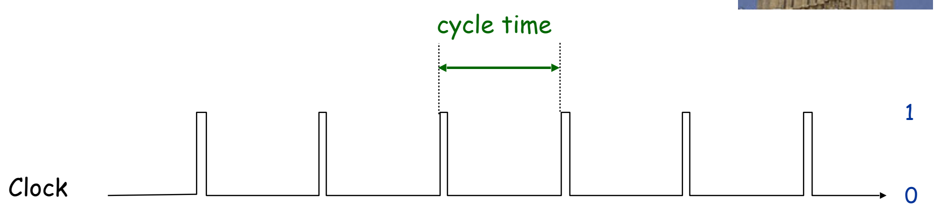
Implementation



Interface

Clock

Clock.
- Fundamental abstraction.
  - regular on-off pulse
- From some oscillating device, possibly external.
- Synchronizes operations of different circuit elements.
- 1 GHz clock means 1 billion pulses per second.



How much does it Hert?

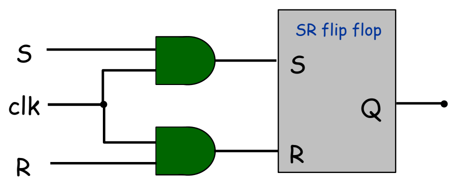
Frequency is inverse of cycle time.
- Expressed in hertz.
- Frequency of 1 Hz means that there is 1 cycle per second.
- Hence:
  - 1 kilohertz (kHz) means 1000 cycles/sec.
  - 1 megahertz (MHz) means 1 million cycles/sec.
  - 1 gigahertz (GHz) means 1 billion cycles/sec.
  - 1 terahertz (THz) means 1 trillion cycles/sec.

By the way, no such thing as 1 "hert" !

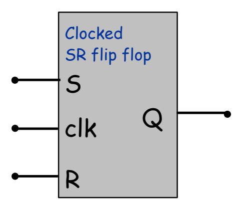
Heinrich Rudolf Hertz
(1857-1894)

Clocked SR Flip-Flop

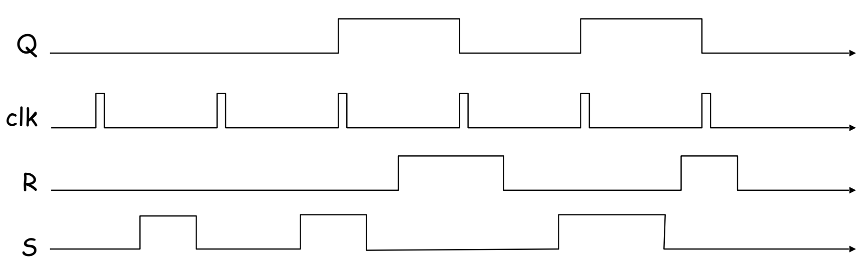
Clocked SR Flip-Flop.
- Same as SR flip-flop except S and R only active when clock is 1.



Implementation

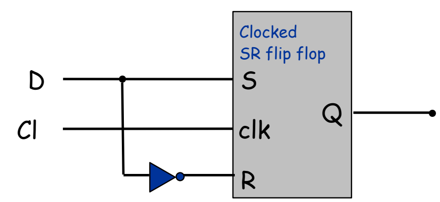


Interface

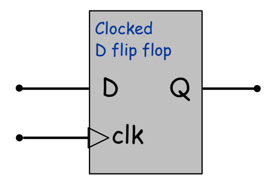


Clocked D Flip-Flop

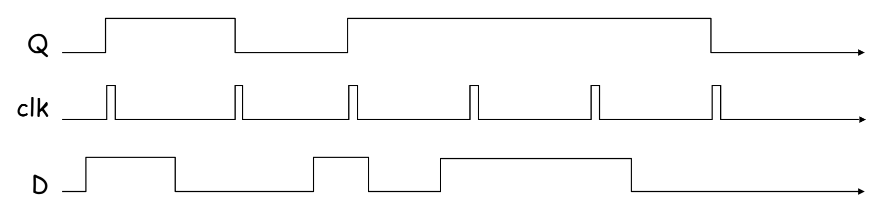
Clocked D Flip-Flop.
- Output follows D input while clock is 1.
- Output is remembered while clock is 0.



Implementation



Interface

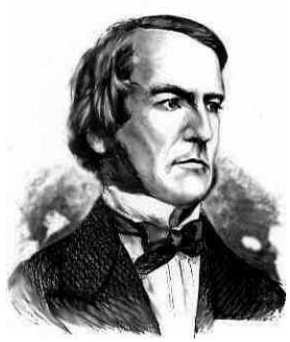


Summary

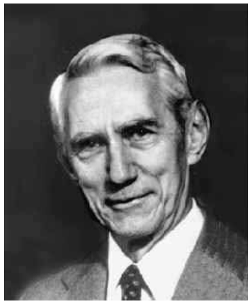
Combinational circuits implement Boolean functions
- Gates and wires   Fundamental building blocks.
- Truth tables.   Describe Boolean functions.
- Sum-of-products.   Systematic method to implement functions.

Sequential circuits add "state" to digital hardware.
- Flip-flop.   Represents 1 bit.
- TOY register.   16 D flip-flops.
- TOY main memory.   256 registers.

Next time: we build a complete TOY computer (oh yes).



George Boole (1815 – 1864)



Claude Shannon (1916 – 2001)