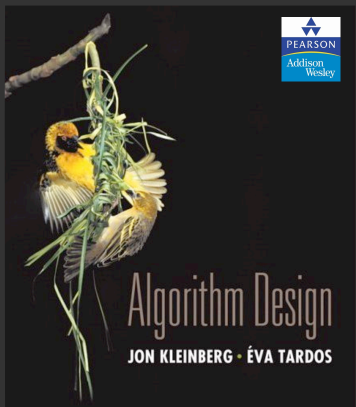


Lecture slides by Kevin Wayne

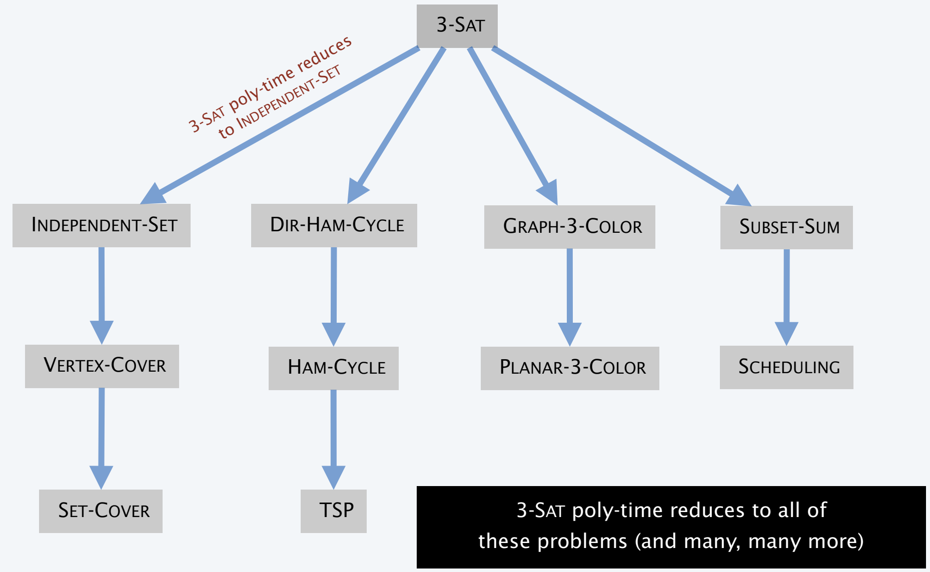
http://www.cs.princeton.edu/~wayne/kleinberg-tardos

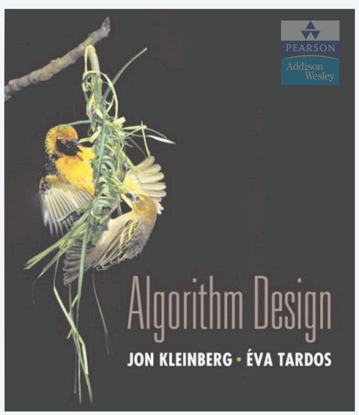
8. Intractability II

- P vs. NP
- NP-complete
- co-NP
- NP-hard

Last updated on May 5, 2013 9:59 AM

Recap





Section 8.3

8. Intractability II

- P vs. NP
- NP-complete
- co-NP
- NP-hard

Decision problems

Decision problem.

- Problem X is a set of strings.
- Instance s is one string.
- Algorithm A solves problem X: $A(s) = yes$ iff $s \in X$.

Def. Algorithm A runs in polynomial time if for every string s, $A(s)$ terminates in at most $p(|s|)$ "steps", where $p(\cdot)$ is some polynomial.

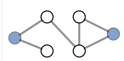
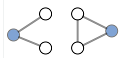
↑ length of s

Ex.

- Problem PRIMES $= \{ 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, \ldots. \}$.
- Instance $s = 592335744548702854681$.
- AKS algorithm PRIMES in $O(|s|^8)$ steps.

Definition of P

P. Decision problems for which there is a poly-time algorithm.

Problem	Description	Algorithm	yes	no
MULTIPLE	Is x a multiple of y ?	grade-school division	51, 17	51, 16
REL-PRIME	Are x and y relatively prime ?	Euclid (300 BCE)	34, 39	34, 51
PRIMES	Is x prime ?	AKS (2002)	53	51
EDIT-DISTANCE	Is the edit distance between x and y less than 5 ?	dynamic programming	niether neither	acgggt ttttta
L-SOLVE	Is there a vector x that satisfies $Ax = b$?	Gauss-Edmonds elimination	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
ST-CONN	Is there a path between s and t in a graph G ?	depth-first search (Theseus)		

NP

Certification algorithm intuition.
- Certifier views things from "managerial" viewpoint.
- Certifier doesn't determine whether $s \in X$ on its own; rather, it checks a proposed proof t that $s \in X$.

Def. Algorithm $C(s, t)$ is a certifier for problem X if for every string s, $s \in X$ iff there exists a string t such that $C(s, t) = yes$.

"certificate" or "witness"

Def. **NP** is the set of problems for which there exists a poly-time certifier.
- $C(s, t)$ is a poly-time algorithm.
- Certificate t is of polynomial size: $|t| \le p(|s|)$ for some polynomial $p(\cdot)$

Remark. **NP** stands for nondeterministic polynomial time.

Certifiers and certificates: composite

COMPOSITES. Given an integer s, is s composite?

Certificate. A nontrivial factor t of s. Such a certificate exists iff s is composite. Moreover $|t| \le |s|$.

Certifier. Check that $1 < t < s$ and that s is a multiple of t.

instance s 437669

certificate t 541 or 809

437,669 = 541 × 809

Conclusion. COMPOSITES ∈ **NP**.

Certifiers and certificates: 3-satisfiability

3-SAT. Given a CNF formula Φ, is there a satisfying assignment?

Certificate. An assignment of truth values to the n boolean variables.

Certifier. Check that each clause in Φ has at least one true literal.

instance s $\Phi = \left(\overline{x_1} \lor x_2 \lor x_3\right) \land \left(x_1 \lor \overline{x_2} \lor x_3\right) \land \left(\overline{x_1} \lor x_2 \lor x_4\right)$

certificate t $x_1 = true,\ x_2 = true,\ x_3 = false,\ x_4 = false$

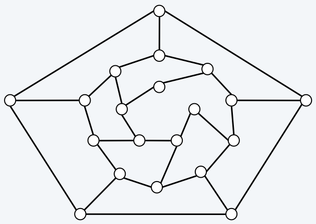
Conclusion. 3-SAT ∈ **NP**.

Certifiers and certificates: Hamilton path

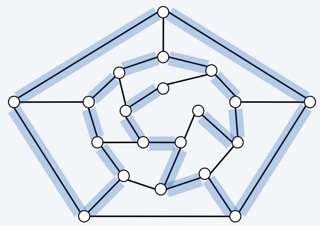
HAM-PATH. Given an undirected graph $G = (V, E)$, does there exist a simple path P that visits every node?

Certificate. A permutation of the n nodes.

Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes.



instance s

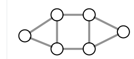
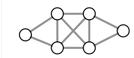
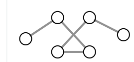
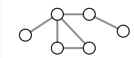


certificate t

Conclusion. HAM-PATH $\in$ **NP**.

Definition of NP

NP. Decision problems for which there is a poly-time certifier.

Problem	Description	Algorithm	yes	no
L-SOLVE	Is there a vector x that satisfies $Ax = b$?	Gauss-Edmonds elimination	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
COMPOSITES	Is x composite ?	AKS (2002)	51	53
FACTOR	Does x have a nontrivial factor less than y ?	?	(56159, 50)	(55687, 50)
SAT	Is there a truth assignment that satisfies the formula ?	?	$\neg x_1 \lor x_2$ $x_1 \lor x_2$	$\neg x_2$ $\neg x_1 \lor x_2$ $x_1 \lor x_2$
3-COLOR	Can the nodes of a graph G be colored with 3 colors?	?		
HAM-PATH	Is there a simple path between s and t that visits every node?	?		

Definition of NP

NP. Decision problems for which there is a poly-time certifier.

> *" NP captures vast domains of computational, scientific, and mathematical endeavors, and seems to roughly delimit what mathematicians and scientists have been aspiring to compute feasibly. "* — *Christos Papadimitriou*

> *" In an ideal world it would be renamed P vs VP. "* — *Clyde Kruskal*

P, NP, and EXP

P. Decision problems for which there is a poly-time algorithm.
NP. Decision problems for which there is a poly-time certifier.
EXP. Decision problems for which there is an exponential-time algorithm.

Claim. **P** $\subseteq$ **NP**.
Pf. Consider any problem $X \in$ **P**.
- By definition, there exists a poly-time algorithm $A(s)$ that solves X.
- Certificate $t = \varepsilon$, certifier $C(s, t) = A(s)$. ▪

Claim. **NP** $\subseteq$ **EXP**.
Pf. Consider any problem $X \in$ **NP**.
- By definition, there exists a poly-time certifier $C(s, t)$ for X.
- To solve input s, run $C(s, t)$ on all strings t with $|t| \le p(|s|)$.
- Return *yes* if $C(s, t)$ returns *yes* for any of these potential certificates. ▪

Remark. Time-hierarchy theorem implies **P** $\subsetneq$ **EXP**.

The main question: P vs. NP

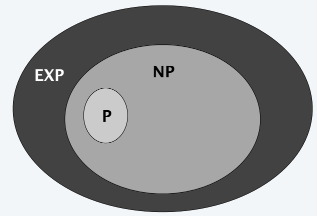
Q. How to solve an instance of 3-SAT with n variables?
A. Exhaustive search: try all 2^n truth assignments.

Q. Can we do anything substantially more clever?
Conjecture. No poly-time algorithm for 3-SAT.

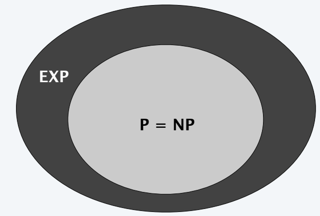
"intractable"

The main question: P vs. NP

Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]
Is the decision problem as easy as the certification problem?



If P ≠ NP



If P = NP

If yes. Efficient algorithms for 3-SAT, TSP, 3-COLOR, FACTOR, …
If no. No efficient algorithms possible for 3-SAT, TSP, 3-COLOR, …

Consensus opinion. Probably no.

Possible outcomes

P ≠ NP.

> *" I conjecture that there is no good algorithm for the traveling salesman problem. My reasons are the same as for any mathematical conjecture: (i) It is a legitimate mathematical possibility and (ii) I do not know."*
> — *Jack Edmonds 1966*

Possible outcomes

P ≠ NP.

> *" In my view, there is no way to even make intelligent guesses about the answer to any of these questions. If I had to bet now, I would bet that P is not equal to NP. I estimate the half-life of this problem at 25–50 more years, but I wouldn't bet on it being solved before 2100. "*
> — *Bob Tarjan*

> *" We seem to be missing even the most basic understanding of the nature of its difficulty…. All approaches tried so far probably (in some cases, provably) have failed. In this sense P =NP is different from many other major mathematical problems on which a gradual progress was being constantly done (sometimes for centuries) whereupon they yielded, either completely or partially. "*
> — *Alexander Razborov*

Possible outcomes

P = NP.

> “ *P = NP. In my opinion this shouldn't really be a hard problem; it's just that we came late to this theory, and haven't yet developed any techniques for proving computations to be hard. Eventually, it will just be a footnote in the books.* ” — *John Conway*

Other possible outcomes

P = NP, but only $\Omega(n^{100})$ algorithm for 3-SAT.

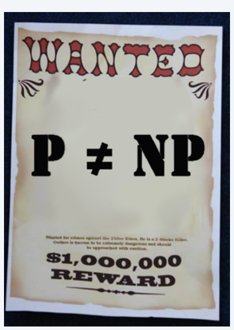
P ≠ NP, but with $O(n^{\log^* n})$ algorithm for 3-SAT.

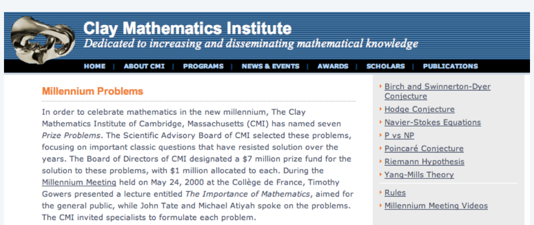
P = NP is independent (of ZFC axiomatic set theory).

> “ *It will be solved by either 2048 or 4096. I am currently somewhat pessimistic. The outcome will be the truly worst case scenario: namely that someone will prove “P = NP because there are only finitely many obstructions to the opposite hypothesis”; hence there will exists a polynomial time solution to SAT but we will never know its complexity!* ” — *Donald Knuth*

Millennium prize

Millennium prize. $1 million for resolution of **P** = **NP** problem.

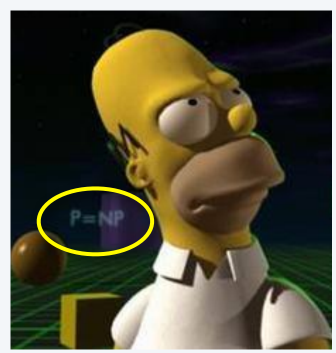




Looking for a job?

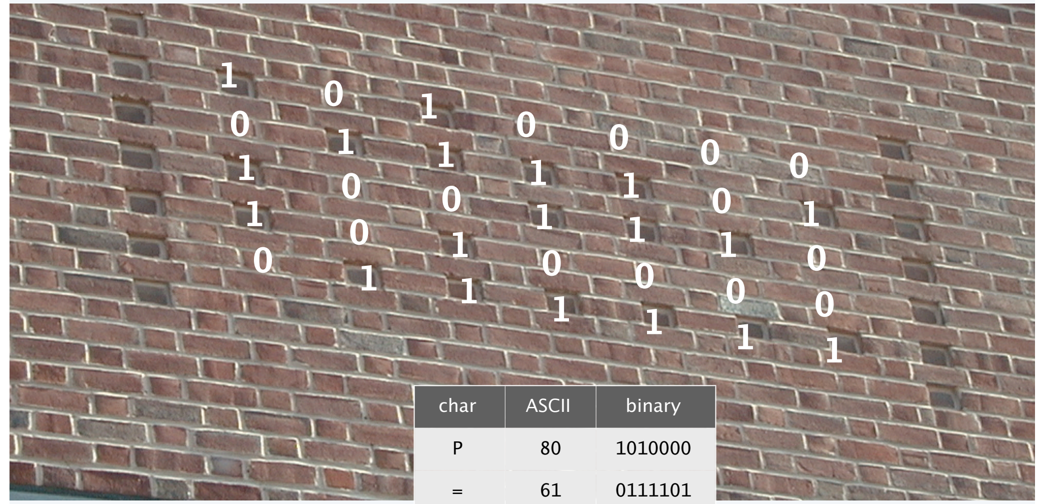
Some writers for the Simpsons and Futurama.

- J. Steward Burns. *M.S. in mathematics* (*Berkeley '93*).
- David X. Cohen. *M.S. in computer science* (*Berkeley '92*).
- Al Jean. *B.S. in mathematics*. (*Harvard '81*).
- Ken Keeler. *Ph.D. in applied mathematics* (*Harvard '90*).
- Jeff Westbrook. *Ph.D. in computer science* (*Princeton '89*).

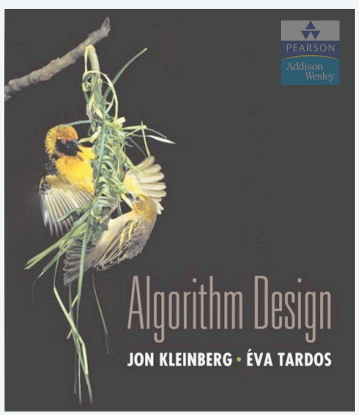


Princeton CS Building, West Wall, Circa 2001

Princeton CS Building, West Wall, Circa 2001



char	ASCII	binary
P	80	1010000
=	61	0111101
N	78	1001110
P	80	1010000
?	63	0111111



SECTION 8.4

8. INTRACTABILITY II

- *P vs. NP*
- *NP-complete*
- *co-NP*
- *NP-hard*

Polynomial transformation

Def. Problem X polynomial (Cook) reduces to problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to oracle that solves problem Y.

Def. Problem X polynomial (Karp) transforms to problem Y if given any input x to X, we can construct an input y such that x is a *yes* instance of X iff y is a *yes* instance of Y.

we require |y| to be of size polynomial in |x|

Note. Polynomial transformation is polynomial reduction with just one call to oracle for Y, exactly at the end of the algorithm for X. Almost all previous reductions were of this form.

Open question. Are these two concepts the same with respect to **NP**?

we abuse notation $\le_p$ and blur distinction

NP-complete

NP-complete. A problem $Y \in$ **NP** with the property that for every problem $X \in$ **NP**, $X \leq_p Y$.

Theorem. Suppose $Y \in$ **NP**-complete. Then $Y \in$ **P** iff **P** = **NP**.

Pf. $\Leftarrow$ If **P** = **NP**, then $Y \in$ **P** because $Y \in$ **NP**.

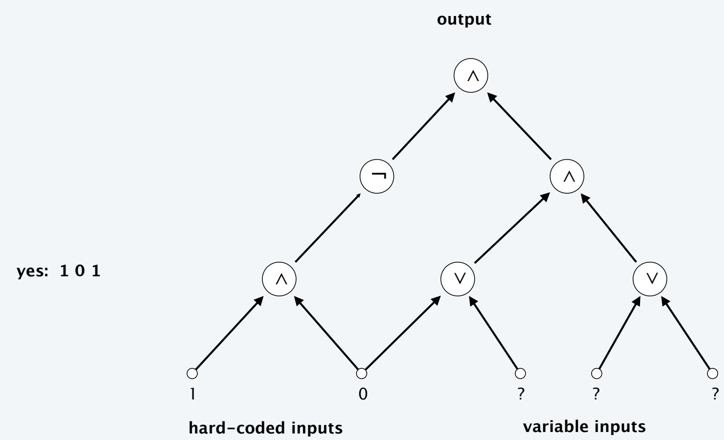
Pf. $\Rightarrow$ Suppose $Y \in$ **P**.

- Consider any problem $X \in$ **NP**. Since $X \leq_p Y$, we have $X \in$ **P**.
- This implies **NP** $\subseteq$ **P**.
- We already know **P** $\subseteq$ **NP**. Thus **P** = **NP**. ▪

Fundamental question. Do there exist "natural" **NP**-complete problems?

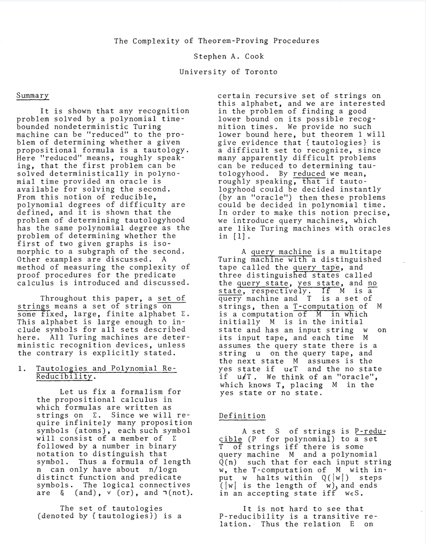
Circuit satisfiability

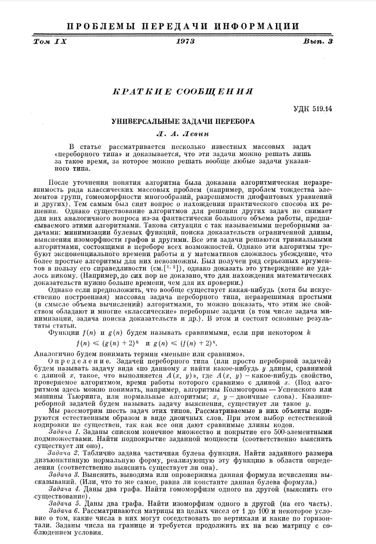
CIRCUIT-SAT. Given a combinational circuit built from AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?



The "first" NP-complete problem

Theorem. CIRCUIT-SAT $\in$ **NP**-complete. [Cook 1971, Levin 1973]





The "first" NP-complete problem

Theorem. CIRCUIT-SAT $\in$ **NP**-complete.

Pf sketch.

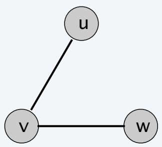
- Clearly, CIRCUIT-SAT $\in$ **NP**.
- Any algorithm that takes a fixed number of bits n as input and produces a *yes* or *no* answer can be represented by such a circuit.
- Moreover, if algorithm takes poly-time, then circuit is of poly-size.

sketchy part of proof; fixing the number of bits is important, and reflects basic distinction between algorithms and circuits

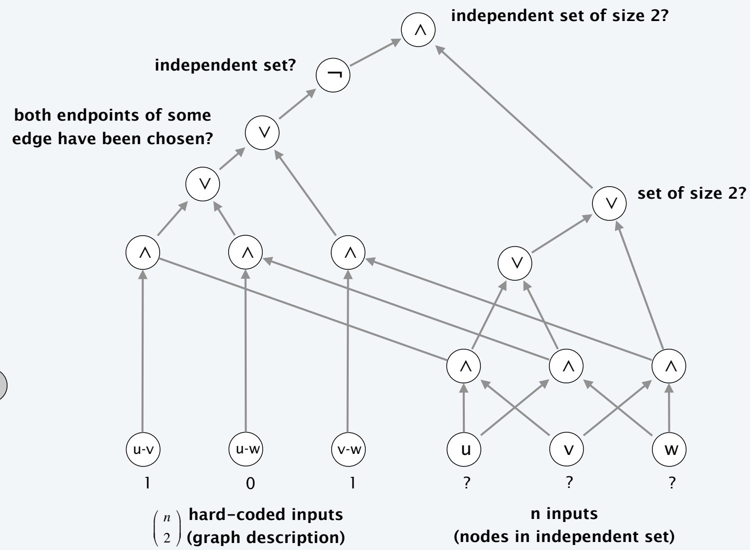
- Consider any problem $X \in$ **NP**. It has a poly-time certifier $C(s, t)$: $s \in X$ iff there exists a certificate t of length $p(|s|)$ such that $C(s, t) = yes$.
- View $C(s, t)$ as an algorithm with $|s| + p(|s|)$ input bits and convert it into a poly-size circuit K.
  - first $|s|$ bits are hard-coded with s
  - remaining $p(|s|)$ bits represent (unknown) bits of t
- Circuit K is satisfiable iff $C(s, t) = yes$.

Example

Ex. Construction below creates a circuit K whose inputs can be set so that it outputs 1 iff graph G has an independent set of size 2.



G = (V, E), n = 3



Establishing NP-completeness

Remark. Once we establish first "natural" **NP**-complete problem, others fall like dominoes.

Recipe. To prove that $Y \in$ **NP**-complete:

- Step 1. Show that $Y \in$ **NP**.
- Step 2. Choose an **NP**-complete problem X.
- Step 3. Prove that $X \leq_p Y$.

Theorem. If $X \in$ **NP**-complete, $Y \in$ **NP**, and $X \leq_p Y$, then $Y \in$ **NP**-complete.

Pf. Consider any problem $W \in$ **NP**. Then, both $W \leq_p X$ and $X \leq_p Y$.

- By transitivity, $W \leq_p Y$.
- Hence $Y \in$ **NP**-complete. ▪

by definition of NP-complete

by assumption

3-satisfiability is NP-complete

Theorem. 3-SAT $\in$ **NP**-complete.

Pf.

- Suffices to show that CIRCUIT-SAT $\leq_P$ 3-SAT since 3-SAT $\in$ **NP**.
- Given a combinational circuit K, we construct an instance Φ of 3-SAT that is satisfiable iff the inputs of K can be set so that it outputs 1.

3-satisfiability is NP-complete

Construction. Let K be any circuit.

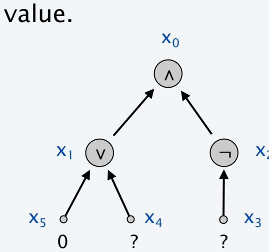
Step 1. Create a 3-SAT variable x_i for each circuit element i.

Step 2. Make circuit compute correct values at each node:

- $x_2 = \neg x_3$ $\Rightarrow$ add 2 clauses: $x_2 \vee x_3$, $\overline{x_2} \vee \overline{x_3}$
- $x_1 = x_4 \vee x_5$ $\Rightarrow$ add 3 clauses: $x_1 \vee \overline{x_4}$, $x_1 \vee \overline{x_5}$, $\overline{x_1} \vee x_4 \vee x_5$
- $x_0 = x_1 \wedge x_2$ $\Rightarrow$ add 3 clauses: $\overline{x_0} \vee x_1$, $\overline{x_0} \vee x_2$, $x_0 \vee \overline{x_1} \vee \overline{x_2}$

Step 3. Hard-coded input values and output value.

- $x_5 = 0$ $\Rightarrow$ add 1 clause: $\overline{x_5}$
- $x_0 = 1$ $\Rightarrow$ add 1 clause: x_0



3-satisfiability is NP-complete

Construction. [continued]

Step 4. Turn clauses of length 1 or 2 into clauses of length 3.

- Create four new variables z_1, z_2, z_3, and z_4.
- Add 8 clauses to force $z_1 = z_2 = false$:

$$(\overline{z_1} \vee z_3 \vee z_4),\ (\overline{z_1} \vee z_3 \vee \overline{z_4}),\ (\overline{z_1} \vee \overline{z_3} \vee z_4),\ (\overline{z_1} \vee \overline{z_3} \vee \overline{z_4})$$
$$(\overline{z_2} \vee z_3 \vee z_4),\ (\overline{z_2} \vee z_3 \vee \overline{z_4}),\ (\overline{z_2} \vee \overline{z_3} \vee z_4),\ (\overline{z_2} \vee \overline{z_3} \vee \overline{z_4})$$

- Replace any clause with a single term (t_i) with $(t_i \vee z_1 \vee z_2)$.
- Replace any clause with two terms $(t_i \vee t_j)$ with $(t_i \vee t_j \vee z_1)$.

3-satisfiability is NP-complete

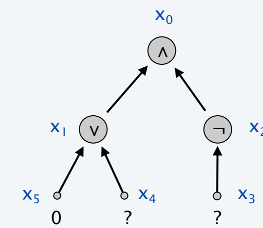
Lemma. Φ is satisfiable iff the inputs of K can be set so that it outputs 1.

Pf. $\Leftarrow$ Suppose there are inputs of K that make it output 1.

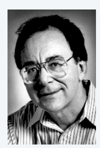
- Can propagate input values to create values at all nodes of K.
- This set of values satisfies Φ.

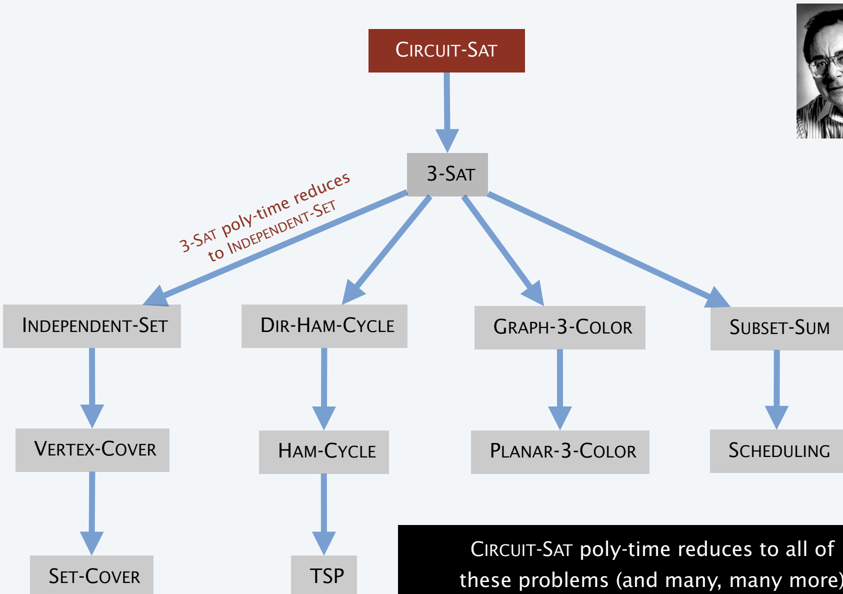
Pf. $\Rightarrow$ Suppose Φ is satisfiable.

- We claim that the set of values corresponding to the circuit inputs constitutes a way to make circuit K output 1.
- The 3-SAT clauses were designed to ensure that the values assigned to all node in K exactly match what the circuit would compute for these nodes. ▪



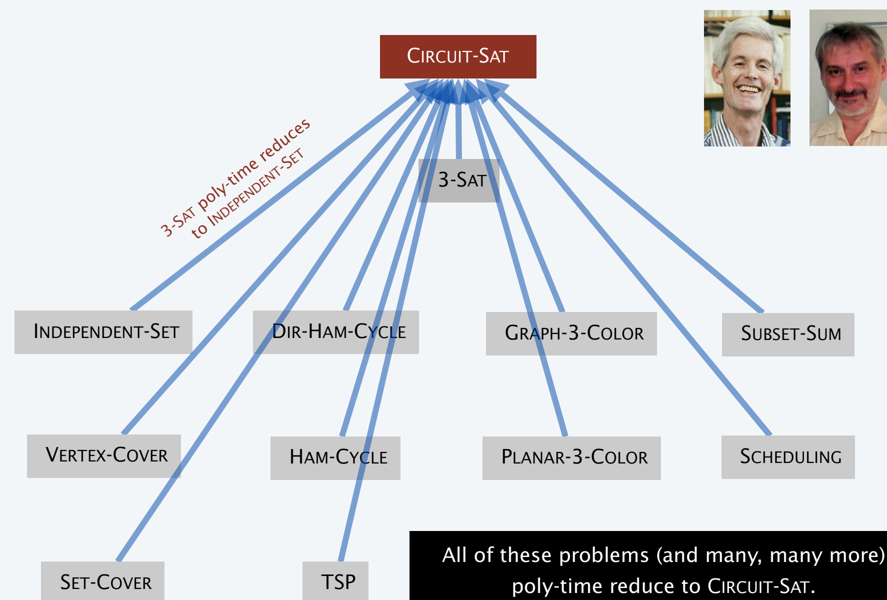
Implications of Karp





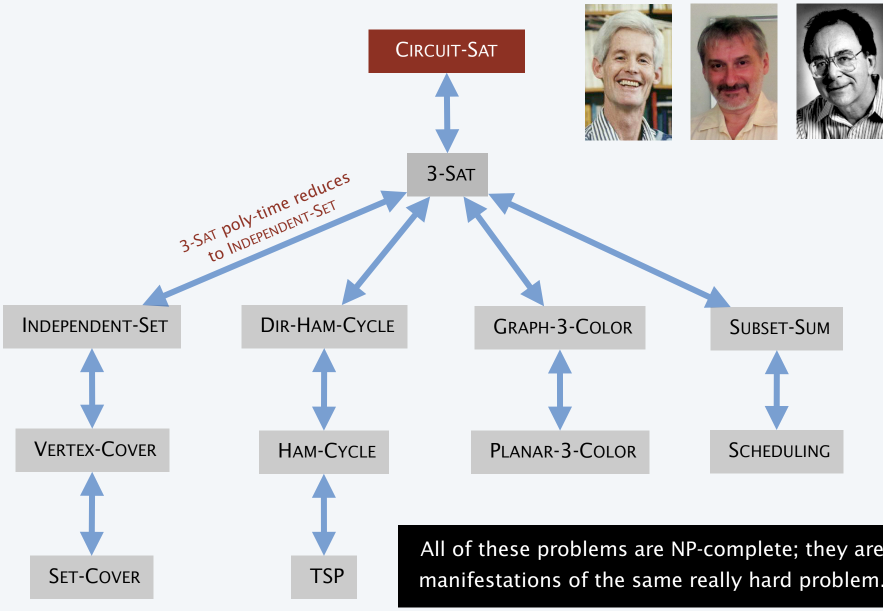
CIRCUIT-SAT poly-time reduces to all of these problems (and many, many more)

Implications of Cook-Levin



All of these problems (and many, many more) poly-time reduce to CIRCUIT-SAT.

Implications of Karp + Cook-Levin



All of these problems are NP-complete; they are manifestations of the same really hard problem.

Some NP-complete problems

Basic genres of NP-complete problems and paradigmatic examples.

- Packing + covering problems: SET-COVER, VERTEX-COVER, INDEPENDENT-SET.
- Constraint satisfaction problems: CIRCUIT-SAT, SAT, 3-SAT.
- Sequencing problems: HAM-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, PARTITION.

Practice. Most **NP** problems are known to be either in **P** or **NP**-complete.

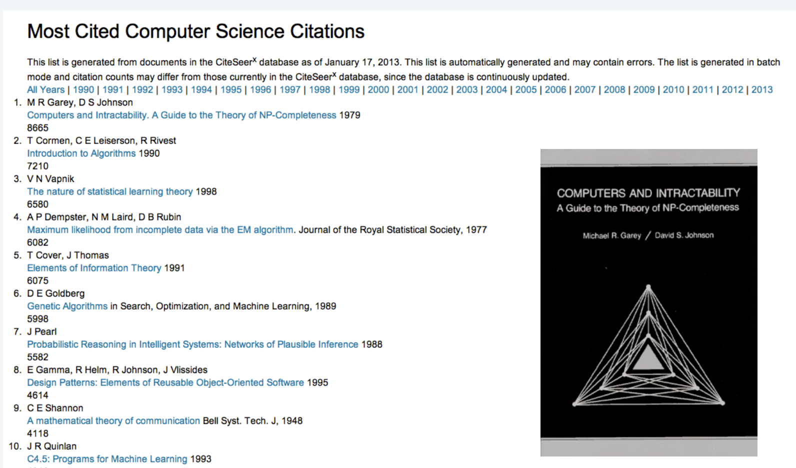
Notable exceptions. FACTOR, GRAPH-ISOMORPHISM, NASH-EQUILIBRIUM.

Theory. [Ladner 1975] Unless **P** = **NP**, there exist problems in **NP** that are neither in **P** nor **NP**-complete.

More hard computational problems

Garey and Johnson. Computers and Intractability.

- Appendix includes over 300 **NP**-complete problems.
- Most cited reference in computer science literature.



Most Cited Computer Science Citations

This list is generated from documents in the CiteSeer[x] database as of January 17, 2013. This list is automatically generated and may contain errors. The list is generated in batch mode and citation counts may differ from those currently in the CiteSeer[x] database, since the database is continuously updated.

All Years | 1990 | 1991 | 1992 | 1993 | 1994 | 1995 | 1996 | 1997 | 1998 | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 | 2005 | 2006 | 2007 | 2008 | 2009 | 2010 | 2011 | 2012 | 2013

1. M R Garey, D S Johnson
   Computers and Intractability. A Guide to the Theory of NP-Completeness 1979
   8665
2. T Cormen, C E Leiserson, R Rivest
   Introduction to Algorithms 1990
   7210
3. V N Vapnik
   The nature of statistical learning theory 1998
   6580
4. A P Dempster, N M Laird, D B Rubin
   Maximum likelihood from incomplete data via the EM algorithm. Journal of the Royal Statistical Society, 1977
   6082
5. T Cover, J Thomas
   Elements of Information Theory 1991
   6075
6. D E Goldberg
   Genetic Algorithms in Search, Optimization, and Machine Learning, 1989
   5998
7. J Pearl
   Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference 1988
   5582
8. E Gamma, R Helm, R Johnson, J Vlissides
   Design Patterns: Elements of Reusable Object-Oriented Software 1995
   4614
9. C E Shannon
   A mathematical theory of communication Bell Syst. Tech. J, 1948
   4118
10. J R Quinlan
    C4.5: Programs for Machine Learning 1993
    4018

More hard computational problems

Aerospace engineering. Optimal mesh partitioning for finite elements.
Biology. Phylogeny reconstruction.
Chemical engineering. Heat exchanger network synthesis.
Chemistry. Protein folding.
Civil engineering. Equilibrium of urban traffic flow.
Economics. Computation of arbitrage in financial markets with friction.
Electrical engineering. VLSI layout.
Environmental engineering. Optimal placement of contaminant sensors.
Financial engineering. Minimum risk portfolio of given return.
Game theory. Nash equilibrium that maximizes social welfare.
Mathematics. Given integer $a_1, \ldots, a_n$, compute $\int_0^{2\pi} \cos(a_1\theta) \times \cos(a_2\theta) \times \cdots \times \cos(a_n\theta)\, d\theta$
Mechanical engineering. Structure of turbulence in sheared flows.
Medicine. Reconstructing 3d shape from biplane angiocardiogram.
Operations research. Traveling salesperson problem.
Physics. Partition function of 3d Ising model.
Politics. Shapley-Shubik voting power.
Recreation. Versions of Sudoko, Checkers, Minesweeper, Tetris.
Statistics. Optimal experimental design.

Extent and impact of NP-completeness

Extent of NP-completeness. [Papadimitriou 1995]

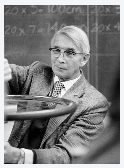
- Prime intellectual export of CS to other disciplines.
- 6,000 citations per year (more than "compiler", "OS", "database").
- Broad applicability and classification power.

NP-completeness can guide scientific inquiry.

- 1926: Ising introduces simple model for phase transitions.
- 1944: Onsager finds closed-form solution to 2D-ISING in tour de force.
- 19xx: Feynman and other top minds seek solution to 3D-ISING.
- 2000: Istrail proves 3D-ISING $\in$ **NP**-complete.

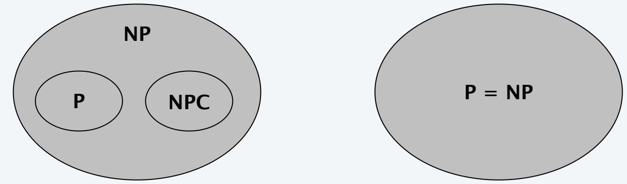
a holy grail of statistical mechanics

search for closed formula appears doomed




P vs. NP revisited

Overwhelming consensus (still). P ≠ NP.



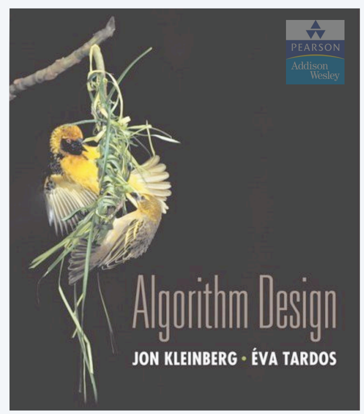
P ≠ NP

P = NP

Why we believe **P** ≠ **NP**.

> " *We admire Wiles' proof of Fermat's last theorem, the scientific theories of Newton, Einstein, Darwin, Watson and Crick, the design of the Golden Gate bridge and the Pyramids, precisely because they seem to require a leap which cannot be made by everyone, let alone a by simple mechanical device.* " — *Avi Wigderson*

You NP-complete me



SECTION 8.9

8. INTRACTABILITY II

- P vs. NP
- NP-complete
- *co-NP*
- NP-hard

Asymmetry of NP

Asymmetry of NP. We only need to have short proofs of *yes* instances.

Ex 1. SAT vs. TAUTOLOGY.

- Can prove a CNF formula is satisfiable by specifying an assignment.
- How could we prove that a formula is not satisfiable?

Ex 2. HAM-CYCLE vs. NO-HAM-CYCLE.

- Can prove a graph is Hamiltonian by specifying a permutation.
- How could we prove that a graph is not Hamiltonian?

Q. How to classify TAUTOLOGY and NO-HAMILTON-CYCLE ?

- SAT ∈ **NP**-complete and SAT $\equiv_P$ TAUTOLOGY.
- HAM-CYCLE ∈ **NP**-complete and HAM-CYCLE $\equiv_P$ NO-HAM-CYCLE.
- But neither TAUTOLOGY nor NO-HAM-CYCLE are known to be in NP.

NP and co-NP

NP. Decision problems for which there is a poly-time certifier.
Ex. SAT, HAMILTON-CYCLE, and COMPOSITE.

Def. Given a decision problem X, its complement $\overline{X}$ is the same problem with the *yes* and *no* answers reverse.

Ex. $X = \{ 0, 1, 4, 6, 8, 9, 10, 12, 14, 15, \ldots \}$
$\overline{X} = \{ 2, 3, 5, 7, 11, 13, 17, 23, 29, \ldots \}$

co-NP. Complements of decision problems in **NP**.
Ex. TAUTOLOGY, NO-HAMILTON-CYCLE, and PRIMES.

NP = co-NP ?

Fundamental open question. Does **NP** = **co-NP**?

- Do *yes* instances have succinct certificates iff *no* instances do?
- Consensus opinion: no.

Theorem. If **NP** ≠ **co-NP**, then **P** ≠ **NP**.
Pf idea.

- **P** is closed under complementation.
- If **P** = **NP**, then **NP** is closed under complementation.
- In other words, **NP** = **co-NP**.
- This is the contrapositive of the theorem.

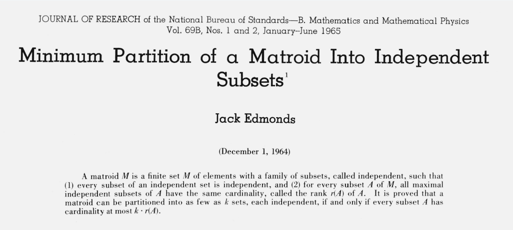
Good characterizations

Good characterization. [Edmonds 1965] **NP** ∩ **co-NP**.

- If problem X is in both **NP** and **co-NP**, then:
  - for *yes* instance, there is a succinct certificate
  - for *no* instance, there is a succinct disqualifier
- Provides conceptual leverage for reasoning about a problem.

Ex. Given a bipartite graph, is there a perfect matching.

- If yes, can exhibit a perfect matching.
- If no, can exhibit a set of nodes S such that $|N(S)| < |S|$.


JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 69B, Nos. 1 and 2, January–June 1965

Minimum Partition of a Matroid Into Independent Subsets[1]

Jack Edmonds

(December 1, 1964)

A matroid M is a finite set M of elements with a family of subsets, called independent, such that (1) every subset of an independent set is independent, and (2) for every subset A of M, all maximal independent subsets of A have the same cardinality, called the rank $r(A)$ of A. It is proved that a matroid can be partitioned into as few as k sets, each independent, if and only if every subset A has cardinality at most $k \cdot r(A)$.

Good characterizations

> We seek a good characterization of the minimum number of independent sets into which the columns of a matrix of M_F can be partitioned. As the criterion of "good" for the characterization we apply the "principle of the absolute supervisor." The good characterization will describe certain information about the matrix which the supervisor can require his assistant to search out along with a minimum partition and which the supervisor can then use with ease to verify with mathematical certainty that the partition is indeed minimum. Having a good characterization does not mean necessarily that there is a good algorithm. The assistant might have to kill himself with work to find the information and the partition.

Good characterizations

Observation. **P** ⊆ **NP** ∩ **co-NP**.

- Proof of max-flow min-cut theorem led to stronger result that max-flow and min-cut are in **P**.
- Sometimes finding a good characterization seems easier than finding an efficient algorithm.

Fundamental open question. Does **P** = **NP** ∩ **co-NP**?

- Mixed opinions.
- Many examples where problem found to have a nontrivial good characterization, but only years later discovered to be in **P**.

Linear programming is in NP ∩ co-NP

Linear programming. Given $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, and $\alpha \in \Re$, does there exist $x \in \Re^n$ such that $Ax \le b$, $x \ge 0$ and $c^T x \ge \alpha$?

Theorem. [Gale-Kuhn-Tucker 1948] LINEAR-PROGRAMMING ∈ **NP** ∩ **co-NP**.
Pf sketch. If (P) and (D) are nonempty, then max = min.

$$(P)\quad \max c^T x \quad \text{s.t. } Ax \le b,\; x \ge 0 \qquad\qquad (D)\quad \min y^T b \quad \text{s.t. } A^T y \ge c,\; y \ge 0$$

CHAPTER XIX

LINEAR PROGRAMMING AND THE THEORY OF GAMES[1]

BY DAVID GALE, HAROLD W. KUHN, AND ALBERT W. TUCKER[2]

The basic "scalar" problem of *linear programming* is to maximize (or minimize) a linear function of several variables constrained by a system of linear inequalities [Dantzig, II]. A more general "vector" problem calls for maximizing (in a sense of partial order) a system of linear functions of several variables subject to a system of linear inequalities and, perhaps, linear equations [Koopmans, III]. The purpose of this chapter is to establish theorems of duality and existence for general "matrix" problems of linear programming which contain the "scalar" and "vector" problems as special cases, and to relate these general problems to the theory of zero-sum two-person games.

Linear programming is in NP ∩ co-NP

Linear programming. Given $A \in \Re^{m \times n}$, $b \in \Re^m$, $c \in \Re^n$, and $\alpha \in \Re$, does there exist $x \in \Re^n$ such that $Ax \le b$, $x \ge 0$ and $c^T x \ge \alpha$?

Theorem. [Khachiyan 1979] LINEAR-PROGRAMMING ∈ **P**.

ЖУРНАЛ
ВЫЧИСЛИТЕЛЬНОЙ МАТЕМАТИКИ И МАТЕМАТИЧЕСКОЙ ФИЗИКИ

Том 20 Январь 1980 Февраль № 1

УДК 519.852

ПОЛИНОМИАЛЬНЫЕ АЛГОРИТМЫ В ЛИНЕЙНОМ ПРОГРАММИРОВАНИИ

Л. Г. ХАЧИЯН

(Москва)

Построены точные алгоритмы линейного программирования, трудоемкость которых ограничена полиномом от длины двоичной записи задачи.

Primality testing is in NP ∩ co-NP

Theorem. [Pratt 1975] PRIMES $\in$ **NP** $\cap$ **co-NP**.

SIAM J. Comput.
Vol. 4, No. 3, September 1975

EVERY PRIME HAS A SUCCINCT CERTIFICATE*

VAUGHAN R. PRATT†

Abstract. To prove that a number n is composite, it suffices to exhibit the working for the multiplication of a pair of factors. This working, represented as a string, is of length bounded by a polynomial in $\log_2 n$. We show that the same property holds for the primes. It is noteworthy that almost no other set is known to have the property that short proofs for membership or nonmembership exist for all candidates without being known to have the property that such proofs are easy to come by. It remains an open problem whether a prime n can be recognized in only $\log_2^\alpha n$ operations of a Turing machine for any fixed α.

The proof system used for certifying primes is as follows.

AXIOM. $(x, y, 1)$.

INFERENCE RULES.

R_1: $(p, x, a), q \vdash (p, x, qa)$ provided $x^{(p-1)/q} \not\equiv 1 \pmod p$ and $q | (p-1)$.

R_2: $(p, x, p-1) \vdash p$ provided $x^{p-1} \equiv 1 \pmod p$.

THEOREM 1. *p is a theorem $\equiv$ p is a prime.*

THEOREM 2. *p is a theorem $\supset$ p has a proof of $\lceil 4 \log_2 p \rceil$ lines.*

Primality testing is in NP ∩ co-NP

Theorem. [Pratt 1975] PRIMES $\in$ **NP** $\cap$ **co-NP**.

Pf sketch. An odd integer s is prime iff there exists an integer $1 < t < s$ s.t.

$$t^{s-1} \equiv 1 \pmod s$$
$$t^{(s-1)/p} \not\equiv 1 \pmod s$$
for all prime divisors p of s-1

instance s	437677
certificate t	17, $2^2 \times 3 \times 36473$

prime factorization of s–1
also need a recursive certificate
to assert that 3 and 36,473 are prime

CERTIFIER (s)

CHECK $s - 1 = 2 \times 2 \times 3 \times 36473$.
CHECK $17^{s-1} = 1 \pmod s$.
CHECK $17^{(s-1)/2} \equiv 437676 \pmod s$.
CHECK $17^{(s-1)/3} \equiv 329415 \pmod s$.
CHECK $17^{(s-1)/36,473} \equiv 305452 \pmod s$.

use repeated squaring

Primality testing is in P

Theorem. [Agrawal-Kayal-Saxena 2004] PRIMES $\in$ **P**.

Annals of Mathematics, **160** (2004), 781–793

PRIMES is in P

By MANINDRA AGRAWAL, NEERAJ KAYAL, and NITIN SAXENA*

Abstract

We present an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite.

Factoring is in NP ∩ co-NP

FACTORIZE. Given an integer x, find its prime factorization.

FACTOR. Given two integers x and y, does x have a nontrivial factor $< y$?

Theorem. FACTOR $\equiv_P$ FACTORIZE.

Pf.

- $\le_P$ trivial.
- $\ge_P$ binary search to find a factor; divide out the factor and repeat. ▪

Theorem. FACTOR $\in$ **NP** $\cap$ **co-NP**.

Pf.

- Certificate: a factor p of x that is less than y.
- Disqualifier: the prime factorization of x (where each prime factor is less than y), along with a Pratt certificate that each factor is prime. ▪

Is factoring in P ?

Fundamental question. Is FACTOR ∈ **P**.

Challenge. Factor this number.

74037563479561712828046796097429573142593188889231289
08493623263897276503402826627689199641962511784399589
43305021275853701189680982867331732731089309005525051
16877063299072396380786710086096962537934650563796359

RSA-704
($30,000 prize if you can factor)

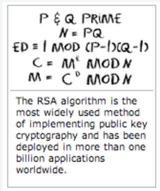
Exploiting intractability

Modern cryptography.

- Ex. Send your credit card to Amazon.
- Ex. Digitally sign an e-document.
- Enables freedom of privacy, speech, press, political association.

RSA. Based on dichotomy between complexity of two problems.

- To use: generate two random n-bit primes and multiply.
- To break: suffices to factor a $2n$-bit integer.



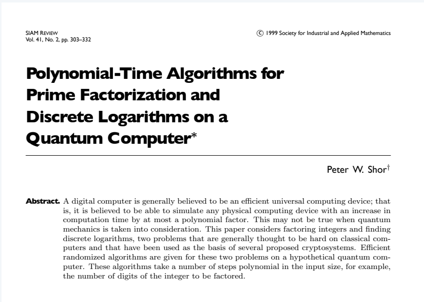
RSA algorithm

RSA sold for $2.1 billion

or design a t-shirt

Factoring on a quantum computer

Theorem. [Shor 1994] Can factor an n-bit integer in $O(n^3)$ steps on a "quantum computer."



SIAM REVIEW
Vol. 41, No. 2, pp. 303–332

Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer*

Peter W. Shor†

Abstract. A digital computer is generally believed to be an efficient universal computing device; that is, it is believed to be able to simulate any physical computing device with an increase in computation time by at most a polynomial factor. This may not be true when quantum mechanics is taken into consideration. This paper considers factoring integers and finding discrete logarithms, two problems that are generally thought to be hard on classical computers and that have been used as the basis of several proposed cryptosystems. Efficient randomized algorithms are given for these two problems on a hypothetical quantum computer. These algorithms take a number of steps polynomial in the input size, for example, the number of digits of the integer to be factored.

2001. Factored $15 = 3 \times 5$ (with high probability) on a quantum computer.
2012. Factored $21 = 3 \times 7$.

Fundamental question. Does **P** = **BQP** ?

8. INTRACTABILITY II

- *P vs. NP*
- *NP-complete*
- *co-NP*
- *NP-hard*

A note on terminology

SIGACT News 12 January 1974

A TERMINOLOGICAL PROPOSAL

D. E. Knuth

While preparing a book on combinatorial algorithms, I felt a strong need for a new technical term, a word which is essentially a one-sided version of polynomial complete. A great many problems of practical interest have the property that they are at least as difficult to solve in polynomial time as those of the Cook-Karp class NP. I needed an adjective to convey such a degree of difficulty, both formally and informally; and since the range of practical applications is so broad, I felt it would be best to establish such a term as soon as possible.

The goal is to find an adjective x that sounds good in sentences like this:

The covering problem is x .
It is x to decide whether a given graph has a Hamiltonian circuit.
It is unknown whether or not primality testing is an x problem.

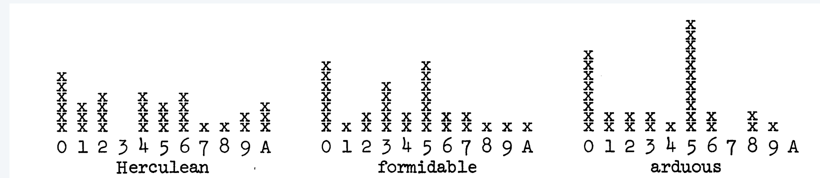
Note. The term x does not necessarily imply that a problem is in **NP**, just that every problem in **NP** poly-time reduces to x.

A note on terminology

Knuth's original suggestions.

- Hard.
- Tough.
- Herculean.
- Formidable.
- Arduous.

so common that it is unclear whether it is being used in a technical sense



assign a real number between 0 and 1 to each choice

A note on terminology

Some English word write-ins.

- Impractical.
- Bad.
- Heavy.
- Tricky.
- Intricate.
- Prodigious.
- Difficult.
- Intractable.
- Costly.
- Obdurate.
- Obstinate.
- Exorbitant.
- Interminable.

A note on terminology

Hard-boiled. [Ken Steiglitz] In honor of Cook.

Hard-ass. [Al Meyer] Hard as satisfiability.

Sisyphean. [Bob Floyd] Problem of Sisyphus was time-consuming.

Ulyssean. [Don Knuth] Ulysses was known for his persistence.

" creative research workers are as full of ideas for new terminology as they are empty of enthusiasm for adopting it. "

— Donald Knuth

A note on terminology: acronyms

PET. [Shen Lin] Probably exponential time.

- If **P** ≠ **NP**, provably exponential time.
- If **P** = **NP**, previously exponential time.

GNP. [Al Meyer] Greater than or equal to **NP** in difficulty.

- And costing more than the GNP to solve.

A note on terminology: made-up words

Exparent. [Mike Paterson] Exponential + apparent.

Perarduous. [Mike Paterson] Through (in space or time) + completely.

Supersat. [Al Meyer] Greater than or equal to satisfiability.

Polychronious. [Ed Reingold] Enduringly long; chronic.

A note on terminology: consensus

NP-complete. A problem in **NP** such that every problem in **NP** poly-time reduces to it.

NP-hard. [Bell Labs, Steve Cook, Ron Rivest, Sartaj Sahni]
A problem such that every problem in **NP** polynomial-time reduces to it.

One final criticism (which applies to all the terms suggested) was stated nicely by Vaughan Pratt: "If the Martians know that P = NP for Turing Machines and they kidnap me, I would lose face calling these problems 'formidable'." Yes; if P = NP , there's no need for any term at all. But I'm willing to risk such an embarrassment, and in fact I'm willing to give a prize of one live turkey to the first person who proves that P = NP .