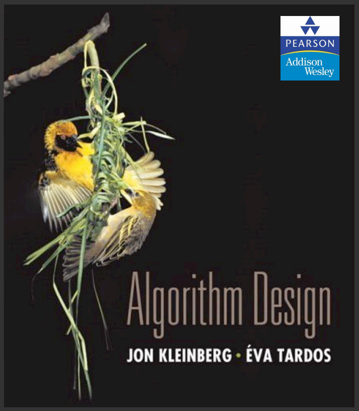


Intractability III

- *special cases*
- *approximation algorithms*
- *exact exponential algorithms*

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

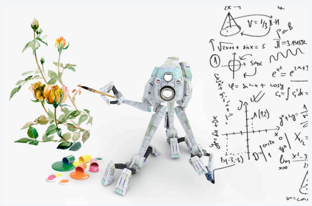
Last updated on May 9, 2013 9:04 AM

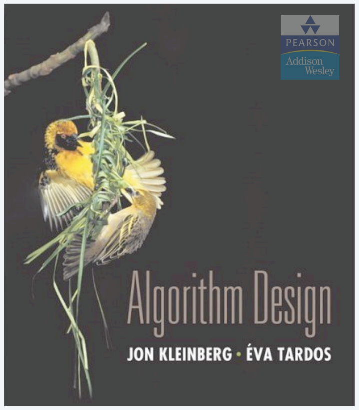
Coping with NP-completeness

Q. Suppose I need to solve an **NP**-hard problem. What should I do?

A. Sacrifice one of three desired features.
i. Solve arbitrary instances of the problem.
ii. Solve problem to optimality.
iii. Solve problem in polynomial time.

Coping strategies.
i. Design algorithms for special cases of the problem.
ii. Design approximation algorithms or heuristics.
iii. Design algorithms that may take exponential time.





Section 10.2

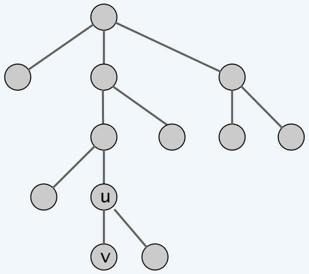
Intractability III

- *special cases: trees*
- *approximation algorithms*
- *exact exponential algorithms*

Independent set on trees

Independent set on trees. Given a tree, find a maximum cardinality subset of nodes such that no two are adjacent.

Fact. A tree has at least one node that is a leaf (degree = 1).



Key observation. If node v is a leaf, there exists a max cardinality independent set containing v.

Pf. [exchange argument]
- Consider a max cardinality independent set S.
- If $v \in S$, we're done.
- Let (u, v) be some edge.
  - if $u \notin S$ and $v \notin S$, then $S \cup \{ v \}$ is independent $\Rightarrow$ S not maximum
  - if $u \in S$ and $v \notin S$, then $S \cup \{ v \} - \{ u \}$ is independent ▪

Independent set on trees: greedy algorithm

Theorem. The following greedy algorithm finds a max cardinality independent set in forests (and hence trees).

Pf. Correctness follows from the previous key observation. ▪

INDEPENDENT-SET-IN-A-FOREST (F)

$S \leftarrow \varnothing$.
WHILE (F has at least 1 edge)
  $e \leftarrow (u, v)$ such that v is a leaf.
  $S \leftarrow S \cup \{ v \}$.
  $F \leftarrow F - \{ u, v \}$. ⟵ delete u and v and all incident edges
RETURN S.

Remark. Can implement in $O(n)$ time by considering nodes in postorder.

Weighted independent set on trees

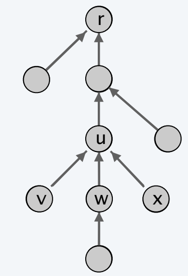
Weighted independent set on trees. Given a tree and node weights $w_v > 0$, find an independent set S that maximizes $\Sigma_{v \in S}\, w_v$.

Dynamic programming solution. Root tree at some node, say r.

- $OPT_{in}(u)$ = max weight independent set of subtree rooted at u, containing u.
- $OPT_{out}(u)$ = max weight independent set of subtree rooted at u, not containing u.
- $OPT = \max \{ OPT_{in}(r),\ OPT_{out}(r) \}$.

$$OPT_{in}(u) \;=\; w_u + \sum_{v \in \text{children}(u)} OPT_{out}(v)$$

$$OPT_{out}(u) \;=\; \sum_{v \in \text{children}(u)} \max \{OPT_{in}(v),\ OPT_{out}(v)\}$$



children(u) = { v, w, x }

Weighted independent set on trees: dynamic programming algorithm

Theorem. The dynamic programming algorithm finds a max weighted independent set in a tree in $O(n)$ time.

can also find independent set itself (not just value)

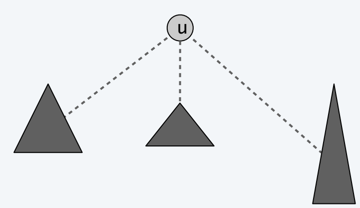
WEIGHTED-INDEPENDENT-SET-IN-A-TREE (T)

Root the tree T at a node r.
$S \leftarrow \varnothing$.
FOREACH (node u of T in postorder)
  IF (u is a leaf)
    $M_{in}[u] = w_u$.
    $M_{out}[u] = 0$.
  ELSE
    $M_{in}[u] \;= w_u + \Sigma_{v \in \text{children}(u)}\, M_{out}[v]$.
    $M_{out}[u] = \Sigma_{v \in \text{children}(u)} \max\,(M_{in}[v], M_{out}[v])$.
RETURN $\max\,(M_{in}[r], M_{out}[r])$.

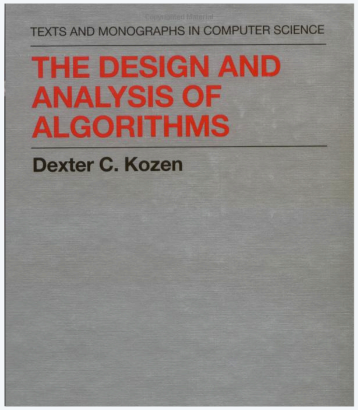
ensures a node is visited after all its children

NP-hard problems on trees: context

Independent set on trees. Tractable because we can find a node that breaks the communication among the subproblems in different subtrees.



Linear-time on trees. VERTEX-COVER, DOMINATING-SET, GRAPH-ISOMORPHISM, ...



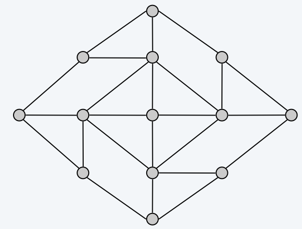
SECTION 23.1

INTRACTABILITY III

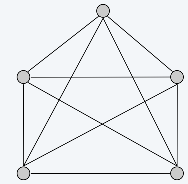
- *special cases: planarity*
- *approximation algorithms*
- *exact exponential algorithms*

Planarity

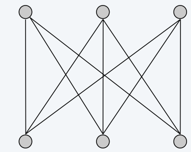
Def. A graph is planar if it can be embedded in the plane in such a way that no two edges cross.



planar



K_5 is nonplanar



$K_{3,3}$ is nonplanar

Applications. VLSI circuit design, computer graphics, ...

Planarity testing

Theorem. [Hopcroft-Tarjan 1974] There exists an $O(n)$ time algorithm to determine whether a graph is planar.

simple planar graph
has at ≤ 3n edges

Efficient Planarity Testing

JOHN HOPCROFT AND ROBERT TARJAN

Cornell University, Ithaca, New York

ABSTRACT. This paper describes an efficient algorithm to determine whether an arbitrary graph G can be embedded in the plane. The algorithm may be viewed as an iterative version of a method originally proposed by Auslander and Parter and correctly formulated by Goldstein. The algorithm uses depth-first search and has $O(V)$ time and space bounds, where V is the number of vertices in G. An ALGOL implementation of the algorithm successfully tested graphs with as many as 900 vertices in less than 12 seconds.

Polynomial time detour

Graph minor theorem. [Robertson-Seymour 1980s]
Pf of theorem. Tour de force.

Corollary. There exist an $O(n^3)$ algorithm to determine if a graph can be embedded in the torus in such a way that no two edges cross.

more than 2↑2↑2↑(n/2)

Mind boggling fact 1. The proof is highly nonconstructive!
Mind boggling fact 2. The constant of proportionality is enormous!

> " *Unfortunately, for any instance G = (V, E) that one could fit into the known universe, one would easily prefer n^{70} to even constant time, if that constant had to be one of Robertson and Seymour's.* " — *David Johnson*

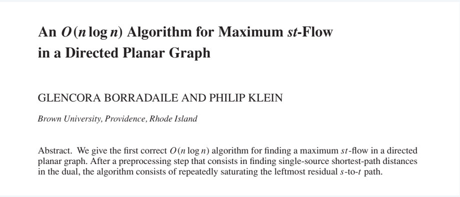
Theorem. There exists an explicit $O(n)$ algorithm.
Practice. LEDA implementation guarantees $O(n^3)$.

Problems on planar graphs

Fact 0. Many graph problems can be solved faster in planar graphs.
Ex. Shortest paths, max flow, MST, matchings, …

Fact 1. Some **NP**-complete problems become tractable in planar graphs.
Ex. MAX-CUT, ISING, CLIQUE, GRAPH-ISOMORPHISM, 4-COLOR, …

Fact 2. Other **NP**-complete problems become easier in planar graphs.
Ex. INDEPENDENT-SET, VERTEX-COVER, TSP, STEINER-TREE, …

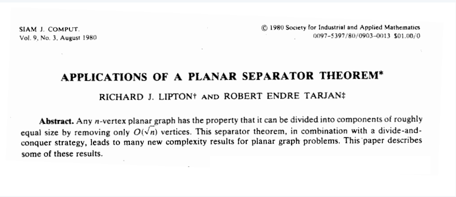


An $O(n \log n)$ Algorithm for Maximum *st*-Flow in a Directed Planar Graph

GLENCORA BORRADAILE AND PHILIP KLEIN

Brown University, Providence, Rhode Island

Abstract. We give the first correct $O(n \log n)$ algorithm for finding a maximum st-flow in a directed planar graph. After a preprocessing step that consists in finding single-source shortest-path distances in the dual, the algorithm consists of repeatedly saturating the leftmost residual s-to-t path.



SIAM J. COMPUT. © 1980 Society for Industrial and Applied Mathematics
Vol. 9, No. 3, August 1980 0097-5397/80/0903-0013 $01.00/0

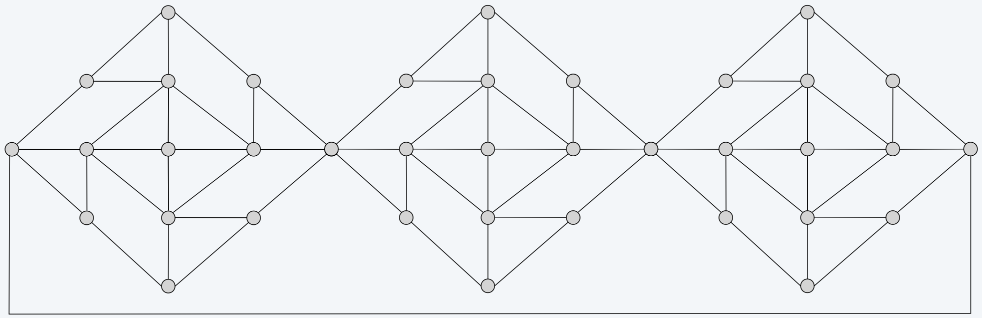
APPLICATIONS OF A PLANAR SEPARATOR THEOREM*

RICHARD J. LIPTON† AND ROBERT ENDRE TARJAN‡

Abstract. Any n-vertex planar graph has the property that it can be divided into components of roughly equal size by removing only $O(\sqrt{n})$ vertices. This separator theorem, in combination with a divide-and-conquer strategy, leads to many new complexity results for planar graph problems. This paper describes some of these results.

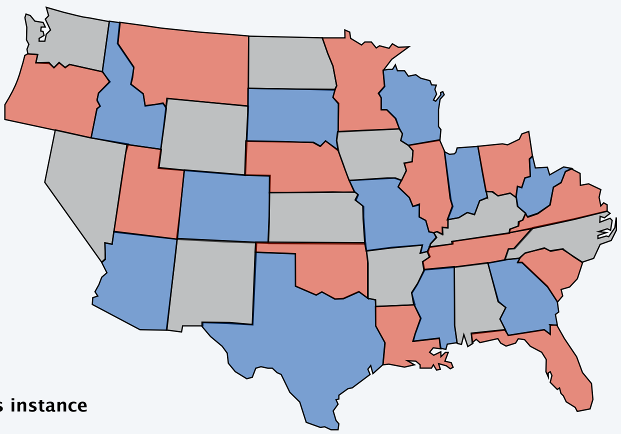
Planar graph 3-colorability

PLANAR-3-COLOR. Given a planar graph, can it be colored using 3 colors so that no two adjacent nodes have the same color?



Planar map 3-colorability

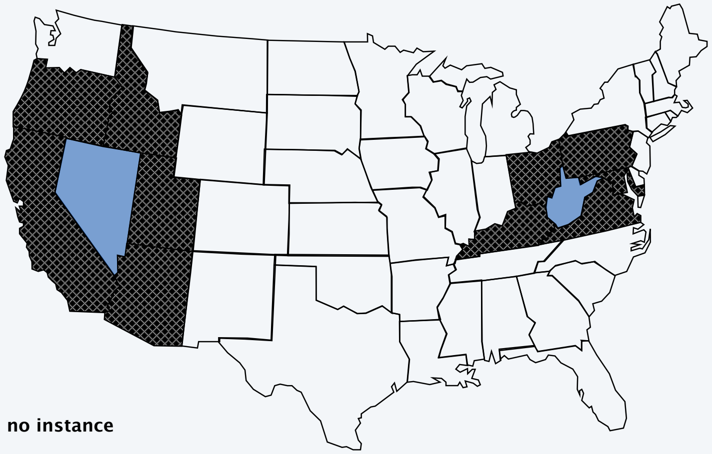
PLANAR-MAP-3-COLOR. Given a planar map, can it be colored using 3 colors so that no two adjacent regions have the same color?



yes instance

Planar map 3-colorability

PLANAR-MAP-3-COLOR. Given a planar map, can it be colored using 3 colors so that no two adjacent regions have the same color?



no instance

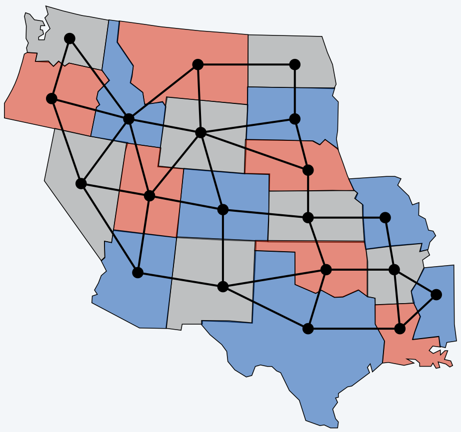
Planar graph and map 3-colorability reduce to one another

Theorem. PLANAR-3-COLOR $\equiv_P$ PLANAR-MAP-3-COLOR.

Pf sketch.

- Nodes correspond to regions.
- Two nodes are adjacent iff they share a nontrivial border.

e.g., not Arizona and Colorado



Planar 3-colorability is NP-complete

Theorem. PLANAR-3-COLOR $\in$ **NP**-complete.

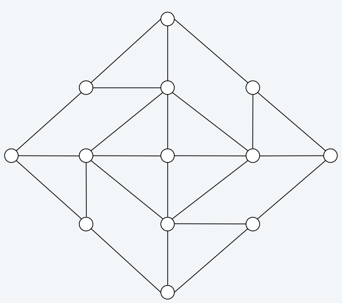
Pf.

- Easy to see that PLANAR-3-COLOR $\in$ **NP**.
- We show 3-COLOR $\le_P$ PLANAR-3-COLOR.
- Given 3-COLOR instance G, we construct an instance of PLANAR-3-COLOR that is 3-colorable iff G is 3-colorable.

Planar 3-colorability is NP-complete

Lemma. W is a planar graph such that:

- In any 3-coloring of W, opposite corners have the same color.
- Any assignment of colors to the corners in which opposite corners have the same color extends to a 3-coloring of W.



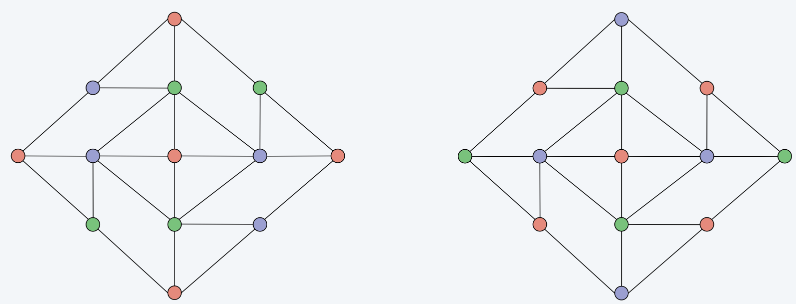
planar gadget W

Planar 3-colorability is NP-complete

Lemma. W is a planar graph such that:

- In any 3-coloring of W, opposite corners have the same color.
- Any assignment of colors to the corners in which opposite corners have the same color extends to a 3-coloring of W.

Pf. The only 3-colorings (modulo permutations) of W are shown below. ▪



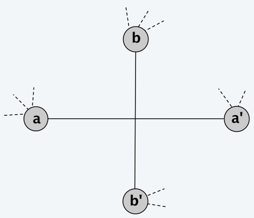
planar gadget W

Planar 3-colorability is NP-complete

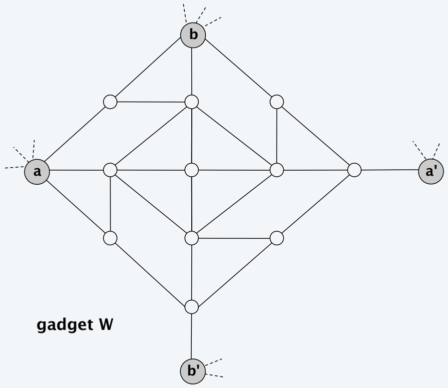
Construction. Given instance G of 3-COLOR, draw G in plane, letting edges cross. Form planar G' by replacing each edge crossing with planar gadget W.

Lemma. G is 3-colorable iff G' is 3-colorable.

- In any 3-coloring of W, $a \neq a'$ and $b \neq b'$.
- If $a \neq a'$ and $b \neq b'$ then can extend to a 3-coloring of W.



a crossing



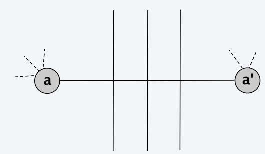
gadget W

Planar 3-colorability is NP-complete

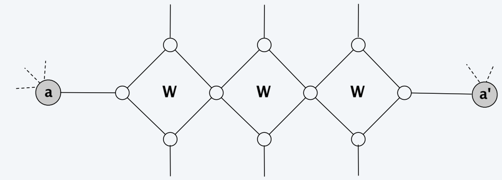
Construction. Given instance G of 3-COLOR, draw G in plane, letting edges cross. Form planar G' by replacing each edge crossing with planar gadget W.

Lemma. G is 3-colorable iff G' is 3-colorable.

- In any 3-coloring of W, $a \neq a'$ and $b \neq b'$.
- If $a \neq a'$ and $b \neq b'$ then can extend to a 3-coloring of W.



multiple crossings

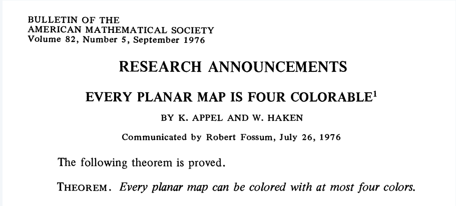


concatenate copies of gadget W

Planar map k-colorability

Theorem. [Appel-Haken 1976] Every planar map is 4-colorable.

- Resolved century-old open problem.
- Used 50 days of computer time to deal with many special cases.
- First major theorem to be proved using computer.


BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 5, September 1976

RESEARCH ANNOUNCEMENTS

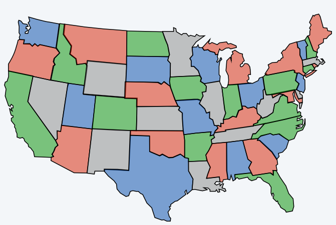
EVERY PLANAR MAP IS FOUR COLORABLE[1]

BY K. APPEL AND W. HAKEN

Communicated by Robert Fossum, July 26, 1976

The following theorem is proved.

THEOREM. *Every planar map can be colored with at most four colors.*



Remarks.

- Appel-Haken yields $O(n^4)$ algorithm to 4-color of a planar map.
- Best known: $O(n^2)$ to 4-color; $O(n)$ to 5-color.
- Determining whether 3 colors suffice is **NP**-complete.

Polynomial-time special cases NP-hard problems

Trees. VERTEX-COVER, INDEPENDENT-SET, DOMINATING-SET, GRAPH-ISOMORPHISM, ...

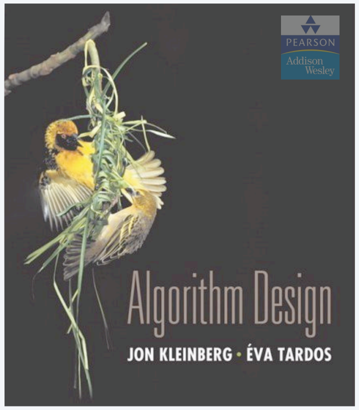
Bipartite graphs. VERTEX-COVER, 2-COLOR, ...

Chordal graphs. K-COLOR, CLIQUE, INDEPENDENT-SET, ...

Planar graphs. MAX-CUT, ISING, CLIQUE, GRAPH-ISOMORPHISM, 4-COLOR, ...

Bounded treewidth. 3-COLOR, HAM-CYCLE, INDEPENDENT-SET, GRAPH-ISOMORPHISM.

Small integers. KNAPSACK, PARTITION, SUBSET-SUM, ...



SECTION 11.8

INTRACTABILITY III

- special cases
- *approximation algorithms*
- exact exponential algorithms

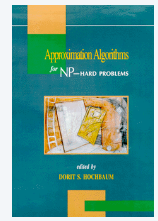
Approximation algorithms

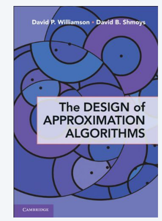
ρ-approximation algorithm.

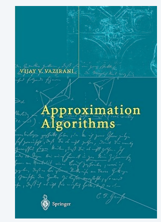
- Guaranteed to run in poly-time.
- Guaranteed to solve arbitrary instances of the problem.
- Guaranteed to find solution within ratio ρ of true optimum.

Ex. Given a graph G, the greedy algorithms finds a VERTEX-COVER that uses $\leq 2\ OPT(G)$ vertices in $O(m + n)$ time.

Challenge. Need to prove a solution's value is close to optimum value, without even knowing what optimum value is!







Knapsack problem

Knapsack problem.

- Given n objects and a knapsack.
- Item i has value $v_i > 0$ and weighs $w_i > 0$. ← we assume $w_i \leq W$ for each i
- Knapsack has weight limit W.
- Goal: fill knapsack so as to maximize total value.

Ex: $\{ 3, 4 \}$ has value 40.

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

original instance (W = 11)

Knapsack is NP-complete

KNAPSACK. Given a set X, weights $w_i \geq 0$, values $v_i \geq 0$, a weight limit W, and a target value V, is there a subset $S \subseteq X$ such that:

$$\sum_{i \in S} w_i \leq W$$
$$\sum_{i \in S} v_i \geq V$$

SUBSET-SUM. Given a set X, values $u_i \geq 0$, and an integer U, is there a subset $S \subseteq X$ whose elements sum to exactly U?

Theorem. SUBSET-SUM $\leq_P$ KNAPSACK.
Pf. Given instance $(u_1, \ldots, u_n, U)$ of SUBSET-SUM, create KNAPSACK instance:

$$v_i = w_i = u_i \qquad \sum_{i \in S} u_i \leq U$$
$$V = W = U \qquad \sum_{i \in S} u_i \geq U$$

Knapsack problem: dynamic programming I

Def. $OPT(i, w)$ = max value subset of items $1,..., i$ with weight limit w.

Case 1. OPT does not select item i.

- OPT selects best of $1, …, i – 1$ using up to weight limit w.

Case 2. OPT selects item i.

- New weight limit $= w – w_i$.
- OPT selects best of $1, …, i – 1$ using up to weight limit $w – w_i$.

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{\, OPT(i-1,w),\ \ v_i + OPT(i-1, w-w_i)\,\} & \text{otherwise} \end{cases}$$

Theorem. Computes the optimal value in $O(n\,W)$ time.

- Not polynomial in input size.
- Polynomial in input size if weights are small integers.

Knapsack problem: dynamic programming II

Def. $OPT(i, v)$ = min weight of a knapsack for which we can obtain a solution of value $\geq v$ using a subset of items $1,..., i$.

Note. Optimal value is the largest value v such that $OPT(i, v) \leq W$.

Case 1. OPT does not select item i.

- OPT selects best of $1, …, i–1$ that achieves value v.

Case 2. OPT selects item i.

- Consumes weight w_i, need to achieve value $v – v_i$.
- OPT selects best of $1, …, i–1$ that achieves value $v – v_i$.

$$OPT(i,v) = \begin{cases} 0 & \text{if } v \leq 0 \\ \infty & \text{if } i = 0 \text{ and } \ v > 0 \\ \min\,\{OPT(i-1,v),\ w_i + OPT(i-1, v-v_i)\} & \text{otherwise} \end{cases}$$

Knapsack problem: dynamic programming II

Theorem. Dynamic programming algorithm II computes the optimal value in $O(n^2\, v_{max})$ time, where v_{max} is the maximum of any value.

Pf.

- The optimal value $V^* \leq \ n\ v_{max}$.
- There is one subproblem for each item and for each value $v \leq V^*$.
- It takes $O(1)$ time per subproblem. ▪

Remark 1. Not polynomial in input size!

Remark 2. Polynomial time if values are small integers.

Knapsack problem: polynomial-time approximation scheme

Intuition for approximation algorithm.

- Round all values up to lie in smaller range.
- Run dynamic programming algorithm II on rounded instance.
- Return optimal items in rounded instance.

item	value	weight
1	934221	1
2	5956342	2
3	17810013	5
4	21217800	6
5	27343199	7

original instance (W = 11)

item	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

rounded instance (W = 11)

Knapsack problem: polynomial-time approximation scheme

Round up all values:

- v_{max} = largest value in original instance.
- ε = precision parameter.
- θ = scaling factor = $\varepsilon\, v_{max} / n$.

$$\bar{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil \theta, \qquad \hat{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil$$

Observation. Optimal solutions to problem with $\bar{v}$ are equivalent to optimal solutions to problem with $\hat{v}$.

Intuition. $\bar{v}$ close to v so optimal solution using $\bar{v}$ is nearly optimal; $\hat{v}$ small and integral so dynamic programming algorithm II is fast.

Knapsack problem: polynomial-time approximation scheme

Round up all values: $\bar{v}_i = \left\lceil \frac{v_i}{\theta} \right\rceil \theta$

Theorem. If S is solution found by rounding algorithm and S^* is any other feasible solution, then $(1+\varepsilon) \sum_{i \in S} v_i \ \geq \ \sum_{i \in S^*} v_i$

Pf. Let S^* be any feasible solution satisfying weight constraint.

$$\begin{aligned}
\sum_{i \in S^*} v_i &\leq \sum_{i \in S^*} \bar{v}_i && \text{always round up} \\
&\leq \sum_{i \in S} \bar{v}_i && \text{solve rounded instance optimally} \\
&\leq \sum_{i \in S} (v_i + \theta) && \text{never round up by more than } \theta \\
&\leq \sum_{i \in S} v_i + n\theta && |S| \leq n \\
&\leq (1+\varepsilon) \sum_{i \in S} v_i && n\theta = \varepsilon v_{max},\ v_{max} \leq \Sigma_{i \in S} v_i
\end{aligned}$$

DP alg can take v_{max}

Knapsack problem: polynomial-time approximation scheme

Theorem. For any $\varepsilon > 0$, the rounding algorithm computes a feasible solution whose value is within a $(1 + \varepsilon)$ factor of the optimum in $O(n^3 / \varepsilon)$ time.

Pf.

- We have already proved the accuracy bound.
- Dynamic program II running time is $O(n^2\, \hat{v}_{\max})$, where

$$\hat{v}_{\max} = \left\lceil \frac{v_{\max}}{\theta} \right\rceil = \left\lceil \frac{n}{\varepsilon} \right\rceil$$

PTAS. $(1 + \varepsilon)$-approximation algorithm for any constant $\varepsilon > 0$.

- Produces arbitrarily high quality solution.
- Trades off accuracy for time.
- But such algorithms are unlikely to exist for certain problems...

Inapproximability

MAX-3-SAT. Given a 3-SAT instance Φ, find an assignment that satisfies the maximum number of clauses.

Theorem. [Karloff-Zwick 1997] There exists a ⅞-approximation algorithm.

Theorem. [Håstad 2001] Unless **P** = **NP**, there does not exist a ρ-approximation for any ρ > ⅞.

A 7/8-Approximation Algorithm for MAX 3SAT?

Howard Karloff *   *Uri Zwick* †

We describe a randomized approximation algorithm which takes an instance of MAX 3SAT as input. If the instance—a collection of clauses each of length at most *three—is satisfiable, then the expected weight of the assignment found is at least 7/8 of optimal. We provide strong evidence (but not a proof) that the algorithm performs equally well on arbitrary MAX 3SAT instances.*

Some Optimal Inapproximability Results

JOHAN HÅSTAD

Royal Institute of Technology, Stockholm, Sweden

Abstract. We prove optimal, up to an arbitrary $\epsilon > 0$, inapproximability results for Max-Ek-Sat for $k \geq 3$, maximizing the number of satisfied linear equations in an over-determined system of linear equations modulo a prime p and Set Splitting. As a consequence of these results we get improved lower bounds for the efficient approximability of many optimization problems studied previously. In particular, for Max-E2-Sat, Max-Cut, Max-di-Cut, and Vertex cover.

Categories and Subject Descriptors: F2.2 [**Analysis of Algorithms and Problem Complexity**]: Non-numerical Algorithms and Problems

Intractability III

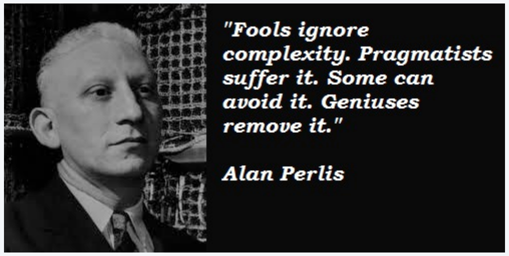
- special cases
- approximation algorithms
- *exact exponential algorithms*

Exact exponential algorithms

Complexity theory deals with worst-case behavior.

- Instances you want to solve may be "easy."

> *" For every polynomial-time algorithm you have, there is an exponential algorithm that I would rather run."* — *Alan Perlis*



Exact algorithms for 3-satisfiability

Brute force. Given a 3-SAT instance with n variables and m clauses, the brute-force algorithm takes $O((m + n)\ 2^n)$ time.

Pf.

- There are 2^n possible truth assignments to the n variables.
- We can evaluate a truth assignment in $O(m + n)$ time. ▪

Exact algorithms for 3-satisfiability

A recursive framework. A 3-SAT formula Φ is either empty or the disjunction of a clause $(\ell_1 \vee \ell_2 \vee \ell_3)$ and a 3-SAT formula Φ' with one fewer clause.

$$\begin{aligned}\Phi &= (\ell_1 \vee \ell_2 \vee \ell_3) \wedge \Phi' \\ &= (\ell_1 \wedge \Phi') \vee (\ell_2 \wedge \Phi') \vee (\ell_3 \wedge \Phi') \\ &= (\Phi' \mid \ell_1 = true) \vee (\Phi' \mid \ell_2 = true) \vee (\Phi' \mid \ell_3 = true)\end{aligned}$$

Notation. $\Phi \mid x = true$ is the simplification of Φ by setting x to $true$.

Ex.

- $\Phi \quad = (x \vee y \vee \neg z) \wedge (x \vee \neg y \vee z) \wedge (w \vee y \vee \neg z) \wedge (\neg x \vee y \vee z)$.
- $\Phi' \quad = (x \vee \neg y \vee z) \wedge (w \vee y \vee \neg z) \wedge (\neg x \vee y \vee z)$.
- $(\Phi' \mid x = true) \quad = (w \vee y \vee \neg z) \wedge (y \vee z)$.

each clause has ≤ 3 literals

Exact algorithms for 3-satisfiability

A recursive framework. A 3-SAT formula Φ is either empty or the disjunction of a clause $(\ell_1 \vee \ell_2 \vee \ell_3)$ and a 3-SAT formula Φ' with one fewer clause.

3-SAT (Φ)

IF Φ is empty RETURN *true*.

$(\ell_1 \vee \ell_2 \vee \ell_3) \wedge \Phi' \leftarrow \Phi$.

IF 3-SAT$(\Phi' \mid \ell_1 = true)$ RETURN *true*.

IF 3-SAT$(\Phi' \mid \ell_2 = true)$ RETURN *true*.

IF 3-SAT$(\Phi' \mid \ell_3 = true)$ RETURN *true*.

RETURN *false*.

Theorem. The brute-force 3-SAT algorithm takes $O(\text{poly}(n)\ 3^n)$ time.

Pf. $T(n) \le 3T(n-1) + \text{poly}(n)$. ▪

Exact algorithms for 3-satisfiability

Key observation. The cases are not mutually exclusive. Every satisfiable assignment containing clause $(\ell_1 \vee \ell_2 \vee \ell_3)$ must fall into one of 3 classes:

- ℓ_1 is *true*.
- ℓ_1 is *false*; ℓ_2 is *true*.
- ℓ_1 is *false*; ℓ_2 is *false*; ℓ_3 is *true*.

3-SAT (Φ)

IF Φ is empty RETURN *true*.

$(\ell_1 \vee \ell_2 \vee \ell_3) \wedge \Phi' \leftarrow \Phi$.

IF 3-SAT$(\Phi' \mid \ell_1 = true)$ RETURN *true*.

IF 3-SAT$(\Phi' \mid \ell_1 = false,\ \ell_2 = true)$ RETURN *true*.

IF 3-SAT$(\Phi' \mid \ell_1 = false,\ \ell_2 = false,\ \ell_3 = true)$ RETURN *true*.

RETURN *false*.

Exact algorithms for 3-satisfiability

Theorem. The brute-force algorithm takes $O(1.84^n)$ time.

Pf. $T(n) \le T(n-1) + T(n-2) + T(n-3) + O(m+n)$. ▪

largest root of $r^3 = r^2 + r + 1$

3-SAT (Φ)

IF Φ is empty RETURN *true*.

$(\ell_1 \vee \ell_2 \vee \ell_3) \wedge \Phi' \leftarrow \Phi$.

IF 3-SAT$(\Phi' \mid \ell_1 = true)$ RETURN *true*.

IF 3-SAT$(\Phi' \mid \ell_1 = false,\ \ell_2 = true)$ RETURN *true*.

IF 3-SAT$(\Phi' \mid \ell_1 = false,\ \ell_2 = false,\ \ell_3 = true)$ RETURN *true*.

RETURN *false*.

Exact algorithms for 3-satisfiability

Theorem. There exists a $O(1.33334^n)$ deterministic algorithm for 3-SAT.

A Full Derandomization of Schöning's k-SAT Algorithm

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August 25, 2010

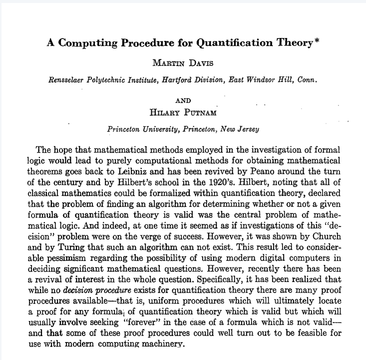
Abstract

Schöning [7] presents a simple randomized algorithm for k-SAT with running time $O(a_k^n \text{poly}(n))$ for $a_k = 2(k-1)/k$. We give a deterministic version of this algorithm running in time $O((a_k + \epsilon)^n \text{poly}(n))$, where $\epsilon > 0$ can be made arbitrarily small.

Exact algorithms for satisfiability

DPPL algorithm. Highly-effective backtracking procedure.

- Splitting rule: assign truth value to literal; solve both possibilities.
- Unit propagation: clause contains only a single unassigned literal.
- Pure literal elimination: if literal appears only negated or unnegated.



A Computing Procedure for Quantification Theory*

MARTIN DAVIS

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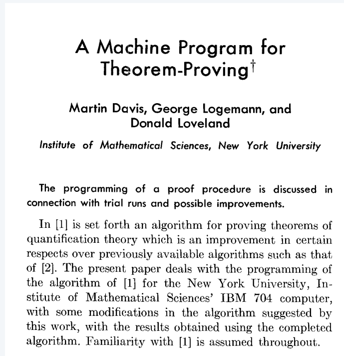
AND

HILARY PUTNAM

Princeton University, Princeton, New Jersey

The hope that mathematical methods employed in the investigation of formal logic would lead to purely computational methods for obtaining mathematical theorems goes back to Leibniz and has been revived by Peano around the turn of the century and by Hilbert's school in the 1920's. Hilbert, noting that all of classical mathematics could be formalized within quantification theory, declared that the problem of finding an algorithm for determining whether or not a given formula of quantification theory is valid was the central problem of mathematical logic. And indeed, at one time it seemed as if investigations of this "decision" problem were on the verge of success. However, it was shown by Church and by Turing that such an algorithm can not exist. This result led to considerable pessimism regarding the possibility of using modern digital computers in deciding significant mathematical questions. However, recently there has been a revival of interest in the whole question. Specifically, it has been realized that while no *decision procedure* exists for quantification theory there are many proof procedures available—that is, uniform procedures which will ultimately locate a proof for any formula of quantification theory which is valid but which will usually involve seeking "forever" in the case of a formula which is not valid—and that some of these proof procedures could well turn out to be feasible for use with modern computing machinery.



A Machine Program for Theorem-Proving†

Martin Davis, George Logemann, and Donald Loveland

Institute of Mathematical Sciences, New York University

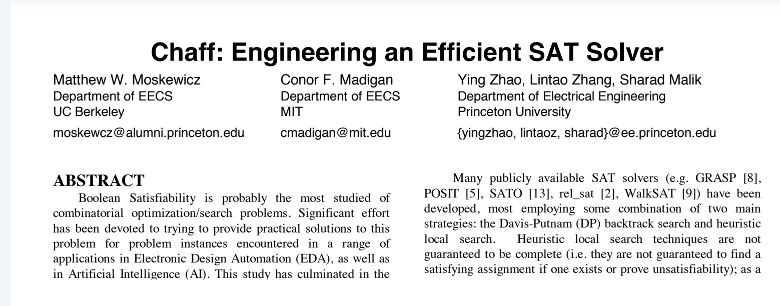
The programming of a proof procedure is discussed in connection with trial runs and possible improvements.

In [1] is set forth an algorithm for proving theorems of quantification theory which is an improvement in certain respects over previously available algorithms such as that of [2]. The present paper deals with the programming of the algorithm of [1] for the New York University, Institute of Mathematical Sciences' IBM 704 computer, with some modifications in the algorithm suggested by this work, with the results obtained using the completed algorithm. Familiarity with [1] is assumed throughout.

Exact algorithms for satisfiability

Chaff. State-of-the-art SAT solver.

- Solves real-world SAT instances with ~ 10K variable.
  Developed at Princeton by undergrads.



Chaff: Engineering an Efficient SAT Solver

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ABSTRACT

Boolean Satisfiability is probably the most studied of combinatorial optimization/search problems. Significant effort has been devoted to trying to provide practical solutions to this problem for problem instances encountered in a range of applications in Electronic Design Automation (EDA), as well as in Artificial Intelligence (AI). This study has culminated in the

Many publicly available SAT solvers (e.g. GRASP [8], POSIT [5], SATO [13], rel_sat [2], WalkSAT [9]) have been developed, most employing some combination of two main strategies: the Davis-Putnam (DP) backtrack search and heuristic local search. Heuristic local search techniques are not guaranteed to be complete (i.e. they are not guaranteed to find a satisfying assignment if one exists or prove unsatisfiability); as a

Exact algorithms for TSP and Hamilton cycle

Theorem. The brute-force algorithm for TSP (or HAM-CYCLE) takes $O(n!)$ time.

Pf.

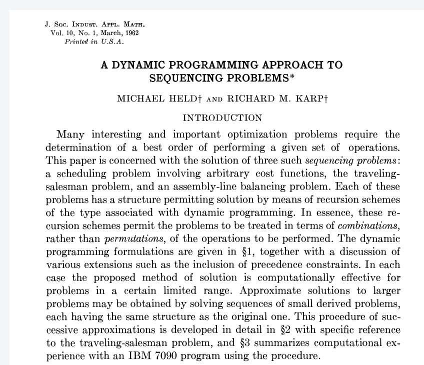
- There are ½ $(n-1)!$ tours.
- Computing the length of a tour takes $O(n)$ time. ▪

Note. The function $n!$ grows exponentially faster than 2^n.

- $2^{40} = 1099511627776 \sim 10^{12}$.
- $40! = 815915283247897734345611269596115894272000000000 \sim 10^{48}$.

Exact algorithms for TSP and Hamilton cycle

Theorem. [Bellman 1962, Held-Karp 1962] There exists a $O(n^2\, 2^n)$ time algorithm for TSP (and HAMILTON-CYCLE).



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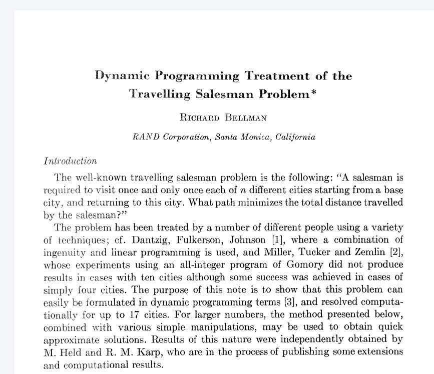
A DYNAMIC PROGRAMMING APPROACH TO SEQUENCING PROBLEMS*

MICHAEL HELD† AND RICHARD M. KARP†

INTRODUCTION

Many interesting and important optimization problems require the determination of a best order of performing a given set of operations. This paper is concerned with the solution of three such *sequencing problems*: a scheduling problem involving arbitrary cost functions, the traveling-salesman problem, and an assembly-line balancing problem. Each of these problems has a structure permitting solution by means of recursion schemes of the type associated with dynamic programming. In essence, these recursion schemes permit the problems to be treated in terms of *combinations*, rather than *permutations*, of the operations to be performed. The dynamic programming formulations are given in §1, together with a discussion of various extensions such as the inclusion of precedence constraints. In each case the proposed method of solution is computationally effective for problems in a certain limited range. Approximate solutions to larger problems may be obtained by solving sequences of small derived problems, each having the same structure as the original one. This procedure of successive approximations is developed in detail in §2 with specific reference to the traveling-salesman problem, and §3 summarizes computational experience with an IBM 7090 program using the procedure.



Dynamic Programming Treatment of the Travelling Salesman Problem*

RICHARD BELLMAN

RAND Corporation, Santa Monica, California

Introduction

The well-known travelling salesman problem is the following: "A salesman is required to visit once and only once each of n different cities starting from a base city, and returning to this city. What path minimizes the total distance travelled by the salesman?"

The problem has been treated by a number of different people using a variety of techniques; cf. Dantzig, Fulkerson, Johnson [1], where a combination of ingenuity and linear programming is used, and Miller, Tucker and Zemlin [2], whose experiments using an all-integer program of Gomory did not produce results in cases with ten cities although some success was achieved in cases of simply four cities. The purpose of this note is to show that this problem can easily be formulated in dynamic programming terms [3], and resolved computationally for up to 17 cities. For larger numbers, the method presented below, combined with various simple manipulations, may be used to obtain quick approximate solutions. Results of this nature were independently obtained by M. Held and R. M. Karp, who are in the process of publishing some extensions and computational results.

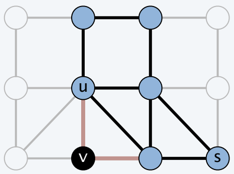
Exact algorithms for TSP and Hamilton cycle

Theorem. [Bellman 1962, Held-Karp 1962] There exists a $O(n^2 2^n)$ time algorithm for TSP (and HAMILTON-CYCLE).

Pf. [dynamic programming]

- Define $c(s, v, X)$ = cost of cheapest path between s and v that visits every node in X exactly once (and uses only nodes in X).
- Observe $OPT = \min_{v \neq s} c(s, v, V) + c(v, s)$.
- There are $n\, 2^n$ subproblems and they satisfy the recurrence:

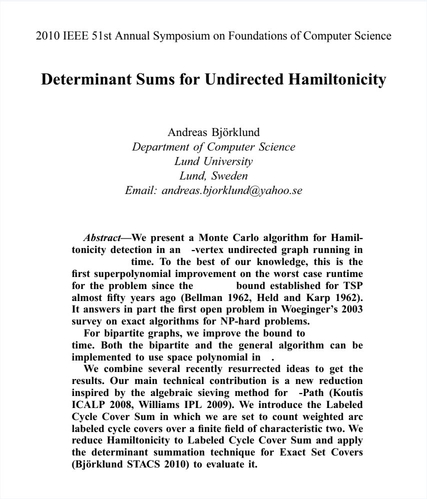
$$c(s,v,X) = \begin{cases} c(s,v) & \text{if } |X| = 2 \\ \min\limits_{u \in X \setminus \{s,v\}} c(s,u,X \setminus \{v\}) + c(u,v) & \text{if } |X| > 2. \end{cases}$$



- The values $c(s, v, X)$ can be computed increasing order of the cardinality of X. ▪

Exact algorithms for Hamilton cycle

Theorem. [Björklund 2010] There exists a $O(1.657^n)$ time randomized algorithm for HAMILTON-CYCLE.



2010 IEEE 51st Annual Symposium on Foundations of Computer Science

Determinant Sums for Undirected Hamiltonicity

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***Abstract*—We present a Monte Carlo algorithm for Hamiltonicity detection in an -vertex undirected graph running in time. To the best of our knowledge, this is the first superpolynomial improvement on the worst case runtime for the problem since the bound established for TSP almost fifty years ago (Bellman 1962, Held and Karp 1962). It answers in part the first open problem in Woeginger's 2003 survey on exact algorithms for NP-hard problems.**

For bipartite graphs, we improve the bound to time. Both the bipartite and the general algorithm can be implemented to use space polynomial in .

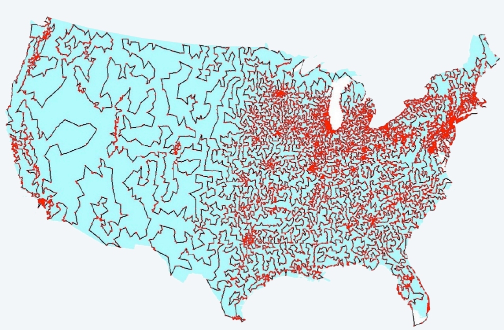
We combine several recently resurrected ideas to get the results. Our main technical contribution is a new reduction inspired by the algebraic sieving method for -Path (Koutis ICALP 2008, Williams IPL 2009). We introduce the Labeled Cycle Cover Sum in which we are set to count weighted arc labeled cycle covers over a finite field of characteristic two. We reduce Hamiltonicity to Labeled Cycle Cover Sum and apply the determinant summation technique for Exact Set Covers (Björklund STACS 2010) to evaluate it.

Euclidean traveling salesperson problem

Euclidean TSP. Given n points in the plane and a real number L, is there a tour that visit every city exactly once that has distance $\leq L$?

Proposition. EUCLIDEAN-TSP is **NP**-hard.
Remark. Not known to be in **NP**.

$$\sqrt{5} + \sqrt{6} + \sqrt{18} \;<\; \sqrt{4} + \sqrt{12} + \sqrt{12}$$
$$8.928198407 \;<\; 8.928203230$$

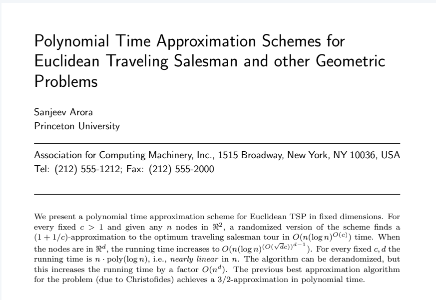


13509 cities in the USA and an optimal tour

Euclidean traveling salesperson problem

Theorem. [Arora 1998, Mitchell 1999] Given n points in the plane, for any constant $\varepsilon > 0$, there exists a poly-time algorithm to find a tour whose length is at most $(1 + \varepsilon)$ times that of the optimal tour.

Pf idea. Structure theorem + dynamic programming.



Polynomial Time Approximation Schemes for Euclidean Traveling Salesman and other Geometric Problems

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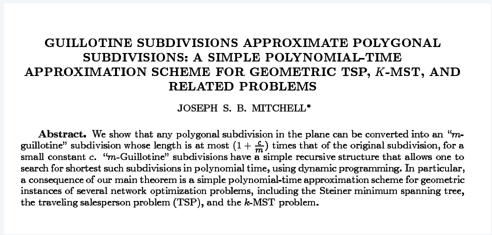
We present a polynomial time approximation scheme for Euclidean TSP in fixed dimensions. For every fixed $c > 1$ and given any n nodes in $\Re^2$, a randomized version of the scheme finds a $(1 + 1/c)$-approximation to the optimum traveling salesman tour in $O(n(\log n)^{O(c)})$ time. When the nodes are in $\Re^d$, the running time increases to $O(n(\log n)^{(O(\sqrt{d}c))^{d-1}})$. For every fixed c, d the running time is $n \cdot \text{poly}(\log n)$, i.e., *nearly linear* in n. The algorithm can be derandomized, but this increases the running time by a factor $O(n^d)$. The previous best approximation algorithm for the problem (due to Christofides) achieves a 3/2-approximation in polynomial time.



GUILLOTINE SUBDIVISIONS APPROXIMATE POLYGONAL SUBDIVISIONS: A SIMPLE POLYNOMIAL-TIME APPROXIMATION SCHEME FOR GEOMETRIC TSP, K-MST, AND RELATED PROBLEMS

JOSEPH S. B. MITCHELL*

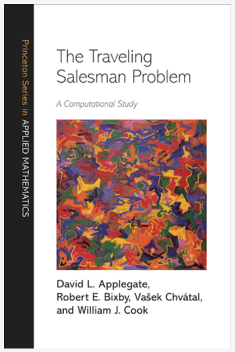
Abstract. We show that any polygonal subdivision in the plane can be converted into an "m-guillotine" subdivision whose length is at most $(1 + \frac{c}{m})$ times that of the original subdivision, for a small constant c. "m-Guillotine" subdivisions have a simple recursive structure that allows one to search for shortest such subdivisions in polynomial time, using dynamic programming. In particular, a consequence of our main theorem is a simple polynomial-time approximation scheme for geometric instances of several network optimization problems, including the Steiner minimum spanning tree, the traveling salesperson problem (TSP), and the k-MST problem.

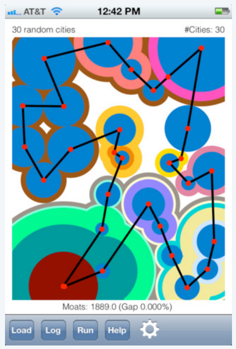
Concorde TSP solver

Concorde TSP solver. [Applegate-Bixby-Chvátal-Cook]

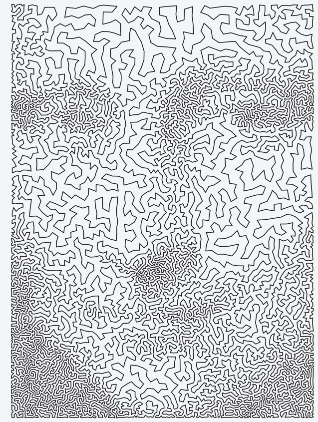
- Linear programming + branch-and-bound + polyhedral combinatorics.
- Greedy heuristics, including Lin-Kernighan.
- MST, Delaunay triangulations, fractional b-matchings, ...

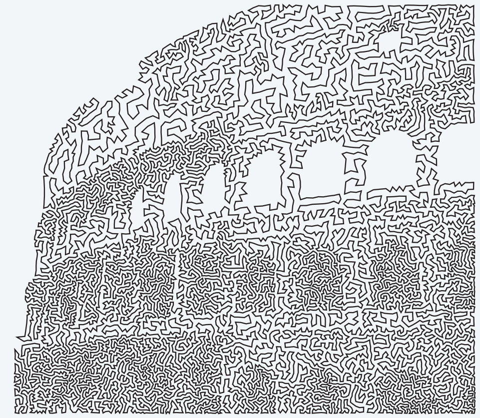
Remarkable fact. Concorde has solved all 110 TSPLIB instances.





TSP line art





Continuous line drawings via the TSP by Robert Bosch and Craig Kaplan

That's all, folks: keep searching!

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!

If you said P is NP tonight,
There would still be papers left to write.
I have a weakness;
I'm addicted to completeness,
And I keep searching for the longest path.

The algorithm I would like to see
Is of polynomial degree.
But it's elusive:
Nobody has found conclusive
Evidence that we can find a longest path.

I have been hard working for so long.
I swear it's right, and he marks it wrong.
Some how I'll feel sorry when it's done: GPA 2.1
Is more than I hope for.

Garey, Johnson, Karp and other men (and women)
Tried to make it order N log N.
Am I a mad fool
If I spend my life in grad school,
Forever following the longest path?

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path.

Written by Dan Barrett in 1988 while a student
at Johns Hopkins during a difficult algorithms take-home final