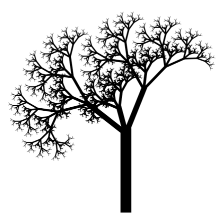
2.3 Recursion

Overview

What is recursion? When one function calls itself directly or indirectly.



Why learn recursion?

- New mode of thinking.
- Powerful programming paradigm.

Many computations are naturally self-referential.

- Binary search, mergesort, FFT, GCD.
- Linked data structures.
- A folder contains files and other folders.

Closely related to mathematical induction.

M. C. Escher, 1956

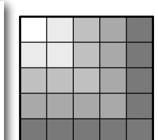
Mathematical Induction

Mathematical induction. Prove a statement involving an integer N by

- base case: Prove it for some specific N (usually 0 or 1).
- induction step: Assume it to be true for all positive integers less than N, use that fact to prove it for N.

Ex. Sum of the first N odd integers is N^2.

1 = 1
1 + 3 = 4
1 + 3 + 5 = 9
1 + 3 + 5 + 7 = 16
1 + 3 + 5 + 7 + 9 = 25
... ...



Base case: True for N = 1.

Induction step:

- Let T(N) be the sum of the first N odd integers: 1 + 3 + 5 + ... + (2N - 1).
- Assume that $T(N-1) = (N-1)^2$.
- $T(N) = T(N-1) + (2N - 1)$
  $= (N-1)^2 + (2N - 1)$
  $= N^2 - 2N + 1 + (2N - 1)$
  $= N^2$

Recursive Program

Recursive Program. Implement a function having integer arguments by

- base case: Do something specific in response to "base" argument values.
- reduction step: Assume the function works for all smaller argument values, and use the function to implement itself for general argument values.

```
public static String convert(int x)
{
   if (x == 1) return "1";
   return convert(x/2) + (x % 2);
}
```

automatic cast to String (either "0" or "1")

Ex 1. Convert positive `int` to binary `String`.

Base case: return "1" for x = 1.

Reduction step:

- convert x/2 to binary
- append "0" if x even
- append "1" if x odd

```
   37          18
"100101" = "10010" + "1"
```


Recursive Program

Recursive Program. Implement a function having integer arguments by

- base case: Implementing it for some specific values of the arguments.
- reduction step: Assume the function works for smaller values of its arguments and use it to implement the function for the given values.

```
public class Binary
{
   public static String convert(int x)
   {
      if (x == 1) return "1";
      return convert(x/2) + (x % 2);
   }

   public static void main(String[] args)
   {
      int x = Integer.parseInt(args[0]);
      System.out.println(convert(x));
   }
}
```

```
% java Binary 6
110
% java Binary 37
100101
% java Binary  999999
11110100001000111111
```


x = 6

environment

convert(6)

```
public static String convert(int x)
{
   if (x == 1) return "1";
   return convert(x / 2) + (x % 2);
}                 "110"
```

```
public class Binary
{
   public static String convert(int x)
   {
      if (x == 0) return "";
      return convert(x/2) + (x % 2);
   }

   public static void main(String[] args)
   {
      int x = Integer.parseInt(args[0]);
      System.out.println(convert(x));
                         "110"
   }
}
```

```
% java Binary 6
110
```

Recursion vs. Iteration

Every program with 1 recursive call corresponds to a loop.

```
public static String convert(int x)
{
   if (x == 1) return "1";
   return convert(x/2) + (x % 2);
}
```

```
public static String convertNR(int x)
{
   String s = "1";
   while (x > 1)
   {
      s = (x % 2) + s;
      x = x/2;
   }
   return s;
}
```

Reasons to use recursion:

- code more compact
- easier to understand
- easier to reason about correctness
- easy to add multiple recursive calls (stay tuned)

Reasons not to use recursion: (stay tuned)

Greatest Common Divisor

Gcd. Find largest integer that evenly divides into p and q.

Ex. gcd(4032, 1272) = 24.

$$4032 = 2^6 \times 3^2 \times 7^1$$
$$1272 = 2^3 \times 3^1 \times 53^1$$
$$\text{gcd} = 2^3 \times 3^1 = 24$$

Applications.

- Simplify fractions: 1272/4032 = 53/168.
- RSA cryptosystem.

Greatest Common Divisor

GCD. Find largest integer that evenly divides into p and q.

Euclid's algorithm. [Euclid 300 BCE]

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \ \% \ q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case

```
gcd(4032, 1272)    = gcd(1272, 216)
                   = gcd(216, 192)
                   = gcd(192, 24)
                   = gcd(24, 0)
                   = 24.
```

4032 = 3 × 1272 + 216

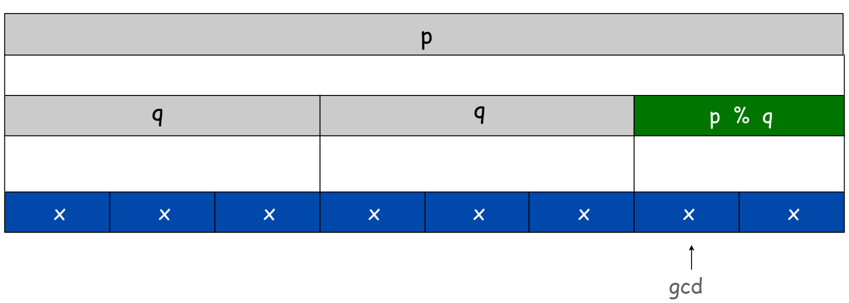
Euclid's Algorithm

GCD. Find largest integer d that evenly divides into p and q.

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \ \% \ q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case



p = 8x
q = 3x

gcd(p, q) = gcd(3x, 2x) = x

Euclid's Algorithm

GCD. Find largest integer d that evenly divides into p and q.

$$\gcd(p, q) = \begin{cases} p & \text{if } q = 0 \\ \gcd(q,\ p \ \% \ q) & \text{otherwise} \end{cases}$$

← base case

← reduction step, converges to base case

Recursive program

```
public static int gcd(int p, int q)
{
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

← base case

← reduction step

p = 1272, q = 216

environment

gcd(1272, 216)

```
static int gcd(int p, int q)
{
   if (q == 0) return p;
   else return gcd(q, p % q);
}
```

24

```
public class Euclid
{
   public static int gcd(int p, int q)
   {
      if (q == 0) return p;
      else return gcd(q, p % q);
   }
   public static void main(String[] args)
   {
      int p = Integer.parseInt(args[0]);
      int q = Integer.parseInt(args[1]);
      System.out.println(gcd(p, q));
   }
}
```

24

```
% java Euclid 1272 216
24
```

Possible debugging challenges with recursion

Missing base case.

```
public static double BAD(int N)
{
   return BAD(N-1) + 1.0/N;
}
```

No convergence guarantee.

```
public static double BAD(int N)
{
   if (N == 1) return 1.0;
   return BAD(1 + N/2) + 1.0/N;
}
```

Both lead to INFINITE RECURSIVE LOOP (bad news).

Try it! ← so that you can recognize and deal with it if it later happens to you

Collatz Sequence

Collatz sequence.

- If n is 1, stop.
- If n is even, divide by 2.
- If n is odd, multiply by 3 and add 1.

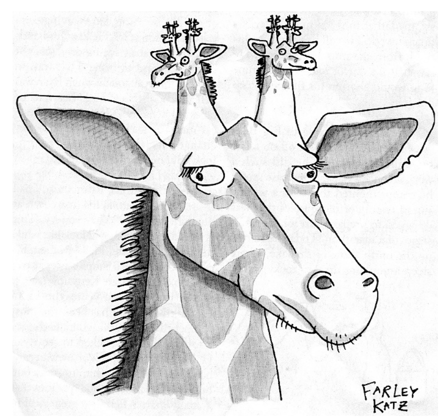
Ex. 35 106 53 160 80 40 20 10 5 16 8 4 2 1.

```
public static void collatz(int N)
{
    StdOut.print(N + " ");
    if (N == 1) return;
    if (N % 2 == 0) collatz(N / 2);
    else collatz(3 * N + 1);
 }
```

No one knows whether or not this function terminates for all N (!)

[usually we decrease N for all recursive calls]

Recursive Graphics



New Yorker Magazine, August 11, 2008


WEEKEND Arts FINE ARTS LEISURE

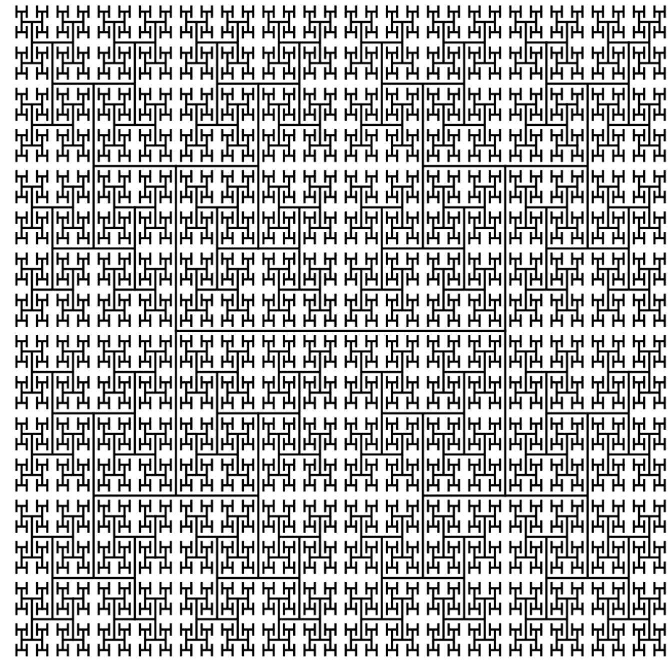
The New York Times

Fruits of Design, Certified Organic

The Gifts to Open Again and Again

Black, White and Read All Over Over

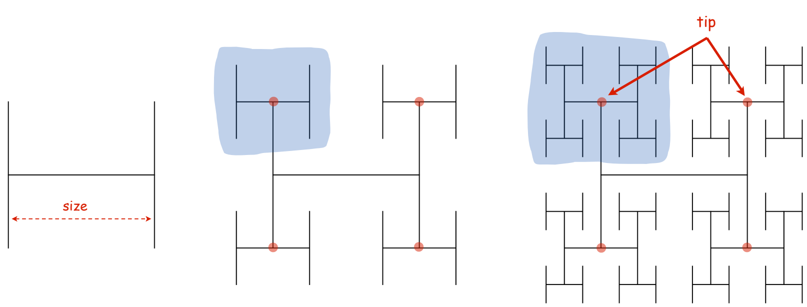
Divine and Devotee Meet Across Hinges



Htree

H-tree of order n.

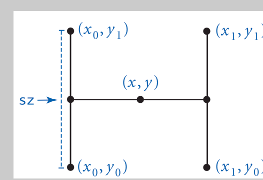
- Draw an H.
- Recursively draw 4 H-trees of order n-1, one connected to each tip. ← and half the size



order 1 order 2 order 3

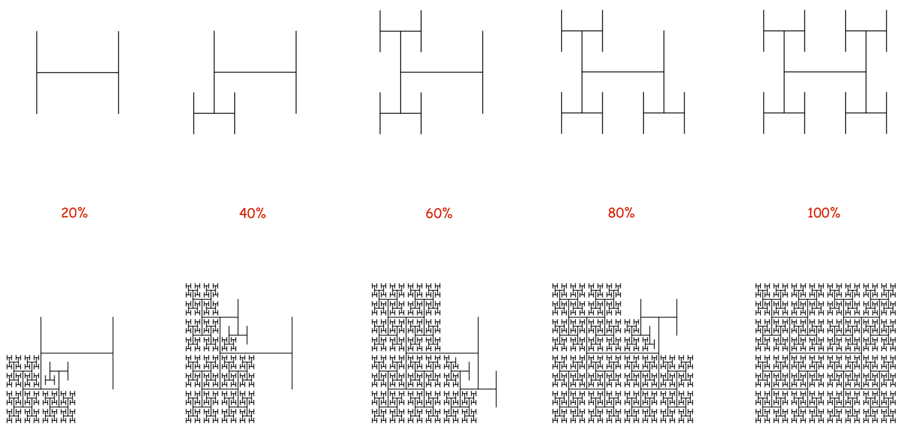
Htree in Java

```
public class Htree
{
   public static void draw(int n, double sz, double x, double y)
   {
      if (n == 0) return;
      double x0 = x - sz/2, x1 = x + sz/2;
      double y0 = y - sz/2, y1 = y + sz/2;

      StdDraw.line(x0,  y, x1,  y);
      StdDraw.line(x0, y0, x0, y1);   ← draw the H, centered on (x, y)
      StdDraw.line(x1, y0, x1, y1);

      draw(n-1, sz/2, x0, y0);
      draw(n-1, sz/2, x0, y1);        ← recursively draw 4 half-size Hs
      draw(n-1, sz/2, x1, y0);
      draw(n-1, sz/2, x1, y1);
   }

   public static void main(String[] args)
   {
      int n = Integer.parseInt(args[0]);
      draw(n, .5, .5, .5);
   }
}
```

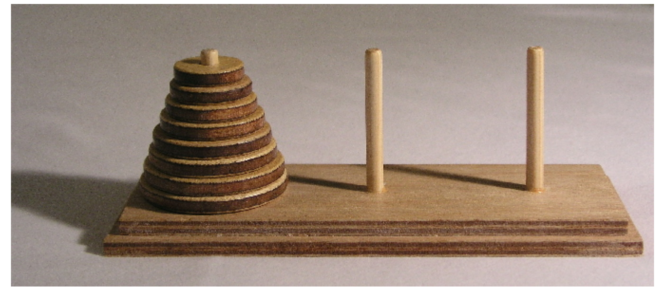


Animated H-tree

Animated H-tree. Pause after drawing each H.



Towers of Hanoi

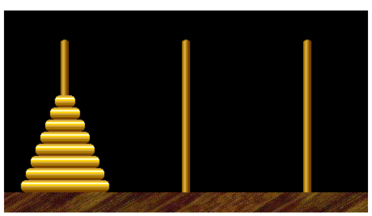


http://en.wikipedia.org/wiki/Image:Hanoiklein.jpg

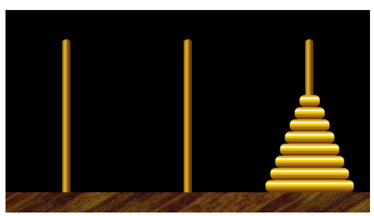
Towers of Hanoi

Move all the discs from the leftmost peg to the rightmost one.

- Only one disc may be moved at a time.
- A disc can be placed either on empty peg or on top of a larger disc.



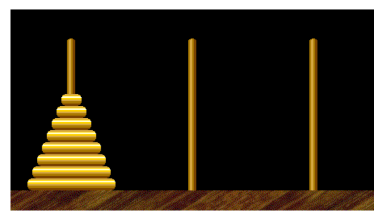
start



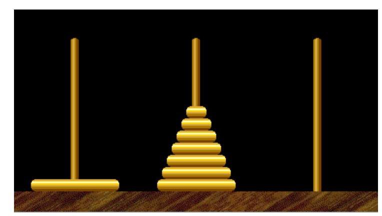
finish

Edouard Lucas (1883)

Towers of Hanoi: Recursive Solution

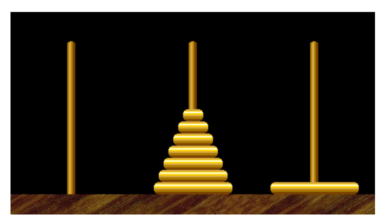


Move n-1 smallest discs right.

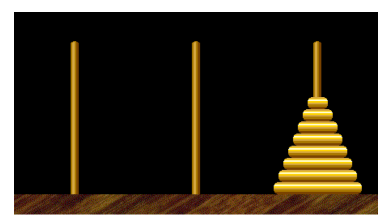


Move largest disc left.

cyclic wrap-around



Move n-1 smallest discs right.



Towers of Hanoi Legend

Q. Is world going to end (according to legend)?

- 64 golden discs on 3 diamond pegs.
- World ends when certain group of monks accomplish task.

Q. Will computer algorithms help?

Towers of Hanoi: Recursive Solution

```
public class TowersOfHanoi
{
   public static void moves(int n, boolean left)
   {
      if (n == 0) return;
      moves(n-1, !left);
      if (left) System.out.println(n + " left");
      else      System.out.println(n + " right");
      moves(n-1, !left);
   }

   public static void main(String[] args)
   {
      int N = Integer.parseInt(args[0]);
      moves(N, true);
   }

}
```

`moves(n, true)` : move discs 1 to n one pole to the left
`moves(n, false)`: move discs 1 to n one pole to the right

smallest disc

Towers of Hanoi: Recursive Solution

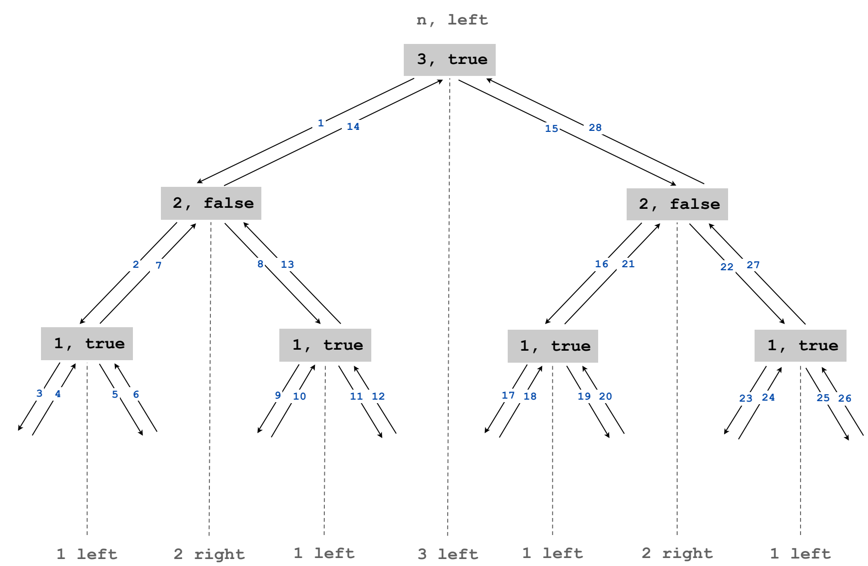
```
% java TowersOfHanoi 3
1 left
2 right
1 left
3 left
1 left
2 right
1 left
```

```
% java TowersOfHanoi 4
1 right
2 left
1 right
3 right
1 right
2 left
1 right
4 left
1 right
2 left
1 right
3 right
1 right
2 left
1 right
```

every other move is smallest disc

subdivisions of ruler

Towers of Hanoi: Recursion Tree



Towers of Hanoi: Properties of Solution

Remarkable properties of recursive solution.

- Takes 2^n - 1 moves to solve n disc problem.
- Sequence of discs is same as subdivisions of ruler.
- Every other move involves smallest disc.

Recursive algorithm yields non-recursive solution!

- Alternate between two moves:
  - move smallest disc to right if n is even (to left if n is odd)
  - make only legal move not involving smallest disc

Recursive algorithm may reveal fate of world.

- Takes 585 billion years for n = 64 (at rate of 1 disc per second).
- Reassuring fact: any solution takes at least this long!

Divide-and-Conquer

Divide-and-conquer paradigm.

- Break up problem into smaller subproblems of same structure.
- Solve subproblems recursively using same method.
- Combine results to produce solution to original problem.

Many important problems succumb to divide-and-conquer.

- Midpoint displacement method for fractional Brownian motion.
- FFT for signal processing.
- Parsers for programming languages.
- Multigrid methods for solving PDEs.
- Quicksort and mergesort for sorting.
- Hilbert curve for domain decomposition.
- Quad-tree for efficient N-body simulation.

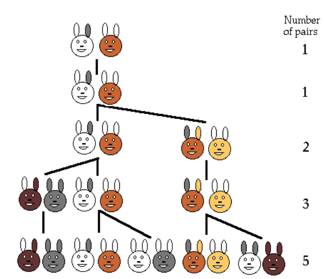
Divide et impera. Veni, vidi, vici. - Julius Caesar

Fibonacci Numbers

Fibonacci Numbers

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

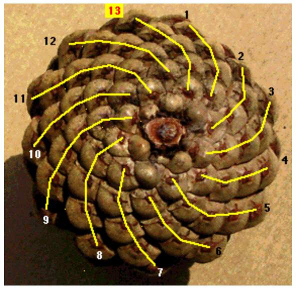
$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$



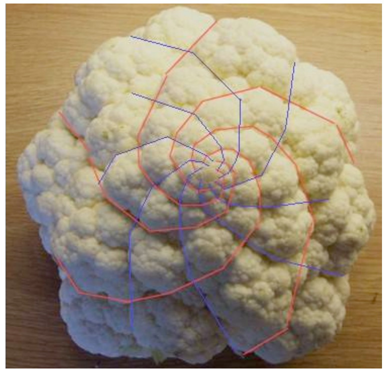
Fibonacci rabbits


L. P. Fibonacci
(1170 - 1250)

Fibonacci Numbers



pinecone



cauliflower

see much, much more at www.youtube.com/user/Vihart

A Possible Pitfall With Recursion

Fibonacci numbers. 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

FYI (classical math):

$$F(n) = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \left\lfloor \phi^n / \sqrt{5} \right\rceil$$

ϕ = golden ratio $\approx$ 1.618

Ex: F(50) $\approx 1.2 \times 10^{10}$

A natural for recursion?

```
public static long F(int n)
{
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```


Recursion Challenge 1 (difficult but important)

Is this an efficient way to compute F(50)?

```
public static long F(int n)
{
   if (n == 0) return 0;
   if (n == 1) return 1;
   return F(n-1) + F(n-2);
}
```


Recursion Challenge 2 (easy and also important)

Is this an efficient way to compute F(50)?

```
long[] F = new long[51];
F[0] = 0; F[1] = 1;
if (n == 1) return 1;
for (int i = 2; i <= 50; i++)
   F[i] = F[i-1] + F[i-2];
```

Summary

How to write simple recursive programs?
- Base case, reduction step.
- Trace the execution of a recursive program.
- Use pictures.

Why learn recursion?
- New mode of thinking.
- Powerful programming tool.

Divide-and-conquer. Elegant solution to many important problems.

Exponential time.
- Easy to specify recursive program that takes exponential time.
- Don't do it unless you plan to (and are working on a small problem).