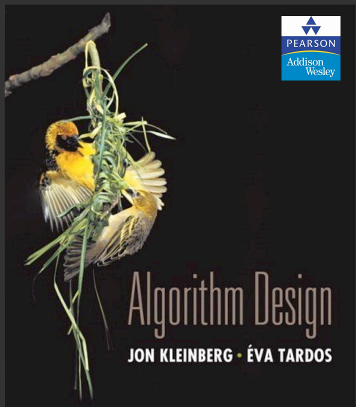

5. Divide And Conquer I

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *median and selection*

Lecture slides by Kevin Wayne
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http://www.cs.princeton.edu/~wayne/kleinberg-tardos

Last updated on May 7, 2013 8:59 AM

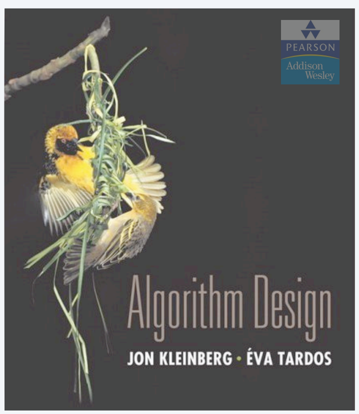
Divide-and-conquer paradigm

Divide-and-conquer.
- Divide up problem into several subproblems.
- Solve each subproblem recursively.
- Combine solutions to subproblems into overall solution.

Most common usage.
- Divide problem of size n into two subproblems of size $n/2$ in linear time.
- Solve two subproblems recursively.
- Combine two solutions into overall solution in linear time.

Consequence.
- Brute force: $\Theta(n^2)$.
- Divide-and-conquer: $\Theta(n \log n)$.

attributed to Julius Caesar



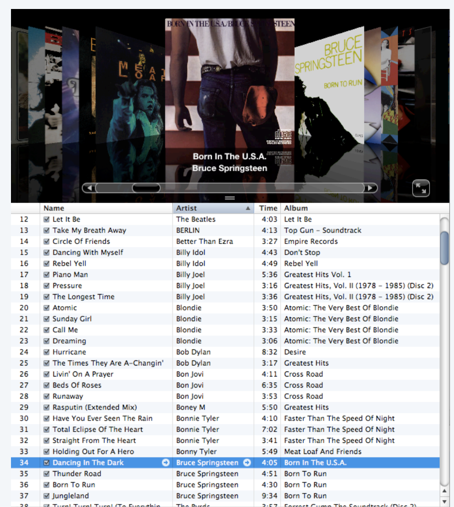
Section 5.1

5. Divide and Conquer

- *mergesort*
- *counting inversions*
- *closest pair of points*
- *median and selection*

Sorting problem

Problem. Given a list of n elements from a totally ordered universe, rearrange them in ascending order.



Sorting applications

Obvious applications.

- Organize an MP3 library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

Some problems become easier once elements are sorted.

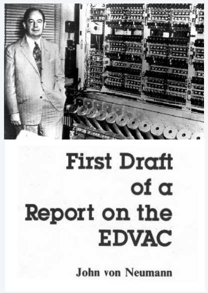
- Identify statistical outliers.
- Binary search in a database.
- Remove duplicates in a mailing list.

Non-obvious applications.

- Convex hull.
- Closest pair of points.
- Interval scheduling / interval partitioning.
- Minimum spanning trees (Kruskal's algorithm).
- Scheduling to minimize maximum lateness or average completion time.
- ...

Mergesort

- Recursively sort left half.
- Recursively sort right half.
- Merge two halves to make sorted whole.



input

A	L	G	O	R	I	T	H	M	S

sort left half

A	G	L	O	R	I	T	H	M	S

sort right half

A	G	L	O	R	H	I	M	S	T

merge results

A	G	H	I	L	M	O	R	S	T

Merging

Goal. Combine two sorted lists A and B into a sorted whole C.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i \leq b_j$, append a_i to C (no larger than any remaining element in B).
- If $a_i > b_j$, append b_j to C (smaller than every remaining element in A).

sorted list A

3	7	10	a_i	18

sorted list B

2	11	b_j	17	23
5	2			

merge to form sorted list C

2	3	7	10	11					

A useful recurrence relation

Def. $T(n)$ = max number of compares to mergesort a list of size $\leq n$.
Note. $T(n)$ is monotone nondecreasing.

Mergesort recurrence.

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n & \text{otherwise} \end{cases}$$

Solution. $T(n)$ is $O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence. Initially we assume n is a power of 2 and replace $\leq$ with $=$.

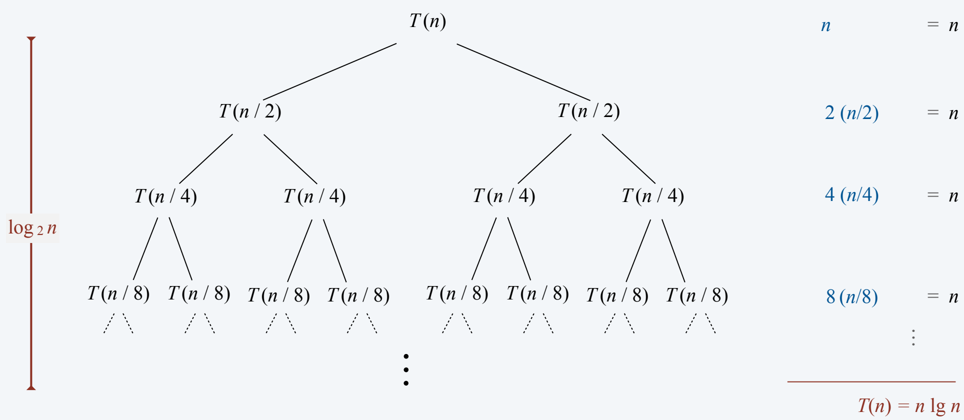
Divide-and-conquer recurrence: proof by recursion tree

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

Pf 1.



Proof by induction

Proposition. If $T(n)$ satisfies the following recurrence, then $T(n) = n \log_2 n$.

assuming n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2\,T(n/2) + n & \text{otherwise} \end{cases}$$

Pf 2. [by induction on n]

- Base case: when $n = 1$, $T(1) = 0$.
- Inductive hypothesis: assume $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

$$\begin{aligned} T(2n) &= 2\,T(n) + 2n \\ &= 2\,n \log_2 n + 2n \\ &= 2\,n\,(\log_2 (2n) - 1) + 2n \\ &= 2\,n \log_2 (2n). \quad \blacksquare \end{aligned}$$

Analysis of mergesort recurrence

Claim. If $T(n)$ satisfies the following recurrence, then $T(n) \le n \lceil \log_2 n \rceil$.

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n & \text{otherwise} \end{cases}$$

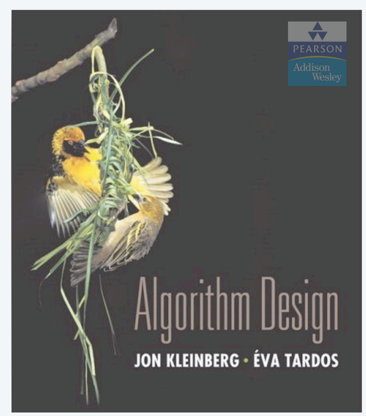
Pf. [by strong induction on n]

- Base case: $n = 1$.
- Define $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$.
- Induction step: assume true for $1, 2, \ldots, n-1$.

$$\begin{aligned} T(n) &\le T(n_1) + T(n_2) + n \\ &\le n_1 \lceil \log_2 n_1 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &\le n_1 \lceil \log_2 n_2 \rceil + n_2 \lceil \log_2 n_2 \rceil + n \\ &= n \lceil \log_2 n_2 \rceil + n \\ &\le n (\lceil \log_2 n \rceil - 1) + n \\ &= n \lceil \log_2 n \rceil. \quad \blacksquare \end{aligned}$$

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\le \left\lceil 2^{\lceil \log_2 n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \log_2 n \rceil} / 2 \end{aligned}$$

$$\log_2 n_2 \le \lceil \log_2 n \rceil - 1$$



SECTION 5.3

5. DIVIDE AND CONQUER

- mergesort
- *counting inversions*
- closest pair of points
- median and selection

Counting inversions

Music site tries to match your song preferences with others.

- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: $1, 2, \ldots, n$.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j are inverted if $i < j$, but $a_i > a_j$.

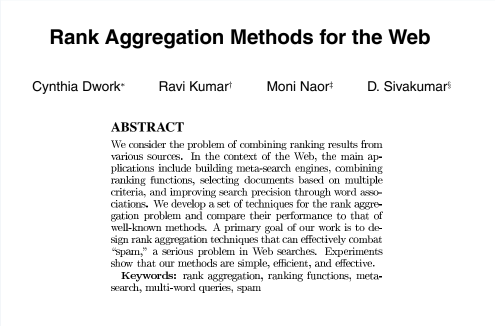
	A	B	C	D	E
me	1	2	3	4	5
you	1	3	4	2	5

2 inversions: 3–2, 4–2

Brute force: check all $\Theta(n^2)$ pairs.

Counting inversions: applications

- Voting theory.
- Collaborative filtering.
- Measuring the "sortedness" of an array.
- Sensitivity analysis of Google's ranking function.
- Rank aggregation for meta-searching on the Web.
- Nonparametric statistics (e.g., Kendall's tau distance).



Rank Aggregation Methods for the Web

Cynthia Dwork* Ravi Kumar† Moni Naor‡ D. Sivakumar§

ABSTRACT

We consider the problem of combining ranking results from various sources. In the context of the Web, the main applications include building meta-search engines, combining ranking functions, selecting documents based on multiple criteria, and improving search precision through word associations. We develop a set of techniques for the rank aggregation problem and compare their performance to that of well-known methods. A primary goal of our work is to design rank aggregation techniques that can effectively combat "spam," a serious problem in Web searches. Experiments show that our methods are simple, efficient, and effective.

Keywords: rank aggregation, ranking functions, meta-search, multi-word queries, spam

Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with $a \in A$ and $b \in B$.
- Return sum of three counts.

input

1	5	4	8	10	2	6	9	3	7

count inversions in left half A

1	5	4	8	10

5–4

count inversions in right half B

2	6	9	3	7

6–3 9–3 9–7

count inversions (a, b) with a ∈ A and b ∈ B

1	5	4	8	10	2	6	9	3	7

4–2 4–3 5–2 5–3 8–2 8–3 8–6 8–7 10–2 10–3 10–6 10–7 10–9

output 1 + 3 + 13 = 17

Counting inversions: how to combine two subproblems?

Q. How to count inversions (a, b) with $a \in A$ and $b \in B$?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element $b \in B$,
  - binary search in A to find how elements in A are greater than b.

list A

7	10	18	3	14

list B

17	23	2	11	16

sort A

3	7	10	14	18

sort B

2	11	16	17	23

binary search to count inversions (a, b) with a ∈ A and b ∈ B

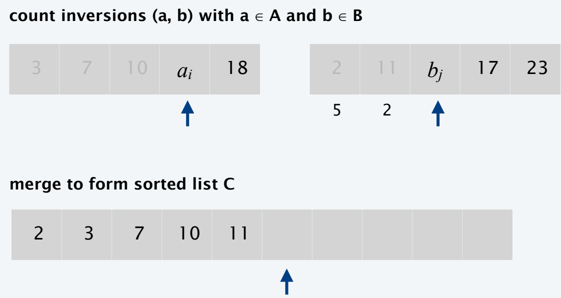
3	7	10	14	18

2	11	16	17	23
5	2	1	1	0

Counting inversions: how to combine two subproblems?

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i < b_j$, then a_i is not inverted with any element left in B.
- If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.



Counting inversions: divide-and-conquer algorithm implementation

Input. List L.
Output. Number of inversions in L and sorted list of elements L'.

SORT-AND-COUNT (L)

IF list L has one element
  RETURN $(0, L)$.

DIVIDE the list into two halves A and B.
$(r_A, A) \leftarrow$ SORT-AND-COUNT(A).
$(r_B, B) \leftarrow$ SORT-AND-COUNT(B).
$(r_{AB}, L') \leftarrow$ MERGE-AND-COUNT(A, B).

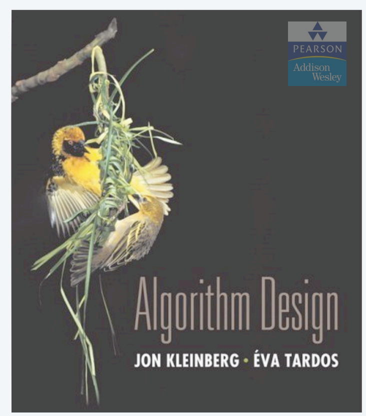
RETURN $(r_A + r_B + r_{AB},\ L')$.

Counting inversions: divide-and-conquer algorithm analysis

Proposition. The sort-and-count algorithm counts the number of inversions in a permutation of size n in $O(n \log n)$ time.

Pf. The worst-case running time $T(n)$ satisfies the recurrence:

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}$$



SECTION 5.4

5. DIVIDE AND CONQUER

- mergesort
- counting inversions
- *closest pair of points*
- median and selection

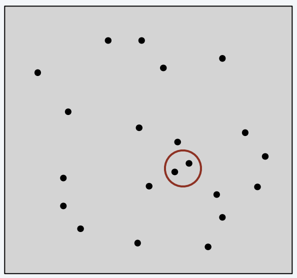
Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems



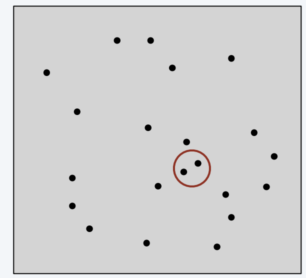
Closest pair of points

Closest pair problem. Given n points in the plane, find a pair of points with the smallest Euclidean distance between them.

Brute force. Check all pairs with $\Theta(n^2)$ distance calculations.

1d version. Easy $O(n \log n)$ algorithm if points are on a line.

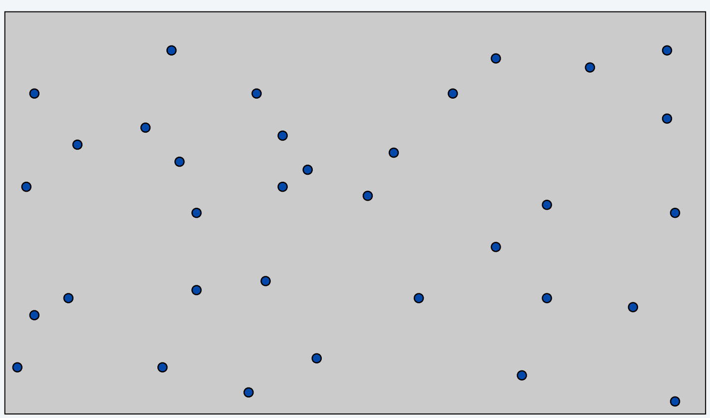
Nondegeneracy assumption. No two points have the same x-coordinate.



Closest pair of points: first attempt

Sorting solution.

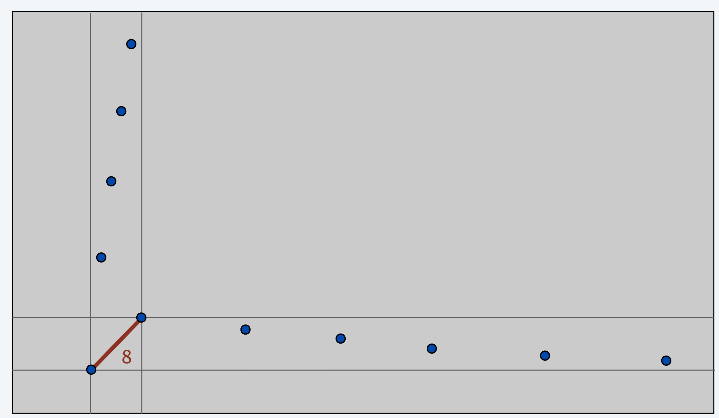
- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.



Closest pair of points: first attempt

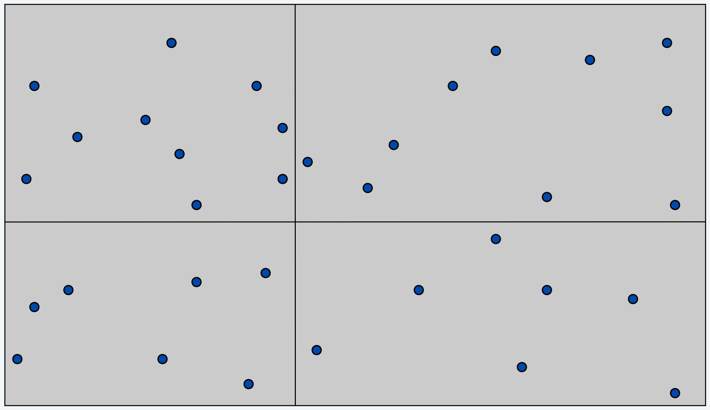
Sorting solution.

- Sort by x-coordinate and consider nearby points.
- Sort by y-coordinate and consider nearby points.



Closest pair of points: second attempt

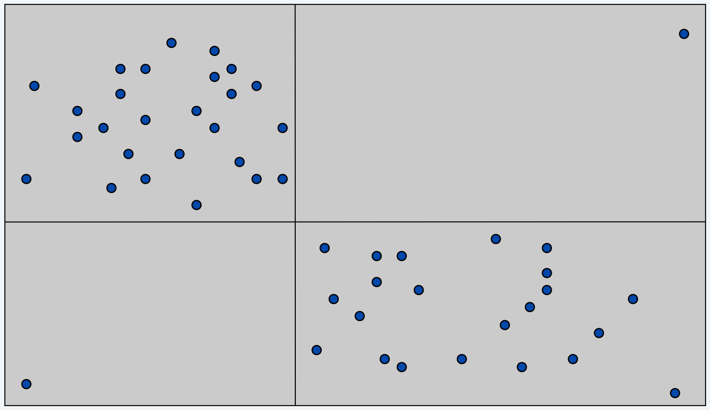
Divide. Subdivide region into 4 quadrants.



Closest pair of points: second attempt

Divide. Subdivide region into 4 quadrants.

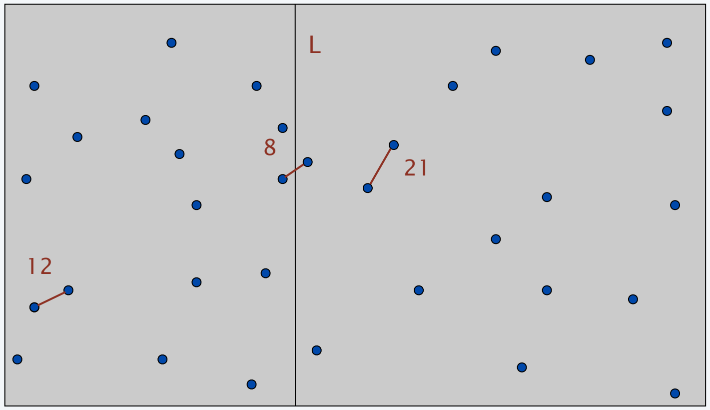
Obstacle. Impossible to ensure $n/4$ points in each piece.



Closest pair of points: divide-and-conquer algorithm

- Divide: draw vertical line L so that $n/2$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side.
- Return best of 3 solutions.

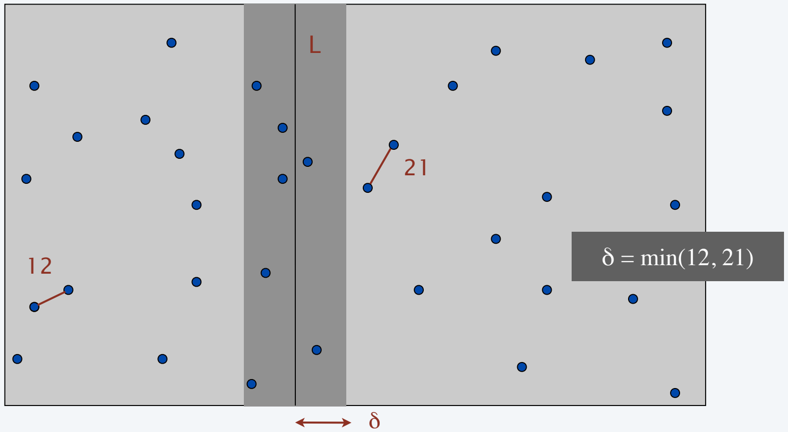
seems like $\Theta(N^2)$



How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance $< \delta$.

- Observation: only need to consider points within δ of line L.

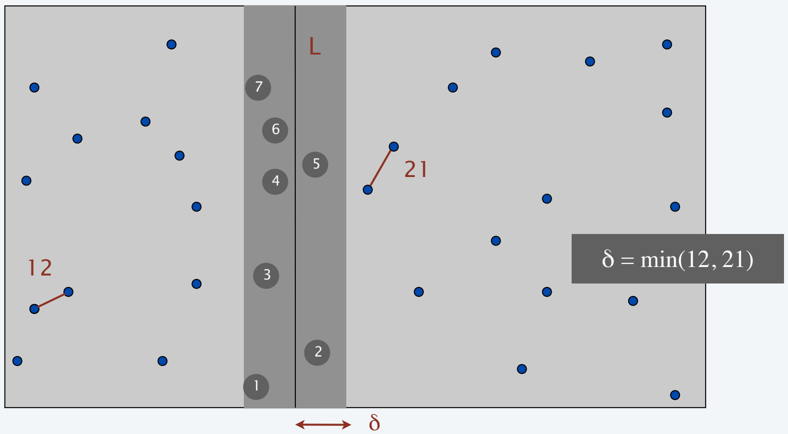


How to find closest pair with one point in each side?

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y-coordinate.
- Only check distances of those within 11 positions in sorted list!

why 11?



How to find closest pair with one point in each side?

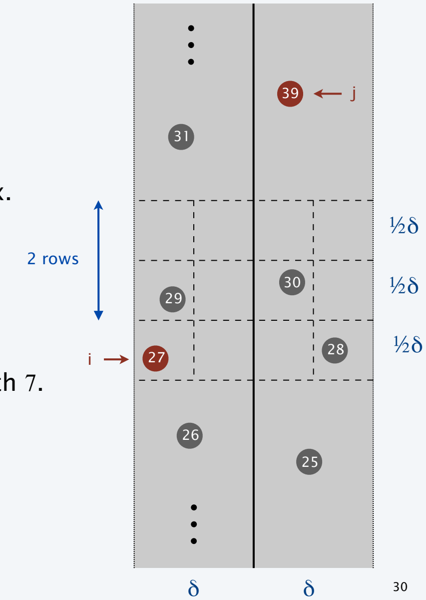
Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Claim remains true if we replace 12 with 7.



Closest pair of points: divide-and-conquer algorithm

CLOSEST-PAIR $(p_1, p_2, \ldots, p_n)$

Compute separation line L such that half the points are on each side of the line. ← $O(n \log n)$

$\delta_1 \leftarrow$ CLOSEST-PAIR (points in left half).

$\delta_2 \leftarrow$ CLOSEST-PAIR (points in right half). ← $2\,T(n/2)$

$\delta \leftarrow \min\{\delta_1, \delta_2\}$.

Delete all points further than δ from line L. ← $O(n)$

Sort remaining points by y-coordinate. ← $O(n \log n)$

Scan points in y-order and compare distance between each point and next 11 neighbors. If any of these distances is less than δ, update δ. ← $O(n)$

RETURN δ.

Closest pair of points: analysis

Theorem. The divide-and-conquer algorithm for finding the closest pair of points in the plane can be implemented in $O(n \log^2 n)$ time.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + O(n \log n) & \text{otherwise} \end{cases}$$

$(x_1 - x_2)^2 + (y_1 - y_2)^2$

Lower bound. In quadratic decision tree model, any algorithm for closest pair (even in 1D) requires $\Omega(n \log n)$ quadratic tests.

Improved closest pair algorithm

Q. How to improve to $O(n \log n)$?

A. Yes. Don't sort points in strip from scratch each time.

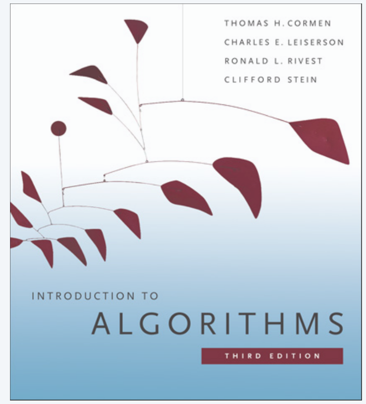
- Each recursive returns two lists: all points sorted by x-coordinate, and all points sorted by y-coordinate.
- Sort by merging two pre-sorted lists.

Theorem. [Shamos 1975] The divide-and-conquer algorithm for finding the closest pair of points in the plane can be implemented in $O(n \log n)$ time.

Pf.
$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{otherwise} \end{cases}$$

Note. See SECTION 13.7 for a randomized $O(n)$ time algorithm.

not subject to lower bound
since it uses the floor function



CHAPTER 9

5. DIVIDE AND CONQUER

- mergesort
- counting inversions
- closest pair of points
- median and selection

Median and selection problems

Selection. Given n elements from a totally ordered universe, find k^{th} smallest.

- Minimum: $k = 1$; maximum: $k = n$.
- Median: $k = \lfloor (n+1)/2 \rfloor$.
- $O(n)$ compares for min or max.
- $O(n \log n)$ compares by sorting.
- $O(n \log k)$ compares with a binary heap.

Applications. Order statistics; find the "top k"; bottleneck paths, …

Q. Can we do it with $O(n)$ compares?

A. Yes! Selection is easier than sorting.

Quickselect

3-way partition array so that:

- Pivot element p is in place.
- Smaller elements in left subarray L.
- Equal elements in middle subarray M.
- Larger elements in right subarray R.

Recur in one subarray—the one containing the k^{th} smallest element.

3-way partitioning
can be done in-place
(using at most n compares)

```
QUICK-SELECT (A, k)
Pick pivot p ∈ A uniformly at random.
(L, M, R) ← PARTITION-3-WAY (A, p).
IF       k ≤ |L|            RETURN QUICK-SELECT (L, k).
ELSE IF  k > |L| + |M|      RETURN QUICK-SELECT (R, k – |L| – |M|)
ELSE                        RETURN p.
```

Quickselect analysis

Intuition. Split candy bar uniformly $\Rightarrow$ expected size of larger piece is ¾.

$$T(n) \le T(\tfrac{3}{4} n) + n \quad \Rightarrow \quad T(n) \le 4n$$

Def. $T(n, k)$ = expected # compares to select k^{th} smallest in an array of size $\le n$.
Def. $T(n) = \max_k T(n, k)$.

Proposition. $T(n) \le 4n$.
Pf. [by strong induction on n]
- Assume true for $1, 2, \ldots, n-1$.
- $T(n)$ satisfies the following recurrence:

can assume we always recur on largest subarray
since T(n) is monotonic and
we are trying to get an upper bound

$$\begin{aligned} T(n) &\le n + 2/n\,[\; T(n/2) + \ldots + T(n-3) + T(n-2) + T(n-1) \;] \\ &\le n + 2/n\,[\; 4\,n/2 + \ldots + 4(n-3) + 4(n-2) + 4(n-1) \;] \\ &= n + 4\,(3/4\,n) \\ &= 4\,n. \quad ▪ \end{aligned}$$

tiny cheat: sum should start at $T(\lfloor n/2 \rfloor)$

Selection in worst case linear time

Goal. Find pivot element p that divides list of n elements into two pieces so that each piece is guaranteed to have $\le 7/10\, n$ elements.

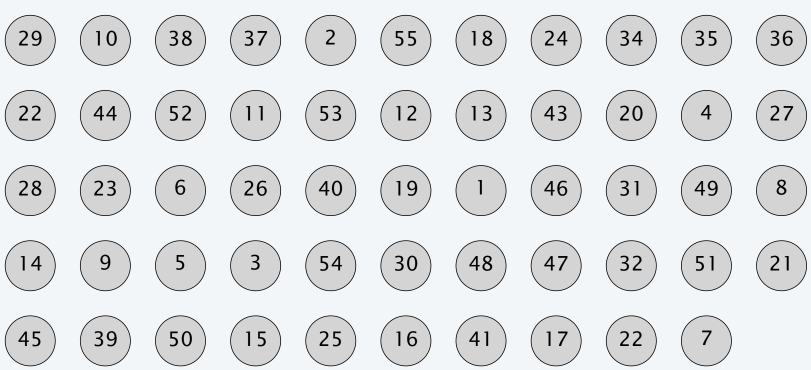
Q. How to find approximate median in linear time?
A. Recursively compute median of sample of $\le 2/10\, n$ elements.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T(7/10\, n) + T(2/10\, n) + \Theta(n) & \text{otherwise} \end{cases}$$

two subproblems
of different sizes!

Choosing the pivot element

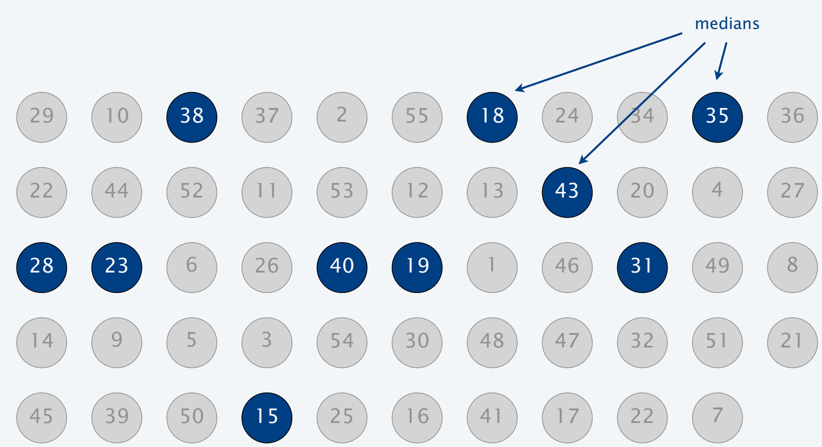
- Divide n elements into $\lfloor n/5 \rfloor$ groups of 5 elements each (plus extra).



N = 54

Choosing the pivot element

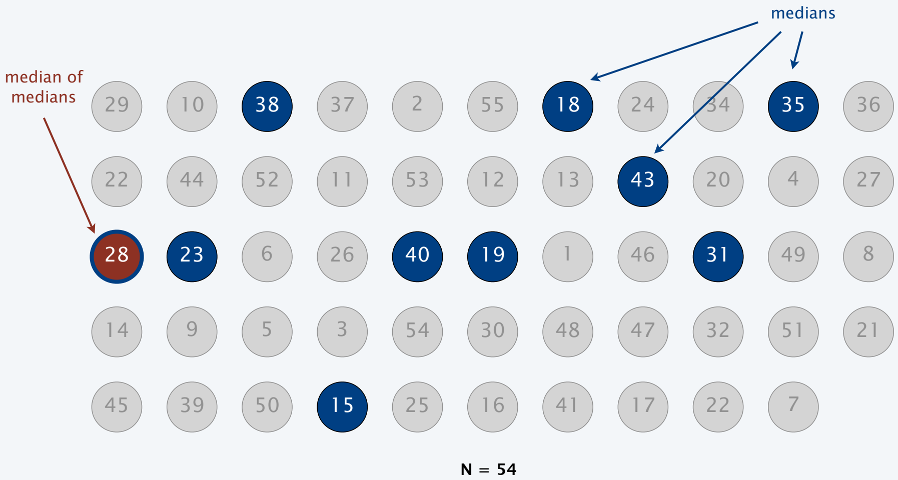
- Divide n elements into $\lfloor n/5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).



N = 54

Choosing the pivot element

- Divide n elements into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).
- Find median of each group (except extra).
- Find median of $\lfloor n / 5 \rfloor$ medians recursively.
- Use median-of-medians as pivot element.



Median-of-medians selection algorithm

MOM-SELECT (A, k)

$n \leftarrow |A|$.

IF $n < 50$ RETURN k^{th} smallest of element of A via mergesort.

Group A into $\lfloor n / 5 \rfloor$ groups of 5 elements each (plus extra).

$B \leftarrow$ median of each group of 5.

$p \leftarrow$ MOM-SELECT$(B, \lfloor n / 10 \rfloor)$ ← median of medians

$(L, M, R) \leftarrow$ PARTITION-3-WAY (A, p).

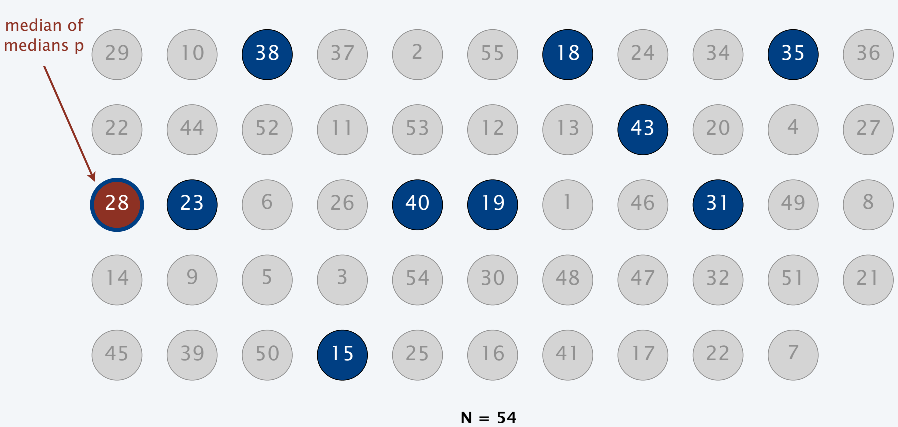
IF $k \le |L|$ RETURN MOM-SELECT (L, k).

ELSE IF $k > |L| + |M|$ RETURN MOM-SELECT $(R, k - |L| - |M|)$

ELSE RETURN p.

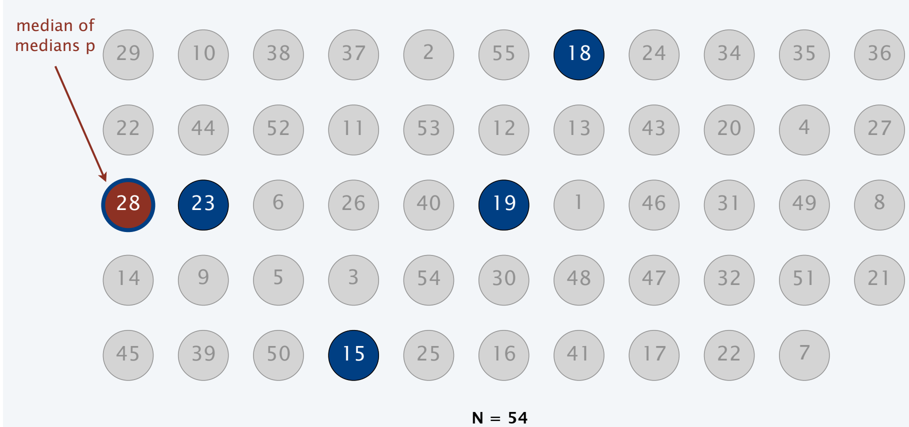
Analysis of median-of-medians selection algorithm

- At least half of 5-element medians $\le p$.



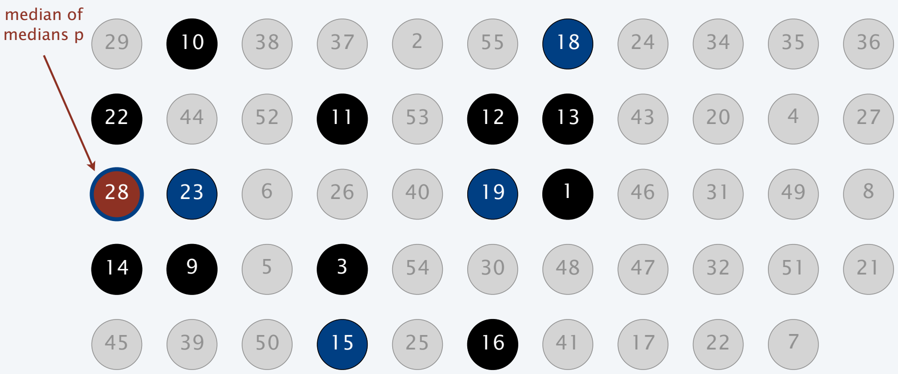
Analysis of median-of-medians selection algorithm

- At least half of 5-element medians $\le p$.
- At least $\lfloor \lfloor n / 5 \rfloor / 2 \rfloor = \lfloor n / 10 \rfloor$ medians $\le p$.



Analysis of median-of-medians selection algorithm

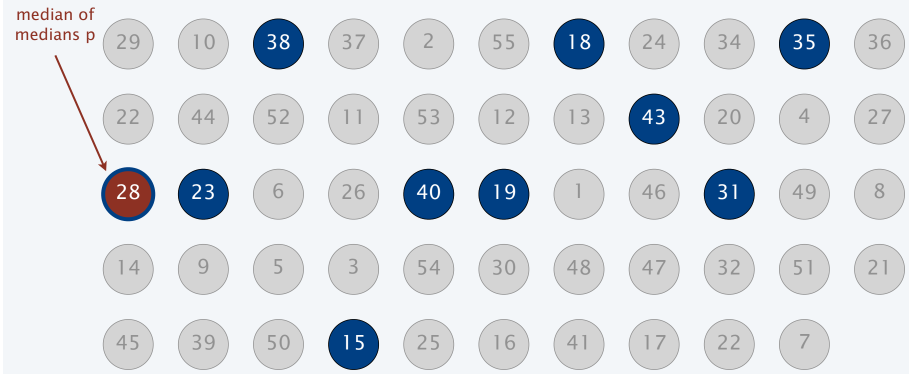
- At least half of 5-element medians $\leq\ p$.
- At least $\lfloor \lfloor n / 5 \rfloor / 2 \rfloor = \lfloor n / 10 \rfloor$ medians $\leq\ p$.
- At least $3\ \lfloor n / 10 \rfloor$ elements $\leq\ p$.



N = 54

Analysis of median-of-medians selection algorithm

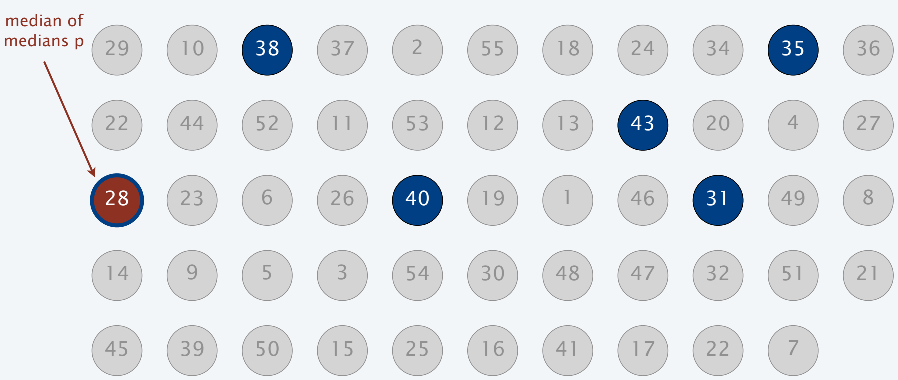
- At least half of 5-element medians $\geq\ p$.



N = 54

Analysis of median-of-medians selection algorithm

- At least half of 5-element medians $\geq\ p$.
- Symmetrically, at least $\lfloor n / 10 \rfloor$ medians $\geq\ p$.



N = 54

Analysis of median-of-medians selection algorithm

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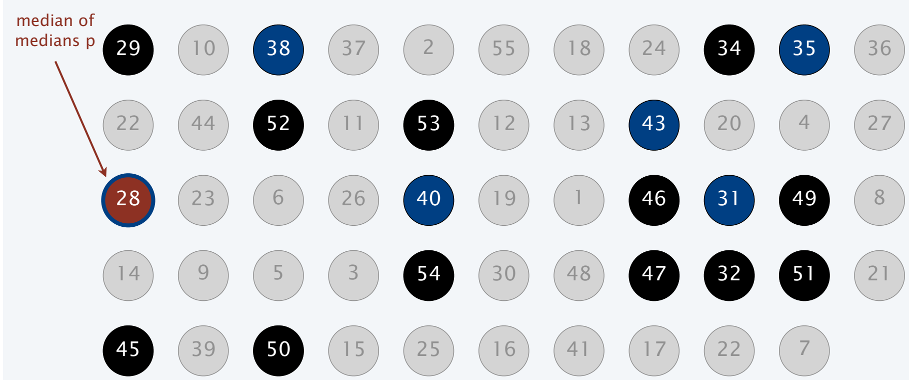

N = 54

Median-of-medians selection algorithm recurrence

Median-of-medians selection algorithm recurrence.

- Select called recursively with $\lfloor n / 5 \rfloor$ elements to compute MOM p.
- At least $3 \lfloor n / 10 \rfloor$ elements $\leq p$.
- At least $3 \lfloor n / 10 \rfloor$ elements $\geq p$.
- Select called recursively with at most $n - 3 \lfloor n / 10 \rfloor$ elements.

Def. $C(n)$ = max # compares on an array of n elements.

$$C(n) \leq \underbrace{C(\lfloor n/5 \rfloor)}_{\text{median of medians}} + \underbrace{C(n - 3 \lfloor n/10 \rfloor)}_{\text{recursive select}} + \underbrace{\tfrac{11}{5}\, n}_{\substack{\text{computing median of 5 (6 compares per group)} \\ \text{partitioning (n compares)}}}$$

Now, solve recurrence.

- Assume n is both a power of 5 and a power of 10?
- Assume $C(n)$ is monotone nondecreasing?

Median-of-medians selection algorithm recurrence

Analysis of selection algorithm recurrence.

- $T(n)$ = max # compares on an array of $\leq n$ elements.
- $T(n)$ is monotone, but $C(n)$ is not!

$$T(n) \leq \begin{cases} 6n & \text{if } n < 50 \\ T(\lfloor n/5 \rfloor) + T(n - 3 \lfloor n/10 \rfloor) + \tfrac{11}{5}\, n & \text{otherwise} \end{cases}$$

Claim. $T(n) \leq 44\, n$.

- Base case: $T(n) \leq 6\, n$ for $n < 50$ (mergesort).
- Inductive hypothesis: assume true for $1, 2, \ldots, n-1$.
- Induction step: for $n \geq 50$, we have:

$$\begin{aligned} T(n) &\leq T(\lfloor n/5 \rfloor) + T(n - 3 \lfloor n/10 \rfloor) + 11/5\ n \\ &\leq 44 (\lfloor n/5 \rfloor) + 44 (n - 3 \lfloor n/10 \rfloor) + 11/5\ n \\ &\leq 44 (n/5) + 44\, n - 44 (n/4) + 11/5\ n \quad \longleftarrow \text{for } n \geq 50,\ 3 \lfloor n/10 \rfloor \geq n/4 \\ &= 44\, n. \quad \blacksquare \end{aligned}$$

Linear-time selection postmortem

Proposition. [Blum-Floyd-Pratt-Rivest-Tarjan 1973] There exists a compare-based selection algorithm whose worst-case running time is $O(n)$.

Time Bounds for Selection

by .

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Theory.

- Optimized version of BFPRT: $\leq 5.4305\, n$ compares.
- Best known upper bound [Dor-Zwick 1995]: $\leq 2.95\, n$ compares.
- Best known lower bound [Dor-Zwick 1999]: $\geq (2 + \varepsilon)\, n$ compares.

Linear-time selection postmortem

Proposition. [Blum-Floyd-Pratt-Rivest-Tarjan 1973] There exists a compare-based selection algorithm whose worst-case running time is $O(n)$.

Time Bounds for Selection

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Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than 5.4305 n comparisons are ever required. This bound is improved for

Practice. Constant and overhead (currently) too large to be useful.

Open. Practical selection algorithm whose worst-case running time is $O(n)$.