**Data structures**

- **Static problems.** Given an input, produce an output.
  - Ex. Sorting, FFT, edit distance, shortest paths, MST, max-flow, ...

- **Dynamic problems.** Given a sequence of operations (given one at a time), produce a sequence of outputs.
  - Ex. Stack, queue, priority queue, symbol table, union-find, ...

**Algorithm.** Step-by-step procedure to solve a problem.

**Data structure.** Way to store and organize data.

- Ex. Array, linked list, binary heap, binary search tree, hash table, ...

### Appetizer

**Goal.** Design a data structure to support all operations in $O(1)$ time.
- INIT($n$): create and return an initialized array (all zero) of length $n$.
- READ($A$, $i$): return $i$th element of array.
- WRITE($A$, $i$, value): set $i$th element of array to value.

**Assumptions.**
- Can MALLOC an uninitialized array of length $n$ in $O(1)$ time.
- Given an array, can read or write $i$th element in $O(1)$ time.

**Remark.** An array does INIT in $O(n)$ time and READ and WRITE in $O(1)$ time.

```
Appetizer
```

### Data structure

- $A[i]$ stores the current value for READ (if initialized).
- $k = \text{number of initialized entries}$.
- $C[j] = \text{index of jth initialized entry for } j = 1, \ldots, k$.
- If $C[j] = i$, then $B[i] = j$ for $j = 1, \ldots, k$.

**Theorem.** $A[i]$ is initialized iff both $1 \leq B[i] \leq k$ and $C[B[i]] = i$.

**Pf.** Ahead.

```
Appetizer
```
**Appetizer**

**Theorem.** \( A[i] \) is initialized iff both \( 1 \leq B[i] \leq k \) and \( C[B[i]] = i \).

**Pf.** \( \Rightarrow \)
- Suppose \( A[i] \) is the \( j \)th entry to be initialized.
- Then \( C[j] = i \) and \( B[i] = j \).
- Thus, \( C[B[i]] = i \).

**Pf.** \( \Leftarrow \)
- Suppose \( A[i] \) is uninitialized.
- If \( B[i] < 1 \) or \( B[i] > k \), then \( A[i] \) clearly uninitialized.
- If \( 1 \leq B[i] \leq k \) by coincidence, then we still can’t have \( C[B[i]] = i \) because none of the entries \( C[1..k] \) can equal \( i \).

**Amortized Analysis**

- binary counter
- multipop stack
- dynamic table

Lecture slides by Kevin Wayne

http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on Apr 8, 2013 6:13 AM
**Amortized analysis**

Worst-case analysis. Determine worst-case running time of a data structure operation as function of the input size. Can be too pessimistic if the only way to encounter an expensive operation is if there were lots of previous cheap operations.

Amortized analysis. Determine worst-case running time of a sequence of data structure operations as a function of the input size.

Ex. Starting from an empty stack implemented with a dynamic table, any sequence of $n$ push and pop operations takes $O(n)$ time in the worst case.

---

**AMORTIZED ANALYSIS**

- binary counter
- multipop stack
- dynamic table

**Binary counter**

**Goal.** Increment a $k$-bit binary counter (mod $2^k$).

**Representation.** $a_j = j^{th}$ least significant bit of counter.

**Counter value**

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</tr>
</tbody>
</table>

**Cost model.** Number of bits flipped.
Binary counter

**Goal.** Increment a \( k \)-bit binary counter (mod \( 2^k \)).

**Representation.** \( a_j = j^{th} \) least significant bit of counter.

The total number of bits flipped is
\[
\sum_{j=0}^{k-1} \left\lfloor \frac{n}{2^j} \right\rfloor < n \sum_{j=0}^{k-1} \frac{1}{2^j} = 2n
\]

**Remark.** Theorem may be false if initial counter is not zero.

Aggregate method (brute force)

**Aggregate method.** Sum up sequence of operations, weighted by their cost.

Accounting method (banker's method)

**Assign different charges to each operation.**
- \( D_i \) = data structure after operation \( i \).
- \( c_i \) = actual cost of operation \( i \).
- \( \hat{c}_i \) = amortized cost of operation \( i = \) amount we charge operation \( i \).
- When \( \hat{c}_i > c_i \), store credits in data structure \( D_0 \) to pay for future ops.
- Initial data structure \( D_0 \) starts with zero credits.

**Key invariant.** The total number of credits in the data structure \( \geq 0 \).
\[
\sum_{i=1}^{k} c_i - \sum_{i=1}^{k} \hat{c}_i \geq 0
\]
Accounting method (banker’s method)

Assign different charges to each operation.
- \( D_i \) = data structure after operation \( i \).
- \( c_i \) = actual cost of operation \( i \).
- \( \hat{c}_i \) = amortized cost of operation \( i \) = amount we charge operation \( i \).
- When \( \hat{c}_i > c_i \), we store credits in data structure \( D_i \) to pay for future ops.
- Initial data structure \( D_0 \) starts with zero credits.

Key invariant. The total number of credits in the data structure \( \geq 0 \).
\[
\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0
\]

Theorem. Starting from the initial data structure \( D_0 \), the total actual cost of any sequence of \( n \) operations is at most the sum of the amortized costs.

Pf. The amortized cost of the sequence of operations is: \( \sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i \).

Intuition. Measure running time in terms of credits (time = money).

Binary counter: accounting method

Credits. One credit pays for a bit flip.

Invariant. Each bit that is set to 1 has one credit.

Accounting.
- Flip bit \( j \) from 0 to 1: charge two credits (use one and save one in bit \( j \)).
- Flip bit \( j \) from 1 to 0: pay for it with saved credit in bit \( j \).

Increment

<table>
<thead>
<tr>
<th>7</th>
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<th>5</th>
<th>4</th>
<th>3</th>
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<th>1</th>
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</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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</tr>
</tbody>
</table>

Binary counter: accounting method

Credits. One credit pays for a bit flip.

Invariant. Each bit that is set to 1 has one credit.

Accounting.
- Flip bit \( j \) from 0 to 1: charge two credits (use one and save one in bit \( j \)).
- Flip bit \( j \) from 1 to 0: pay for it with saved credit in bit \( j \).
Binary counter: **accounting method**

**Credits.** One credit pays for a bit flip.

**Invariant.** Each bit that is set to 1 has one credit.

**Accounting.**
- Flip bit $j$ from 0 to 1: charge two credits (use one and save one in bit $j$).
- Flip bit $j$ from 1 to 0: pay for it with saved credit in bit $j$.

**Theorem.** Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(n)$ bits.

**Pf.** The algorithm maintains the invariant that any bit that is currently set to 1 has one credit $\Rightarrow$ number of credits in each bit $\geq 0$. $lacksquare$

---

**Potential method (physicist's method)**

**Potential function.** $\Phi(D_i)$ maps each data structure $D_i$ to a real number s.t.:
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each data structure $D_i$.

**Actual and amortized costs.**
- $c_i = \text{actual cost of } i^{th} \text{ operation}$.
- $\check{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = \text{amortized cost of } i^{th} \text{ operation}$.

**Theorem.** Starting from the initial data structure $D_0$, the total actual cost of any sequence of $n$ operations is at most the sum of the amortized costs.

**Pf.** The amortized cost of the sequence of operations is:

$$\sum_{i=1}^{n} \check{c}_i = \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1}))$$

$$= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

$$\geq \sum_{i=1}^{n} c_i \quad \blacksquare$$

---

**Potential method (physicist's method)**

**Potential function.** Let $\Phi(D) = \text{number of 1 bits in the binary counter } D$.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**increment**

<table>
<thead>
<tr>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
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<th>1</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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</tr>
</tbody>
</table>
Potential function. Let $\Phi(D) = \text{number of 1 bits in the binary counter } D$.

- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

Theorem. Starting from the zero counter, a sequence of $n$ INCREMENT operations flips $O(n)$ bits.

Pf.
- Suppose that the $i^{th}$ increment operation flips $t_i$ bits from 1 to 0.
- The actual cost $c_i \leq t_i + 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$
  \[ \leq c_i + 1 - t_i \]
  \[ \leq 2. \]

Famous potential functions

**Fibonacci heaps.** $\Phi(H) = \text{trees}(H) + 2 \text{ marks}(H)$.

**Splay trees.** $\Phi(T) = \sum_{x \in T} \lceil \log_2 \text{size}(x) \rceil$

**Move-to-front.** $\Phi(L) = 2 \times \text{inversions}(L, L^*)$.

**Preflow-push.** $\Phi(f) = \sum_{v : \text{excess}(v) > 0} \text{height}(v)$

**Red-black trees.** $\Phi(T) = \sum_{x \in T} w(x)$

\[ w(x) = \begin{cases} 
0 & \text{if } x \text{ is red} \\
1 & \text{if } x \text{ is black and has no red children} \\
0 & \text{if } x \text{ is black and has one red child} \\
2 & \text{if } x \text{ is black and has two red children} 
\end{cases} \]
**Amortized Analysis**

- binary counter
- multipop stack
- dynamic table

**Multipop stack**

**Goal.** Support operations on a set of $n$ elements:
- $\text{PUSH}(S, x)$: push object $x$ onto stack $S$.
- $\text{POP}(S)$: remove and return the most-recently added object.
- $\text{MULTIPOP}(S, k)$: remove the most-recently added $k$ objects.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ $\text{PUSH}$, $\text{POP}$, and $\text{MULTIPOP}$ operations takes $O(n^2)$ time.

**Pf.**
- Use a singly-linked list.
- $\text{POP}$ and $\text{PUSH}$ take $O(1)$ time each.
- $\text{MULTIPOP}$ takes $O(n)$ time.

---

**Exceptions.** We assume $\text{POP}$ throws an exception if stack is empty.

---

**Multipop stack: aggregate method**

**Goal.** Support operations on a set of $n$ elements:
- $\text{PUSH}(S, x)$: push object $x$ onto stack $S$.
- $\text{POP}(S)$: remove and return the most-recently added object.
- $\text{MULTIPOP}(S, k)$: remove the most-recently added $k$ objects.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ $\text{PUSH}$, $\text{POP}$, and $\text{MULTIPOP}$ operations takes $O(n)$ time.

**Pf.**
- An object is popped at most once for each time it is pushed onto stack.
- There are $\leq n$ $\text{PUSH}$ operations.
- Thus, there are $\leq n$ $\text{POP}$ operations (including those made within $\text{MULTIPOP}$).
**Multipop stack: accounting method**

**Credits.** One credit pays for a push or pop.

**Accounting.**
- **PUSH**$(S, x)$: charge two credits.
  - use one credit to pay for pushing $x$ now
  - store one credit to pay for popping $x$ at some point in the future
- No other operation is charged a credit.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTIPOP operations takes $O(n)$ time.

**Pf.** The algorithm maintains the invariant that every object remaining on the stack has 1 credit $\Rightarrow$ number of credits in data structure $\geq 0$. ●

---

**Multipop stack: potential method**

**Potential function.** Let $\Phi(D) =$ number of objects currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTIPOP operations takes $O(n)$ time.

**Pf.** [Case 1: push]
- Suppose that the $i^{th}$ operation is a PUSH.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$.

---

**Multipop stack: potential method**

**Potential function.** Let $\Phi(D) =$ number of objects currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTIPOP operations takes $O(n)$ time.

**Pf.** [Case 2: pop]
- Suppose that the $i^{th}$ operation is a POP.
- The actual cost $c_i = 1$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 - 1 = 0$.

---

**Multipop stack: potential method**

**Potential function.** Let $\Phi(D) =$ number of objects currently on the stack.
- $\Phi(D_0) = 0$.
- $\Phi(D_i) \geq 0$ for each $D_i$.

**Theorem.** Starting from an empty stack, any intermixed sequence of $n$ PUSH, POP, and MULTIPOP operations takes $O(n)$ time.

**Pf.** [Case 3: multipop]
- Suppose that the $i^{th}$ operation is a MULTIPOP of $k$ objects.
- The actual cost $c_i = k$.
- The amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = k - k = 0$. ●
**Amortized Analysis**

- binary counter
- multipop stack
- dynamic table

**Goal.** Store items in a table (e.g., for hash table, binary heap).
- Two operations: INSERT and DELETE.
  - too many items inserted ⇒ expand table.
  - too many items deleted ⇒ contract table.
- Requirement: if table contains $m$ items, then space = $\Theta(m)$.

**Theorem.** Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n^2)$ time.

**Pf.** A single INSERT or DELETE takes $O(n)$ time.

---

**Dynamic table: insert only**

- Initialize table to be size 1.
- INSERT: if table is full, first copy all items to a table of twice the size.

<table>
<thead>
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<th>insert</th>
<th>old size</th>
<th>new size</th>
<th>cost</th>
</tr>
</thead>
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<td>5</td>
<td>4</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>8</td>
<td>–</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>8</td>
<td>–</td>
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<tr>
<td>8</td>
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<td>8</td>
<td>–</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
</tr>
</tbody>
</table>

**Cost model.** Number of items that are copied.

**Dynamic table: insert only**

**Theorem.** [via aggregate method] Starting from an empty dynamic table, any sequence of $n$ INSERT operations takes $O(n)$ time.

**Pf.** Let $c_i$ denote the cost of the $i$th insertion.

$$c_i = \begin{cases} 
  i & \text{if } i-1 \text{ is an exact power of } 2 \\
  1 & \text{otherwise}
\end{cases}$$

Starting from empty table, the cost of a sequence of $n$ INSERT operations is:

$$\sum_{i=1}^{n} c_i \leq n + \sum_{j=0}^{\lfloor \log n \rfloor} 2^j$$

$$< n + 2n$$

$$= 3n \quad \blacksquare$$
Dynamic table: insert only

1 2 3 4

Dynamic table: insert only

1 2 3 4 5 6 7 8

Dynamic table: insert only

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

Dynamic table: insert only

Accounting.

- **INSERT**: charge 3 credits (use 1 credit to insert; save 2 with new item).

**Theorem.** [via accounting method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

**Pf.** The algorithm maintains the invariant that there are 2 credits with each item in right half of table.

- When table doubles, one-half of the items in the table have 2 credits.
- This pays for the work needed to double the table.

---

**Theorem.** [via potential method] Starting from an empty dynamic table, any sequence of \( n \) INSERT operations takes \( O(n) \) time.

**Pf.** Let \( \Phi(D_i) = 2 \cdot \text{size}(D_i) - \text{capacity}(D_i) \).

**Case 1.** [does not trigger expansion] \( \text{size}(D_i) \leq \text{capacity}(D_{i-1}) \).

- Actual cost \( c_i = 1 \).
- \( \Phi(D_i) - \Phi(D_{i-1}) = 2 \).
- Amortized costs \( \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 2 = 3 \).

**Case 2.** [triggers expansion] \( \text{size}(D_i) = 1 + \text{capacity}(D_{i-1}) \).

- Actual cost \( c_i = 1 + \text{capacity}(D_{i-1}) \).
- \( \Phi(D_i) - \Phi(D_{i-1}) = 2 - \text{capacity}(D_i) + \text{capacity}(D_{i-1}) = 2 - \text{capacity}(D_{i-1}) \).
- Amortized costs \( \hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 2 = 3 \).
Dynamic table: doubling and halving

Thrashing.
- Initialize table to be of fixed size, say 1.
- \text{INSERT}: if table is full, expand to a table of twice the size.
- \text{DELETE}: if table is \frac{1}{2}-full, contract to a table of half the size.

Efficient solution.
- Initialize table to be of fixed size, say 1.
- \text{INSERT}: if table is full, expand to a table of twice the size.
- \text{DELETE}: if table is \frac{1}{4}-full, contract to a table of half the size.

Memory usage. A dynamic table uses $O(n)$ memory to store $n$ items.

\textbf{Pf.} Table is always at least \frac{1}{4}-full (provided it is not empty).

Dynamic table: insert and delete

\textbf{Theorem.} [via aggregate method] Starting from an empty dynamic table, any intermixed sequence of $n$ \texttt{INSERT} and \texttt{DELETE} operations takes $O(n)$ time.

\textbf{Pf.}
- In between resizing events, each \texttt{INSERT} and \texttt{DELETE} takes $O(1)$ time.
- Consider total amount of work between two resizing events.
  - Just after the table is doubled to size $m$, it contains $m/2$ items.
  - Just after the table is halved to size $m$, it contains $m/2$ items.
  - Just before the next resizing, it contains either $m/4$ or $2m$ items.
  - After resizing to $m$, we must perform $\Omega(m)$ operations before we resize again (either $\geq m$ insertions or $\geq m/4$ deletions).
- Resizing a table of size $m$ requires $O(m)$ time.

\textbf{Accounting.}
- \texttt{INSERT}: charge 3 credits (1 credit for insert; save 2 with new item).
- \texttt{DELETE}: charge 2 credits (1 credit to delete, save 1 in emptied slot).

\textbf{Theorem.} [via accounting method] Starting from an empty dynamic table, any intermixed sequence of $n$ \texttt{INSERT} and \texttt{DELETE} operations takes $O(n)$ time.

\textbf{Pf.} The algorithm maintains the invariant that there are 2 credits with each item in the right half of table; 1 credit with each empty slot in the left half.
- When table doubles, each item in right half of table has 2 credits.
- When table halves, each empty slot in left half of table has 1 credit.

\begin{center}
\begin{tabular}{cccccccccccc}
\texttt{insert} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\end{tabular}
\end{center}
Dynamic table: insert and delete

Theorem. [via potential method] Starting from an empty dynamic table, any intermixed sequence of $n$ INSERT and DELETE operations takes $O(n)$ time.

Proof sketch.

\begin{itemize}
  \item Let $\alpha(D) = \frac{\text{size}(D)}{\text{capacity}(D)}$.
  
  \begin{align*}
    \Phi(D) &= \begin{cases} 
    2 \text{size}(D) - \text{capacity}(D) & \text{if } \alpha \geq 1/2 \\
    \frac{1}{2} \text{capacity}(D) - \text{size}(D) & \text{if } \alpha < 1/2
    \end{cases}
  \end{align*}

  \item When $\alpha(D) = 1/2$, $\Phi(D) = 0$. [zero potential after resizing]
  \item When $\alpha(D) = 1$, $\Phi(D) = \text{size}(D)$. [can pay for expansion]
  \item When $\alpha(D) = 1/4$, $\Phi(D) = \text{size}(D)$. [can pay for contraction]
  \end{itemize}

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